Research Article

Poincaré-Type Inequalities for the Composite Operator in $L^{\mathcal{A}}$ -Averaging Domains

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We first establish the local Poincaré inequality with $L^{\mathcal{A}}$ -averaging domains for the composition of the sharp maximal operator and potential operator, applied to the nonhomogeneous A -harmonic equation. Then, according to the definition of $L^{\mathcal{A}}$ -averaging domains and relative properties, we demonstrate the global Poincaré inequality with $L^{\mathcal{A}}$ -averaging domains. Finally, we give some illustrations for these theorems.

1. Introduction

Poincaré inequality applied to differential forms has a vital role in PDEs, nonlinear analysis, and other related fields. With the further research conducted, we have established various versions of Poincaré inequality under different conditions. From [1–8], we have obtained the Poincaré inequality for the solution to the A -harmonic equation in uniformly bounded domain, John domains, and L^s -averaging domains. Nevertheless, most of these Poincaré inequalities are developed in L^s -averaging domains. In this paper, we will establish the Poincaré inequality for the composition of the sharp maximal operator and potential operator in $L^{\mathcal{A}}$ -averaging domains. As we all know, both the uniformly bounded domain and John domains are special L^s -averaging domains, and the L^s -averaging domains are also particular $L^{\mathcal{A}}$ -averaging domains, so the following results are the generalizations of the Poincaré inequality in L^s -averaging domains.

For convenience, we firstly introduce some notations and terminologies. Except for special instructions, $E \subseteq \mathbb{R}^m$ is a bounded domain, $|E|$ denotes the Lebesgue measure of E , and $m \geq 2$. The constant K and C can be varied at each step of the proof. Suppose that B_x^r is a ball, with a radius r , centered at x . For any $\rho > 0$, $B \subseteq E$ and $\rho B \subseteq E$ have the same center and satisfy $\text{diam}(\rho B) = \rho \text{diam}(B)$. Let $\Lambda^l(\mathbb{R}^m)$ be the space of all

l -forms in $\mathbb{R}^m$, which is expanded by the exterior product of $e^{\mathcal{B}} = e^{i_1} \wedge e^{i_2} \wedge \cdots \wedge e^{i_l}$, where $\mathcal{B} = (i_1, \dots, i_l)$, $1 \leq i_1 < \cdots < i_l \leq m$, $l = 1, 2, \dots, m$. $C^\infty(\Lambda^l E)$ is the space of a smooth l -form on E . We use $D'(E, \Lambda^l)$ to denote the space of all differential l -forms on E ; that is, $w(x)$ belongs to $D'(E, \Lambda^l)$ if and only if there exist some l th-differential functions $w_{\mathcal{B}}$ in E such that $w(x) = \sum_{\mathcal{B}} w_{\mathcal{B}}(x) dx_{\mathcal{B}} = \sum w_{i_1 i_2 \dots i_l}(x) dx_{i_1} \wedge dx_{i_2} \wedge \cdots \wedge dx_{i_l}$. $L^p(E, \Lambda^l)$ is a Banach space with the norm equipped by $\|w(x)\|_{p,E} = (\int_E |w(x)|^p dx)^{1/p}$, where $w(x) \in D'(E, \Lambda^l)$ and every coefficient function $w_{\mathcal{B}} \in L^p(E)$, $0 < p < \infty$. In fact, $w(x)$ on E is the Schwartz distribution. If $\omega(x) > 0$ a.e. and $\omega(x) \in L^1_{\text{loc}}(\mathbb{R}^m)$, $\omega(x)$ is called a weight. Let $d\mu = \omega(x)dx$; then $L^p(E, \Lambda^l, \omega)$ is a weighted Banach space with the norm expressed by $\|w(x)\|_{p,E,\omega} = (\int_E |w(x)|^p \omega(x) dx)^{1/p}$. In this notation, the exterior derivative is denoted by d and Hodge codifferential operator is expressed by d^* . Search [9] for more details.

Considering our purpose, we intend to give a brief discussion about the A -harmonic equation for the differential form. The following equation is called a nonhomogeneous A -harmonic equation:

$$d^* A(x, dw) = B(x, dw), \quad (1)$$

where $A : E \times \wedge^l(\mathbb{R}^m) \rightarrow \wedge^l(\mathbb{R}^m)$ and $B : E \times \wedge^l(\mathbb{R}^m) \rightarrow \wedge^{l-1}(\mathbb{R}^m)$ satisfy the conditions:

$$\begin{aligned} |A(x, \xi)| &\leq a|\xi|^{p-1}, & A(x, \xi) \cdot \xi &\geq |\xi|^p, \\ |B(x, \xi)| &\leq b|\xi|^{p-1}, \end{aligned} \quad (2)$$

for almost every $x \in E$ and all $\xi \in \wedge^l(\mathbb{R}^m)$. Here, $a, b > 0$ are constants and $1 < p < \infty$ is a fixed exponent associated with (1). If $B = 0$, the equation $d^*A(x, dw) = 0$ is called a homogenous A -harmonic equation. See [9] for more information.

In order to describe it easily, we first give some definitions in this part.

Definition 1. Let $E \subseteq \mathbb{R}^m$ be a bounded domain and $w(x) \in L^p(E, \wedge^l)$; the sharp maximal operator $\mathbb{M}_s^\sharp$ is equipped with

$$\begin{aligned} \mathbb{M}_s^\sharp(w) &= \mathbb{M}_s^\sharp w = \mathbb{M}_s^\sharp w(x) \\ &= \sup_{r>0} \left(\frac{1}{|B_x^r|} \int_{B_x^r} |w(t) - w_{B_x^r}|^s dt \right)^{1/s}, \end{aligned} \quad (3)$$

where B_x^r is the ball of radius r , centered at x , $1 \leq s \leq p$, $p \geq 1$.

Especially, if we take $s = 1$, denote $\mathbb{M}_s^\sharp \triangleq \mathbb{M}^\sharp$.

Definition 2 (see [10]). Suppose that $w(x)$ is a differential l -form; the potential operator P is expressed by

$$Pw(x) = \sum_{\mathcal{B}} \int_E K(x, y) w_{\mathcal{B}}(y) dy dx_{\mathcal{B}}, \quad (4)$$

where the nonnegative and measurable function $K(x, y)$, defined on the set $\{(x, y) \mid x \neq y, x, y \in \mathbb{R}^m\}$, is a kernel function, and the summation is over all ordered l -tuple $\mathcal{B}$.

Definition 3. Take an increasingly continuous function $\mathcal{A} : [0, +\infty) \rightarrow [0, +\infty)$ as a convex function with $\mathcal{A}(0) = 0$, and $E \subseteq \mathbb{R}^m$ is a bounded domain, for any $w(x) \in L^p(E)$; the Orlicz norm for differential form is denoted by

$$\|w\|_{L_{E,\mu}^{\mathcal{A}}} = \inf \left\{ \lambda > 0 \mid \frac{1}{\mu(E)} \int_E \mathcal{A}(\lambda^{-1}|u|) d\mu < 1 \right\}, \quad (5)$$

where measure μ satisfies $d\mu = \omega(x)dx$, $\omega(x)$ is a weight.

We call $\mathcal{A}$ an Orlicz function if $\mathcal{A} : [0, +\infty) \rightarrow [0, +\infty)$ is an increasingly continuous function and satisfies $\mathcal{A}(0) = 0$ and $\mathcal{A}(\infty) = \infty$. Meanwhile, if the Orlicz function $\mathcal{A}(t)$ is a convex function, it is called a Young function.

Based on the above definition, we get the notation of $L^{\mathcal{A}}$ -averaging domains.

Definition 4 (see [3]). Let $\mathcal{A}$ be a Young function; the proper domain $E \subseteq \mathbb{R}^m$ is called the $L^{\mathcal{A}}$ -averaging domains if $\mu(E) <$

∞ and there exists a constant $C > 0$ such that for any $B_0 \subseteq E$ and $\mathcal{A}(|w|) \in L_{\text{loc}}^1(E, \mu)$, w satisfies

$$\begin{aligned} \frac{1}{\mu(E)} \int_E \mathcal{A}(\tau |w - w_{B_0}|) d\mu \\ \leq C \sup_{4B \subset E} \frac{1}{\mu(B)} \int_B \mathcal{A}(\sigma |w - w_B|) d\mu, \end{aligned} \quad (6)$$

where the measure μ is denoted by $d\mu = \omega(x)dx$, $\omega(x)$ is a weight, σ and τ are constants with $0 < \tau, \sigma < 1$, and the supremum is over all balls $B \subset E$ with $4B \subset E$.

Notice that if we let $\mathcal{A}(t) = t^s$, $L^{\mathcal{A}}$ -averaging domains become the L^s -averaging domains, so $L^{\mathcal{A}}$ -averaging domains are the generalization of L^s -averaging domains.

Definition 5 (see [11]). We call $w(x) \in D^l(E, \wedge^l)$ belongs to the WRH($\wedge^l, E$)-class, $l = 0, 1, \dots, m$, if for any constants $0 < s, t < \infty$ and any ball $B \subset E$ with $\rho B \subset E$, there exists a constant $C > 0$ such that $w(x)$ satisfies

$$\|w\|_{s,B} \leq C|B|^{(t-s)/st} \|w\|_{t,\rho B}, \quad (7)$$

where $\rho > 1$ is a constant.

Remark 6. If $w(x)$ is a solution to the A -harmonic equation, we can prove that $w(x)$ belongs to the WRH($\wedge^l, E$)-class.

2. Main Results

Before the main results are given, we need to impose some restrictions on the kernel function $K(x, y)$ and Young function $\mathcal{A}$. Firstly, let the kernel function satisfy the standard estimates; it is equal to say that if there exist $0 < \delta < 1$ and a constant $c > 0$ such that for any point $z \in \{z : |x - z| < (1/2)|x - y|, x, y \in \mathbb{R}^m\}$, the kernel function $K(x, y)$ satisfies that

- (1) $K(x, y) \leq c|x - y|^{-m}, x \neq y$;
- (2) $|K(x, y) - K(z, y)| \leq c|x - z|^\delta |x - y|^{-m-\delta}, x \neq y$;
- (3) $|K(y, x) - K(y, z)| \leq c|x - z|^\delta |x - y|^{-m-\delta}, x \neq y$,

where function $K(x, y) : \mathbb{R}^m \times \mathbb{R}^m \rightarrow \mathbb{R}, m \geq 1$.

With regard to the Young function $\mathcal{A}$, we let the Young function $\mathcal{A}$ belong to the $G(p, q, C)$ -class ($1 \leq p < q < \infty, C \geq 1$); that is, for any $t > 0$, the Young function $\mathcal{A}$ satisfies that

- (1) $1/C \leq \mathcal{A}(t^{1/p})/f(t) \leq C$;
- (2) $1/C \leq \mathcal{A}(t^{1/q})/g(t) \leq C$,

where f and g are the increasingly convex and concave functions defined on $[0, \infty]$, respectively.

Now, we establish these two important theorems based on the above conditions.

Theorem 7. Suppose that the Young function $\mathcal{A}$ belongs to the $G(p, q, C)$ -class, $w \in C^\infty(\wedge^l E)$ is a solution to the nonhomogenous A -harmonic equation, the sharp maximal

operator is noted by $\mathbb{M}_s^\sharp$, P is the potential operator with its kernel function $K(x, y)$ satisfying the standard estimates, $1 \leq s < p$, $q < \infty$, and the bounded subset $E \subseteq \mathbb{R}^m$ is the $L^{\mathcal{A}}$ -averaging domains. Then, for any ball $B \subseteq E$, one gets

$$\left\| \mathbb{M}_s^\sharp(P(w)) - \left(\mathbb{M}_s^\sharp(P(w)) \right)_B \right\|_{L_{\rho B}^{\mathcal{A}}} \leq K \|w\|_{L_{\rho B}^{\mathcal{A}}}, \quad (8)$$

where B and $\rho B \subseteq E$ and the constant $\rho > 1$.

Based on the above theorem, we can establish the following theorem for the global Poincaré inequality in $L^{\mathcal{A}}$ -averaging domains.

Theorem 8. Suppose that the Young function $\mathcal{A}$ belongs to the $G(p, q, C)$ -class, $w \in C^\infty(\Lambda^1 E)$ is a solution to the nonhomogeneous A -harmonic equation, the sharp maximal operator is denoted by $\mathbb{M}_s^\sharp$, P is the potential operator with its kernel function $K(x, y)$ satisfying the standard estimates, $1 \leq s < p$, $q < \infty$, and the bounded subset $E \subseteq \mathbb{R}^m$ is the $L^{\mathcal{A}}$ -averaging domains. Then, one has

$$\left\| \mathbb{M}_s^\sharp(P(w)) - \left(\mathbb{M}_s^\sharp(P(w)) \right)_{B_0} \right\|_{L_E^{\mathcal{A}}} \leq K \|w\|_{L_E^{\mathcal{A}}}, \quad (9)$$

where $B_0 \subseteq B$ is a fixed ball, which appears in Definition 4.

3. Preliminary Results

For proving the theorems in Section 2, we will show and demonstrate some lemmas in this part.

Lemma 9 (see [9]). Let $0 < p, q < \infty$, and $1/t = (1/p) + (1/q)$, if f and g are the measurable functions defined on $\mathbb{R}^m$, then

$$\|fg\|_{t,I} \leq \|f\|_{p,I} \cdot \|g\|_{q,I}, \quad (10)$$

for any $I \subseteq \mathbb{R}^m$.

Lemma 10 (see [5]). Let P be the potential operator applied on a differential form with $E \subseteq \mathbb{R}^m$, $w(x) \in WRH(\Lambda^1, E)$, and assume that the weight $\omega(x)$ belongs to $A(\alpha, \beta, E)$ with $\alpha, \beta > 0$. Then, there exists a constant C , independent of $w(x)$ such that

$$\|P(w) - (P(w))_B\|_{s,B,\omega} \leq C |B| \text{diam}(B) \|w\|_{s,B,\omega}, \quad (11)$$

for any $B \subseteq E$, where $s > 1$ is a constant.

Remark 11. If we take $\omega(x) \equiv 1$, we get

$$\|P(w) - (P(w))_B\|_{s,B} \leq C |B| \text{diam}(B) \|w\|_{s,B}. \quad (12)$$

Lemma 12 (see [3]). Take Ψ defined on $[0, +\infty)$ to be a strictly increasing convex function, $\Psi(0) = 0$, and $E \subseteq \mathbb{R}^m$ is a domain. Assume that $w(x) \in D'(E, \Lambda^1)$ satisfies $\Psi(|w|) \in L^1(E, \mu)$ and, for any constant c ,

$$\mu \{x \in E : |w - c| > 0\} > 0, \quad (13)$$

where μ is a Radon measure defined by $d\mu(x) = \omega(x)dx$ with a weight $\omega(x)$; then for any $a > 0$, one obtains

$$\int_E \Psi\left(\frac{a}{2} |w - w_E|\right) d\mu \leq \int_E \Psi(a |w|) d\mu. \quad (14)$$

Lemma 13. If $\omega(x) \in A_r(E)$, then there exist constants $\alpha > 1$ and K , not dependent on ω , such that

$$\|\omega\|_{\alpha,B} \leq K |B|^{(1/\alpha)-1} \|\omega\|_{1,B}, \quad (15)$$

for all balls B contained in E .

Lemma 14. The sharp maximal operator $\mathbb{M}_s^\sharp$ is denoted by Definition 1, and the potential operator P is defined by Definition 2 with the kernel function $K(x, y)$ satisfying the standard estimates, $w(x) \in L^t(E, \Lambda^1) \cap C^\infty(\Lambda^1 E)$ ($l = 1, 2, \dots, m$), $t \geq 1$. Then, there exists a constant $K > 0$, independent of w , such that

$$\int_B \left| \mathbb{M}_s^\sharp(P(w)) - \left(\mathbb{M}_s^\sharp(P(w)) \right)_B \right|^t dx \leq K |B|^{1+(1/m)} \|w\|_{t,B}, \quad (16)$$

for all balls $B \subseteq E$.

Proof. Let B be a ball in E , using Lemma 10 on any $B_x^r \subset B$, we have

$$\begin{aligned} & \left(\frac{1}{|B_x^r|} \int_{B_x^r} |P(w) - (P(w))_{B_x^r}|^s dx \right)^{1/s} \\ & \leq K |B_x^r| \text{diam}(B_x^r) |B_x^r|^{-1/s} \|w\|_{s,B_x^r} \\ & \leq K |B_x^r|^{1-(1/s)} \text{diam}(B_x^r) \|w\|_{s,B} \\ & \leq K |B|^{1-(1/s)+(1/m)} \|w\|_{s,B}. \end{aligned} \quad (17)$$

From Lemma 14 in [7], it follows that

$$\|w\|_{s,B} \leq |B|^{(1/s)-(1/t)} \|w\|_{t,B}, \quad (18)$$

where $0 < s \leq t < \infty$. Substituting (18) into (17) yields

$$\begin{aligned} & \left(\frac{1}{|B_x^r|} \int_{B_x^r} |P(w) - (P(w))_{B_x^r}|^s dx \right)^{1/s} \\ & \leq K |B|^{1-(1/t)+(1/m)} \|w\|_{t,B}. \end{aligned} \quad (19)$$

Taking the supremum for r , we get that

$$\begin{aligned} & \sup_{r>0} \left(\frac{1}{|B_x^r|} \int_{B_x^r} |P(w) - (P(w))_{B_x^r}|^s dx \right)^{1/s} \\ & \leq \sup_{r>0} K |B|^{1-(1/t)+(1/m)} \|w\|_{t,B} \\ & = K |B|^{1-(1/t)+(1/m)} \|w\|_{t,B}. \end{aligned} \quad (20)$$

That is,

$$\mathbb{M}_s^\sharp(P(w)) \leq K |B|^{1-(1/t)+(1/m)} \|w\|_{t,B}. \quad (21)$$

According to the definition of $L^t(E)$ norm and formula (21), it yields

$$\begin{aligned} \|\mathbb{M}_s^\#(P(w))\|_{t,B} &= \left(\int_B |\mathbb{M}_s^\#(P(w))|^t dx \right)^{1/t} \\ &\leq \left(\int_B |K|B|^{1-(1/t)+(1/m)} \|w\|_{t,B}^t dx \right)^{1/t} \quad (22) \\ &= K|B|^{1+(1/m)} \|w\|_{t,B}. \end{aligned}$$

Choosing $\Psi(t) = 2^t$, $a = 2$, and $\omega(x) \equiv 1$ in Lemma 12, we have

$$\begin{aligned} &\|\mathbb{M}_s^\#(P(w)) - (\mathbb{M}_s^\#(P(w)))_B\|_{t,B} \\ &= \left(\int_B |\mathbb{M}_s^\#(P(w)) - (\mathbb{M}_s^\#(P(w)))_B|^t dx \right)^{1/t} \quad (23) \\ &\leq \left(\int_B 2^t |\mathbb{M}_s^\#(P(w))|^t dx \right)^{1/t} \\ &\leq K|B|^{1+(1/m)} \|w\|_{t,B}. \end{aligned}$$

The proof of Lemma 14 has been completed. $\square$

Lemma 15. Suppose that $w(x) \in C^\infty(\Lambda^1 E)$ is a solution to the A -harmonic equation, $E \subset \mathbb{R}^m$ is a bounded domain, P is a potential operator with the kernel function $K(x, y)$ satisfying the standard estimates, and the sharp maximal operator $\mathbb{M}_s^\#$ is expressed by Definition 1, $1 \leq s < p$, $q < \infty$. Then, there exists a constant $K > 0$, such that

$$\|\mathbb{M}_s^\#(P(w)) - (\mathbb{M}_s^\#(P(w)))_B\|_{q,B} \leq K\|w\|_{p,\rho B}, \quad (24)$$

where the ball $B \subset E$ with $\rho B \subset E$, constant $\rho > 1$, the measure μ is defined by $d\mu = \omega(x)dx$, weight $\omega(x) \in A_r(E)$, $\omega(x) \geq \delta > 0$, for some $r > 1$ and a constant δ .

Proof. Because $1/q = ((\alpha - 1)/\alpha q) + (1/\alpha q)$, for any B with ρB contained in E , using Lemmas 9 and 14, we have

$$\begin{aligned} &\left(\int_B |\mathbb{M}_s^\#(P(w)) - (\mathbb{M}_s^\#(P(w)))_B|^q d\mu \right)^{1/q} \\ &= \left(\int_B (|\mathbb{M}_s^\#(P(w)) - (\mathbb{M}_s^\#(P(w)))_B| \omega^{1/q})^q dx \right)^{1/q} \\ &\leq \left(\int_B (|\mathbb{M}_s^\#(P(w)) - (\mathbb{M}_s^\#(P(w)))_B|^{\alpha q/(\alpha-1)}) dx \right)^{(\alpha-1)/\alpha q} \\ &\quad \times \left(\int_B \omega^\alpha dx \right)^{1/q\alpha} \\ &\leq K|B|^{(1-\alpha)/\alpha q} \|\omega\|_{1,B}^{1/q} \|\mathbb{M}_s^\#(P(w)) - (\mathbb{M}_s^\#(P(w)))_B\|_{q\alpha/(\alpha-1),B}. \quad (25) \end{aligned}$$

According to Lemma 14 and Definition 5, letting $p = r \times z$, we get

$$\begin{aligned} &\|\mathbb{M}_s^\#(P(w)) - (\mathbb{M}_s^\#(P(w)))_B\|_{\alpha q/(\alpha-1),B} \\ &\leq K|B|^{1+(1/m)} \|w\|_{\alpha q/(\alpha-1),B} \quad (26) \\ &= K|B|^{1+(1/m)+((z(\alpha-1)-\alpha q)/\alpha q z)} \|w\|_{z,\rho B}. \end{aligned}$$

Therefore, we know that

$$\begin{aligned} &\left(\int_B |\mathbb{M}_s^\#(P(w)) - (\mathbb{M}_s^\#(P(w)))_B|^q d\mu \right)^{1/q} \\ &\leq K|B|^{(1-\alpha)/q\alpha} \|\omega\|_{1,B}^{1/q} |B|^{1+(1/m)+((z(\alpha-1)-\alpha q)/\alpha q z)} \|w\|_{z,\rho B} \quad (27) \\ &= K|B|^{1+(1/m)-(1/z)} (\mu(B))^{1/q} \|w\|_{z,\rho B}. \end{aligned}$$

Because of $1/z = (1/p) + ((p-z)/zp)$, and using generalized Hölder's inequality, we get

$$\begin{aligned} \|w\|_{z,\rho B} &= \left(\int_{\rho B} (|w|\omega^{1/p} \omega^{-1/p})^z dx \right)^{1/z} \\ &\leq \left(\int_{\rho B} |w|^p \omega dx \right)^{1/p} \cdot \left(\int_{\rho B} \omega^{1/(r-1)} dx \right)^{(r-1)/p} \quad (28) \\ &= \|w\|_{p,\rho B,\omega} \|\omega^{-1}\|_{1/(r-1),\rho B}^{1/p}. \end{aligned}$$

In the light of $\omega \in A_r(E)$, finding details in [9], we know

$$\sup_{B \subset E} \left(\frac{1}{|B|} \int_B \omega dx \right) \left(\frac{1}{|B|} \int_B \left(\frac{1}{\omega} \right)^{1/(r-1)} dx \right)^{1/(r-1)} < \infty. \quad (29)$$

Therefore, we can see that

$$\begin{aligned} \left\| \frac{1}{\omega} \right\|_{1/(r-1),\rho B}^{1/p} &= \|\omega\|_{1,\rho B}^{-1/p} \cdot \|\omega\|_{1,\rho B}^{1/p} \cdot \left\| \frac{1}{\omega} \right\|_{1/(r-1),\rho B}^{1/p} \\ &\leq K(\mu(\rho B))^{-1/p} |\rho B|^{1/z} \quad (30) \\ &\leq K(\mu(\rho B))^{-1/p} |B|^{1/z}. \end{aligned}$$

In addition, considering $\omega \geq \delta > 0$, so we have that

$$\mu(\rho B) = \int_{\rho B} d\mu \geq \int_{\rho B} \delta dx = \delta |\rho B|. \quad (31)$$

Combining (27), (28), and (31), we obtain

$$\begin{aligned} &\left(\int_B |\mathbb{M}_s^\#(P(w)) - (\mathbb{M}_s^\#(P(w)))_B|^q d\mu \right)^{1/q} \\ &\leq K|\rho B|^{1+(1/m)-(1/p)} (\mu(B))^{1/q} \left(\int_{\rho B} |w|^{1/p} d\mu \right)^{1/q} \quad (32) \\ &\leq K|E|^{1+(1/m)-(1/p)} (\mu(E))^{1/q} \left(\int_{\rho B} |w|^{1/p} d\mu \right)^{1/q} \\ &\leq K\|w\|_{p,\rho B}. \end{aligned}$$

Therefore, we finish the proof of this lemma. $\square$

4. Demonstration of Main Results

According to the above definitions and lemmas, we will prove these two theorems in detail. Firstly, let us prove Theorem 7.

Proof of Theorem 7. Let B and $\rho B \subseteq E$, f and g are, respectively, convex and concave increasing function, use Lemma 15, and take $\omega(x) \equiv 1$; then

$$\begin{aligned} & \mathcal{A} \left(\lambda^{-1} \left(\int_B \left| \mathbb{M}_s^\#(P(w)) - (\mathbb{M}_s^\#(P(w)))_B \right|^q dx \right) \right)^{1/q} \\ & \leq \mathcal{A} \left(\lambda^{-1} K \left(\int_{\rho B} |w|^p dx \right)^{1/p} \right) \\ & = \mathcal{A} \left(\left(\lambda^{-p} K^p \int_{\rho B} |w|^p dx \right)^{1/p} \right) \\ & \leq C f \left(\lambda^{-p} K^p \int_{\rho B} |w|^p dx \right) \\ & \leq C \int_{\rho B} f \left(\lambda^{-p} K^p |w|^p \right) dx. \end{aligned} \quad (33)$$

Because $f(t) \leq C\mathcal{A}(t^{1/p})$, we know that

$$\int_{\rho B} f \left(\lambda^{-p} K^p |w|^p \right) dx \leq C \int_{\rho B} \mathcal{A} \left(K \lambda^{-1} |w| \right) dx. \quad (34)$$

Furthermore, we obtain

$$\begin{aligned} & \mathcal{A} \left(\lambda^{-1} \left(\int_B \left| \mathbb{M}_s^\#(P(w)) - (\mathbb{M}_s^\#(P(w)))_B \right|^q dx \right) \right)^{1/q} \\ & \leq C \int_{\rho B} \mathcal{A} \left(K \lambda^{-1} |w| \right) dx. \end{aligned} \quad (35)$$

For function g , using Jensen's inequality, we get

$$\begin{aligned} & \int_B \mathcal{A} \left(\lambda^{-1} \left| \mathbb{M}_s^\#(P(w)) - (\mathbb{M}_s^\#(P(w)))_B \right| \right) dx \\ & \leq g \left(\int_B g^{-1} \left(\mathcal{A} \left(\lambda^{-1} \left| \mathbb{M}_s^\#(P(w)) - (\mathbb{M}_s^\#(P(w)))_B \right| \right) \right) dx \right) \\ & \leq g \left(C \int_B \left(\lambda^{-1} \left| \mathbb{M}_s^\#(P(w)) - (\mathbb{M}_s^\#(P(w)))_B \right| \right)^q dx \right) \\ & \leq C \mathcal{A} \left(\left(C \int_B \left(\lambda^{-1} \left| \mathbb{M}_s^\#(P(w)) - (\mathbb{M}_s^\#(P(w)))_B \right| \right)^q dx \right)^{1/q} \right) \\ & = C \mathcal{A} \left(\lambda^{-1} \left(C \int_B \left| \mathbb{M}_s^\#(P(w)) - (\mathbb{M}_s^\#(P(w)))_B \right|^q dx \right)^{1/q} \right). \end{aligned} \quad (36)$$

Using the doubling property of $\mathcal{A}$ for the above the formula, we have

$$\begin{aligned} & \int_B \mathcal{A} \left(\lambda^{-1} \left| \mathbb{M}_s^\#(P(w)) - (\mathbb{M}_s^\#(P(w)))_B \right| \right) dx \\ & \leq K \mathcal{A} \left(\lambda^{-1} \left(\int_B \left| \mathbb{M}_s^\#(P(w)) - (\mathbb{M}_s^\#(P(w)))_B \right|^q dx \right)^{1/q} \right) \\ & \leq K \int_{\rho B} \mathcal{A} \left(\lambda^{-1} |w| \right) dx. \end{aligned} \quad (37)$$

The proof of Theorem 7 has been finished. $\square$

Now, we will use Definition 4 and Theorem 7 to prove Theorem 8.

Proof of Theorem 8. According to Definition 4, we can know

$$\begin{aligned} & \int_E \mathcal{A} \left(\lambda^{-1} \left| \mathbb{M}_s^\#(P(w)) - (\mathbb{M}_s^\#(P(w)))_{B_0} \right| \right) dx \\ & \leq K \sup_{B \subseteq E} \int_B \mathcal{A} \left(\lambda^{-1} \left| \mathbb{M}_s^\#(P(w)) - (\mathbb{M}_s^\#(P(w)))_B \right| \right) dx \\ & \leq K \sup_{B \subseteq E} \int_{\rho B} \mathcal{A} \left(\lambda^{-1} |w| \right) dx \\ & \leq K \sup_{B \subseteq E} \int_E \mathcal{A} \left(\lambda^{-1} |w| \right) dx. \end{aligned} \quad (38)$$

Because $\sup_{B \subseteq E} \int_E \mathcal{A}(\lambda^{-1}|w|)dx$ is independent on the ball B , we obtain that

$$\left\| \mathbb{M}_s^\#(P(w)) - (\mathbb{M}_s^\#(P(w)))_{B_0} \right\|_{L_E^\mathcal{A}} \leq K \|w\|_{L_E^\mathcal{A}}. \quad (39)$$

We finish the proof of Theorem 8. $\square$

5. Applications

In this part, we firstly use Theorem 8 to do an estimate for a solution to the Laplace equation $\Delta u = 0$.

Example 16. Let u be a differential 2-form in $\mathbb{R}^m$, and

$$u = \frac{1}{9} (x_1 dx_2 \wedge dx_3 + x_2 dx_1 \wedge dx_3 + x_3 dx_1 \wedge dx_2), \quad (40)$$

where $\vartheta = \sqrt{x_1^2 + x_2^2 + x_3^2}$. It is very easy to obtain that $|u| = 1$ and $du = 0$, so u is a solution for the Laplace equation $\Delta u = 0$. If we take

$$\mathcal{A}(t) = t \log_+^t = \begin{cases} t, & t \leq e \\ t \log^t, & t > e, \end{cases} \quad (41)$$

then $\mathcal{A}(t)$ is a Young function and belongs to the $G(p, q, C)$ -class, with $\mathcal{A}(|u|) \in L^1(E)$. According to Theorem 8, we get

that, for any fixed $B_0 \subset E$, there exists a constant $K > 0$ such that

$$\left\| \mathbb{M}_s^\#(P(u)) - \left(\mathbb{M}_s^\#(P(u)) \right)_{B_0} \right\|_{L_E^\mathcal{A}} \leq K \|1\|_{L_E^\mathcal{A}}, \quad (42)$$

where $\|1\|_{L_E^\mathcal{A}} = \inf\{\lambda > 0 \mid (1/|E|) \int_E \mathcal{A}(\lambda^{-1} \log^+ \lambda) dt < 1\}$.

Now, our aim is to prove the following corollary by using Theorem 7.

Corollary 17. *Suppose that the Young function $\mathcal{A}$ belongs to the $G(p, q, C)$ -class, and $w \in C^\infty(\Lambda^1 E)$ is a solution to the nonhomogenous A -harmonic equation. The sharp maximal operator is noted by $M_s^\#$, P is the potential operator with its kernel function $K(x, y)$ satisfying the standard estimates ($1 \leq s < p, q < \infty$), and the bounded $E \subseteq \mathfrak{R}^m$ is the $L^\mathcal{A}$ -averaging domains. Then, there exists a constant $K > 0$, such that*

$$\left\| \mathbb{M}_s^\#(P(w)) \right\|_{L_B^\mathcal{A}} \leq K \|w\|_{L_{\rho B}^\mathcal{A}}, \quad (43)$$

where $B \subset E$ with $\rho B \subseteq E$, and the constant $\rho > 1$.

Proof. By using Minkowski inequality, we know that

$$\begin{aligned} \left\| \mathbb{M}_s^\#(P(w)) \right\|_{L_B^\mathcal{A}} &\leq \left\| \mathbb{M}_s^\#(P(w)) - \left(\mathbb{M}_s^\#(P(w)) \right)_B \right\|_{L_B^\mathcal{A}} \\ &\quad + \left\| \left(\mathbb{M}_s^\#(P(w)) \right)_B \right\|_{L_B^\mathcal{A}}. \end{aligned} \quad (44)$$

From [12] and formula (22), we have

$$\begin{aligned} &\int_B \mathcal{A} \left(\lambda^{-1} \left| \left(\mathbb{M}_s^\#(P(w)) \right)_B \right| \right) dt \\ &= f \cdot f^{-1} \left(\int_B \mathcal{A} \left(\lambda^{-1} \left| \left(\mathbb{M}_s^\#(P(w)) \right)_B \right| \right) dt \right) \\ &\leq f \left(K \int_B f^{-1} \left(\mathcal{A} \left(\lambda^{-1} \left| \left(\mathbb{M}_s^\#(P(w)) \right)_B \right| \right) \right) dt \right) \\ &\leq f \left(K \int_B \lambda^{-q} \left| \left(\mathbb{M}_s^\#(P(w)) \right)_B \right|^q dt \right) \\ &\leq f \left(K \int_B \lambda^{-q} \left| \left(\mathbb{M}_s^\#(P(w)) \right) \right|^q dt \right) \\ &\leq f \left(K \lambda^{-q} \int_B |w|^q dt \right) \\ &\leq K \int_B f \left(\lambda^{-q} |w|^q \right) dt \\ &\leq K \int_B \mathcal{A} \left(\lambda^{-1} |w| \right) dt \\ &\leq K \|w\|_{L_{\rho B}^\mathcal{A}}, \end{aligned} \quad (45)$$

where $\rho > 1$. In addition, according to Theorem 7, there exists a constant K such that

$$\left\| \mathbb{M}_s^\#(P(w)) - \left(\mathbb{M}_s^\#(P(w)) \right)_B \right\|_{L_B^\mathcal{A}} \leq K \|w\|_{L_{\rho B}^\mathcal{A}}. \quad (46)$$

Substituting (45) and (46) into (44), we conclude that there exists a constant $K > 0$, independent of w , such that

$$\left\| \mathbb{M}_s^\#(P(w)) \right\|_{L_B^\mathcal{A}} \leq K \|w\|_{L_{\rho B}^\mathcal{A}}. \quad (47)$$

The proof of Corollary 17 has been completed. $\square$

Virtually, we can obtain a global estimate about the composition operator by using Definition 4.

Corollary 18. *Suppose that the Young function $\mathcal{A}$ belongs to the $G(p, q, C)$ -class, and $w \in C^\infty(\Lambda^1 E)$ is a solution to the nonhomogenous A -harmonic equation. Let the sharp maximal operator be noted by $M_s^\#$, P is the potential operator with its kernel function $K(x, y)$ satisfying standard estimates ($1 \leq s < p, q < \infty$), and the bounded $E \subseteq \mathfrak{R}^m$ is the $L^\mathcal{A}$ -averaging domains. Then, there exists a constant $K > 0$ such that*

$$\left\| \mathbb{M}_s^\#(P(w)) \right\|_{L_E^\mathcal{A}} \leq K \|w\|_{L_E^\mathcal{A}}, \quad (48)$$

where B_0 is a fixed ball.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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