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Research Article

Implicit Approximation Scheme for the Solution of K-Positive Definite Operator Equation

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We construct an implicit sequence suitable for the approximation of solutions of K-positive definite operator equations in real Banach spaces. Furthermore, implicit error estimate is obtained and the convergence is shown to be faster in comparsion to the explicit error estimate obtained by Osilike and Udomene (2001).

1. Introduction

Let E be a real Banach space and let J denote the normalized duality mapping from E to 2^{E^*} defined by

$$J(x) = \left\{ f^* \in E^* : \langle x, f^* \rangle = \|x\|^2,\ \|f^*\| = \|x\| \right\}, \quad (1)$$

where E^* denotes the dual space of E and $\langle \cdot, \cdot \rangle$ denotes the generalized duality pairing. It is well known that if E^* is strictly convex, then J is single valued. We will denote the single-valued duality mapping by j.

Let E be a Banach space. The *modulus of smoothness* of E is the function.

$\rho_E : [0, \infty) \to [0, \infty)$ defined by

$$\rho_E(t) = \sup \left\{ \frac{1}{2} \left(\|x + y\| + \|x - y\| \right) - 1 : \|x\| \le 1, \|y\| \le t \right\}. \quad (2)$$

The Banach space E is called *uniformly smooth* if

$$\lim_{t \to 0} \frac{\rho_E(t)}{t} = 0. \quad (3)$$

A Banach space E is said to be *strictly convex* if for two elements $x, y \in E$ which are linearly independent we have that $\|x + y\| < \|x\| + \|y\|$.

Let E_1 be a dense subspace of a Banach space E. An operator T with domain $D(T) \supseteq E_1$ is called continuously E_1-invertible if the range of T, $R(T)$, with T in E considered as an operator restricted to E_1, is dense in E and T has a bounded inverse on $R(T)$.

Let E be a Banach space and let A be a linear unbounded operator defined on a dense domain, $D(A)$, in E. An operator A will be called K positive definite (Kpd) [1] if there exist a continuously $D(A)$-invertible closed linear operator K with $D(A) \subset D(K)$ and a constant $c > 0$ such that $j(Kx) \in J(Kx)$,

$$\langle Ax, j(Kx) \rangle \ge c\|Kx\|^2, \quad \forall x \in D(A). \quad (4)$$

Without loss of generality, we assume that $c \in (0, 1)$.

In [1], Chidume and Aneke established the extension of Kpd operators of Martynjuk [2] and Petryshyn [3, 4] from Hilbert spaces to arbitrary real Banach spaces. They proved the following result.

Theorem 1. *Let E be a real separable Banach space with a strictly convex dual E and let A be a Kpd operator with $D(A) = D(K)$. Suppose*

$$\langle Ax, j(Ky) \rangle = \langle Kx, j(Ay) \rangle, \quad \forall x, y \in D(A). \quad (5)$$

Then, there exists a constant $\alpha > 0$ such that for all $x \in D(A)$

$$\|Ax\| \le \alpha \|Kx\|. \quad (6)$$

Furthermore, the operator A is closed, $R(A) = E$, and the equation $Ax = f$ has a unique solution for any given $f \in E$.

As the special case of Theorem 1 in which $E = L_p$ (l_p) spaces, $2 \le p < \infty$, Chidume and Aneke [1] introduced an iteration process which converges strongly to the unique solution of the equation $Ax = f$, where A and K are commuting. Recently, Chidume and Osilike [5] extended the results of Chidume and Aneke [1] to the more general real separable q-uniformly smooth Banach spaces, $1 < q < \infty$, by removing the commutativity assumption on A and K. Later on, Chuanzhi [6] proved convergence theorems for the iterative approximation of the solution of the Kpd operator equation $Ax = f$ in more general separable uniformly smooth Banach spaces.

In [7], Osilike and Udomene proved the following result.

Theorem 2. *Let E be a real separable Banach space with a strictly convex dual and let $A : D(A) \subseteq E \to E$ be a Kpd operator with $D(A) = D(K)$. Suppose $\langle Ax, j(Ky)\rangle = \langle Kx, j(Ay)\rangle$ for all $x, y \in D(A)$. Choose any $\epsilon_1 \in (0, c^2/(1 + \alpha(1-c) + \alpha^2))$ and define $T_\epsilon : D(A) \subseteq E \to E$ by*

$$T_\epsilon x = x + \epsilon K^{-1} f - \epsilon K^{-1} Ax. \tag{7}$$

Then the Picard iteration scheme generated from an arbitrary $x_0 \in D(A)$ by

$$x_{n+1} = T_\epsilon x_n = T_\epsilon^n x_0 \tag{8}$$

converges strongly to the solution of the equation $Ax = f$. Moreover, if x^* denotes the solution of the equation $Ax = f$, then

$$\left\| x_{n+1} - x^* \right\| \le (1 - c\epsilon(1-c))^n \beta^{-1} \left\| Kx_0 - Kx^* \right\|. \tag{9}$$

The most general iterative formula for approximating solutions of nonlinear equation and fixed point of nonlinear mapping is the Mann iterative method [8] which produces a sequence $\{x_n\}$ via the recursive approach $x_{n+1} = \alpha_n x_n + (1 - \alpha_n)Tx_n$, for nonlinear mapping $T : C = D(T) \to C$, where the initial guess $x_0 \in C$ is chosen arbitrarily. For convergence results of this scheme and related iterative schemes, see, for example, [9–15].

In [16], Xu and Ori introduced the implicit iteration process $\{x_n\}$, which is the modification of Mann, generated by $x_0 \in C$, $x_n = \alpha_n x_{n-1} + (1 - \alpha_n)T_n x_n$, for $T_i, i = 1, 2, \ldots, N$, nonexpansive mappings, and $T_n = T_n \pmod N$ and $\{\alpha_n\} \subset (0, 1)$. They proved the weak convergence of this process to a common fixed point of the finite family of nonexpansive mappings in Hilbert spaces. Since then fixed point problems and solving (or approximating) nonlinear equations based on implicit iterative processes have been considered by many authors (see, e.g., [17–21]).

It is our purpose in this paper to introduce implicit scheme which converges strongly to the solution of the Kpd operator equation $Ax = f$ in a separable Banach space. Even though our scheme is implicit, the error estimate obtained indicates that the convergence of the implicit scheme is faster in comparison to the explicit scheme obtained by Osilike and Udomene [7].

2. Main Results

We need the following results.

Lemma 3 (see [10]). *If E^* is uniformly convex then there exists a continuous nondecreasing function $b : [0, \infty) \to [0, \infty)$ such that $b(0) = 0$, $b(\delta t) \le \delta b(t)$ for all $\delta \ge 1$ and*

$$\left\| x + y \right\|^2 \le \left\| x \right\|^2 + 2\left\langle y, j(x)\right\rangle + \max\{\left\| x \right\|, 1\} \left\| y \right\| b\left(\left\| y \right\|\right), \tag{R}$$

for all $x, y \in E$.

Lemma 4 (see [22]). *If there exists a positive integer N such that for all $n \ge N$, $n \in \mathbb{N}$ (the set of all positive integers),*

$$\rho_{n+1} \le (1 - \theta_n)\rho_n + b_n, \tag{10}$$

then

$$\lim_{n\to\infty} \rho_n = 0, \tag{11}$$

where $\theta_n \in [0, 1)$, $\sum_{n=1}^{\infty} \theta_n = \infty$ and $b_n = o(\theta_n)$.

Remark 5 (see [6]). Since K is continuously $D(A)$ invertible, there exists a constant $\beta > 0$ such that

$$\left\| Kx \right\| \ge \beta \left\| x \right\|, \quad \forall x \in D(K) = D(A). \tag{12}$$

In the continuation $c \in (0, 1)$, α and β are the constants appearing in (4), (6), and (12), respectively. Furthermore, $\epsilon > 0$ is defined by

$$\epsilon = \frac{c - \eta}{\alpha(1 - \eta)}, \quad \eta \in (0, c). \tag{13}$$

With these notations, we now prove our main results.

Theorem 6. *Let E be a real separable Banach space with a strictly convex dual and let $A : D(A) \subseteq E \to E$ be a Kpd operator with $D(A) = D(K)$. Suppose $\langle Ax, j(Ky)\rangle = \langle Kx, j(Ay)\rangle$ for all $x, y \in D(A)$. Let x^* denote a solution of the equation $Ax = f$. For arbitrary $x_0 \in E$, define the sequence $\{x_n\}_{n=0}^{\infty}$ in E by*

$$x_n = x_{n-1} + \epsilon K^{-1} f - \epsilon K^{-1} Ax_n, \quad n \ge 0. \tag{14}$$

Then, $\{x_n\}_{n=0}^{\infty}$ converges strongly to x^* with

$$\left\| x_n - x^* \right\| \le \rho^n \beta^{-1} \left\| Kx_0 - Kx^* \right\|, \tag{15}$$

where $\rho = 1 - ((c - \eta)/(\alpha(1 - \eta) + c - \eta))\eta \in (0, 1)$. Thus, the choice $\eta = c/2$ yields $\rho = 1 - (c^2/(4\alpha(1 - c/2) + 2c))$. Moreover, x^* is unique.

Proof. The existence of the unique solution to the equation $Ax = f$ comes from Theorem 1. From (4) we have

$$\left\langle Ax - cKx, j(Kx)\right\rangle \ge 0, \tag{16}$$

and from Lemma 1.1 of Kato [23], we obtain that

$$\left\| Kx \right\| \le \left\| Kx + \gamma(Ax - cKx) \right\|, \tag{17}$$

for all $x \in D(A)$ and $\gamma > 0$. Now, from (14), linearity of K and the fact that $Ax^* = f$ we obtain that

$$\begin{aligned} Kx_n &= Kx_{n-1} + \epsilon f - \epsilon Ax_n \\ &= Kx_{n-1} + \epsilon Ax^* - \epsilon Ax_n, \end{aligned} \tag{18}$$

which implies that

$$Kx_{n-1} = Kx_n - \epsilon Ax^* + \epsilon Ax_n, \tag{19}$$

so that

$$Kx_{n-1} - Kx^* = Kx_n - Kx^* - \epsilon Ax^* + \epsilon Ax_n. \tag{20}$$

With the help of (14) and Theorem 1, we have the following estimate:

$$\begin{aligned} \|Ax_n - Ax^*\| &= \|A(x_n - x^*)\| \le \alpha \|K(x_n - x^*)\| \\ &= \alpha \|Kx_n - Kx^*\| \\ &= \alpha \|Kx_{n-1} - Kx^* - \epsilon (Ax_n - Ax^*)\| \\ &\le \alpha \|Kx_{n-1} - Kx^*\| + \alpha\epsilon \|Ax_n - Ax^*\|, \end{aligned} \tag{21}$$

which gives

$$\|Ax_n - Ax^*\| \le \frac{\alpha}{1-\alpha\epsilon} \|Kx_{n-1} - Kx^*\|. \tag{22}$$

Furthermore, inequality (20) can be rewritten as

$$\begin{aligned} &Kx_{n-1} - Kx^* \\ &\quad = (1+\epsilon)(Kx_n - Kx^*) \\ &\qquad + \epsilon (Ax_n - Ax^* - c(Kx_n - Kx^*)) \\ &\qquad - \epsilon (1-c)(Kx_n - Kx^*) \\ &\quad = (1+\epsilon)\Big[Kx_n - Kx^* \\ &\qquad\qquad + \frac{\epsilon}{1+\epsilon}(Ax_n - Ax^* - c(Kx_n - Kx^*))\Big] \\ &\qquad - \epsilon (1-c)(Kx_n - Kx^*) \\ &\quad = (1+\epsilon)\Big[Kx_n - Kx^* \\ &\qquad\qquad + \frac{\epsilon}{1+\epsilon}(Ax_n - Ax^* - c(Kx_n - Kx^*))\Big] \\ &\qquad - \epsilon (1-c)(Kx_{n-1} - Kx^*) + \epsilon^2 (1-c)(Ax_n - Ax^*). \end{aligned} \tag{23}$$

In addition, from (17) and (22), we get that

$$\begin{aligned} &\|Kx_{n-1} - Kx^*\| \\ &\quad \ge (1+\epsilon)\Big\|Kx_n - Kx^* \\ &\qquad\qquad + \frac{\epsilon}{1+\epsilon}(Ax_n - Ax^* - c(Kx_n - Kx^*))\Big\| \\ &\qquad - \epsilon (1-c)\|Kx_{n-1} - Kx^*\| - \epsilon^2 (1-c)\|Ax_n - Ax^*\| \\ &\quad \ge (1+\epsilon)\|Kx_n - Kx^*\| - \epsilon (1-c)\|Kx_{n-1} - Kx^*\| \\ &\qquad - \epsilon^2 (1-c)\frac{\alpha}{1-\alpha\epsilon}\|Kx_{n-1} - Kx^*\|, \end{aligned} \tag{24}$$

which implies that

$$\begin{aligned} &\|Kx_n - Kx^{**}\| \\ &\quad \le \frac{1 + \epsilon(1-c) + \epsilon^2(1-c)(\alpha/(1-\alpha\epsilon))}{1+\epsilon}\|Kx_{n-1} - Kx^*\| \\ &\quad = \rho \|Kx_{n-1} - Kx^*\|, \end{aligned} \tag{25}$$

where

$$\begin{aligned} \rho &= \frac{1 + \epsilon(1-c) + \epsilon^2(1-c)(\alpha/(1-\alpha\epsilon))}{1+\epsilon} \\ &= 1 - \frac{\epsilon}{1+\epsilon}\left(c - \epsilon(1-c)\frac{\alpha}{1-\alpha\epsilon}\right) \\ &= 1 - \frac{\epsilon}{1+\epsilon}\eta \\ &= 1 - \frac{c-\eta}{\alpha(1-\eta) + c - \eta}\eta \\ &= 1 - \frac{c^2}{4\alpha(1-c/2) + 2c}. \end{aligned} \tag{26}$$

From (25) and (26), we have that

$$\|Kx_n - Kx^*\| \le \rho \|Kx_{n-1} - Kx^*\| \le \cdots \le \rho^n \|K(x_0 - x^*)\|. \tag{27}$$

Hence by Remark 5, we get that

$$\begin{aligned} &\|x_n - x^*\| \\ &\quad \le \beta^{-1}\|Kx_n - Kx^*\| \le \cdots \le \rho^n \beta^{-1}\|Kx_0 - Kx^*\| \longrightarrow 0, \end{aligned} \tag{28}$$

as $n \to \infty$. Thus, $x_n \to x^*$ as $n \to \infty$. □

In [6], Chuanzhi provided the following result.

Theorem 7. *Let E be a real uniformly smooth separable Banach space, and let $A : D(A) \subseteq E \to E$ be a Kpd operator with $D(A) = D(K)$. Suppose $\langle Ax, j(Ky)\rangle = \langle Kx, j(Ay)\rangle$ for all $x, y \in D(A)$. For arbitrary $f \in E$ and $x_0 \in D(A)$, define the sequence $\{x_n\}_{n=0}^{\infty}$ by*

$$\begin{gathered} x_{n+1} = x_n + t_n\gamma_n, \\ \gamma_n = K^{-1}f - K^{-1}Ax_n, \\ 0 \le t_n \le \frac{1}{2c}, \\ \sum t_n = 0, \qquad \lim_{n\to\infty} t_n = 0, \\ b(\alpha t_n) \le \frac{2c}{B\alpha}, \quad n \ge 0, \end{gathered} \tag{29}$$

where $b(t)$ is as in (R), α is the constant appearing in inequality (6), c is the constant appearing in inequality (4), and

$$B = \max\{\|K\gamma_0\|, 1\}. \tag{30}$$

Then, $\{x_n\}_{n=0}^{\infty}$ converges strongly to the unique solution of $Ax = f$.

However, its implicit version is as follows.

Theorem 8. *Let E be a real uniformly smooth separable Banach space, and let $A : D(A) \subseteq E \to E$ be a Kpd operator with $D(A) = D(K)$. Suppose $\langle Ax, j(Ky)\rangle = \langle Kx, j(Ay)\rangle$ for all $x, y \in D(A)$. For arbitrary $f \in E$ and $x_0 \in D(A)$, define the sequence $\{x_n\}_{n=0}^{\infty}$ by*

$$x_n = x_{n-1} + t_n\gamma_n, \tag{31}$$

$$\gamma_n = K^{-1}f - K^{-1}Ax_n, \tag{32}$$

$$\sum t_n = \infty, \quad \lim_{n\to\infty} t_n = 0, \quad n \geq 0. \tag{33}$$

Then, $\{x_n\}_{n=0}^{\infty}$ converges strongly to the unique solution of $Ax = f$.

Proof. The existence of the unique solution to the equation $Ax = f$ comes from Theorem 1. Using (31) and (32) we obtain

$$K\gamma_n = K\gamma_{n-1} - t_nA\gamma_n. \tag{34}$$

Consider

$$\begin{aligned}\|K\gamma_n\|^2 &= \langle K\gamma_n, j(K\gamma_n)\rangle = \langle K\gamma_{n-1} - t_nA\gamma_n, j(K\gamma_n)\rangle \\ &= \langle K\gamma_{n-1}, j(K\gamma_n)\rangle - t_n\langle A\gamma_n, j(K\gamma_n)\rangle \\ &\leq \|K\gamma_{n-1}\|\|K\gamma_n\| - ct_n\|K\gamma_n\|^2,\end{aligned} \tag{35}$$

which implies that

$$\|K\gamma_n\| \leq \|K\gamma_{n-1}\| - ct_n\|K\gamma_n\|. \tag{36}$$

Hence, $\{K\gamma_n\}_{n=0}^{\infty}$ is bounded. Let

$$M_1 = \sup_{n\geq 0}\|K\gamma_n\|. \tag{37}$$

Also from (6) it can be easily seen that $\{A\gamma_n\}_{n=0}^{\infty}$ is also bounded. Let

$$M_2 = \sup_{n\geq 0}\|A\gamma_n\|. \tag{38}$$

Denote $M = M_1 + M_2$; then $M < \infty$.

By using (34) and Lemma 3, we have

$$\begin{aligned}\|K\gamma_n\|^2 &= \|K\gamma_{n-1} - t_nA\gamma_n\|^2 \\ &\leq \|K\gamma_{n-1}\|^2 - 2t_n\langle A\gamma_n, j(K\gamma_{n-1})\rangle \\ &\quad + \max\{\|K\gamma_{n-1}\|, 1\}\|t_nA\gamma_n\|b(\|t_nA\gamma_n\|) \\ &= \|K\gamma_{n-1}\|^2 - 2t_n\langle A\gamma_{n-1}, j(K\gamma_{n-1})\rangle \\ &\quad + 2t_n\langle A\gamma_{n-1} - A\gamma_n, j(K\gamma_{n-1})\rangle \\ &\quad + \max\{\|K\gamma_{n-1}\|, 1\}t_n\|A\gamma_n\|b(t_n\|A\gamma_n\|) \\ &\leq (1 - 2ct_n)\|K\gamma_{n-1}\|^2 + 2t_n\|A\gamma_{n-1} - A\gamma_n\|\|K\gamma_{n-1}\| \\ &\quad + \max\{\|K\gamma_{n-1}\|, 1\}\alpha t_n\|K\gamma_n\|b(\alpha t_n\|K\gamma_n\|) \\ &\leq (1 - 2ct_n)\|K\gamma_{n-1}\|^2 + 2Mt_n\eta_n \\ &\quad + \max\{M, 1\}\alpha^2M^2t_nb(t_n),\end{aligned} \tag{39}$$

where

$$\eta_n = \|A\gamma_{n-1} - A\gamma_n\|. \tag{40}$$

By using (6) and (34) we obtain that

$$\begin{aligned}\|A\gamma_{n-1} - A\gamma_n\| &= \|A(\gamma_{n-1} - \gamma_n)\| \leq \alpha\|K(\gamma_{n-1} - \gamma_n)\| \\ &= \alpha t_n\|A\gamma_n\| \leq M\alpha t_n \longrightarrow 0, \quad \text{as } n \longrightarrow \infty.\end{aligned} \tag{41}$$

Thus,

$$\eta_n \longrightarrow 0 \quad \text{as } n \longrightarrow \infty. \tag{42}$$

Denote

$$\begin{aligned}\rho_n &= \|x_n - p\|, \\ \theta_n &= 2ct_n, \\ \sigma_n &= 2Mt_n\eta_n + \max\{M, 1\}\alpha^2M^2t_nb(t_n).\end{aligned} \tag{43}$$

Condition (33) assures the existence of a rank $n_0 \in \mathbb{N}$ such that $\theta_n = 2ct_n \leq 1$, for all $n \geq n_0$. Since $b(t)$ is continuous, so $\lim_{n\to\infty} b(t_n) = 0$ (by condition (33)). Now with the help of (33), (42), and Lemma 4, we obtain from (39) that

$$\lim_{n\to\infty}\|K\gamma_n\| = 0. \tag{44}$$

At last by Remark 5, $\gamma_n \to 0$ as $n \to \infty$; that is $Ax_n \to f$ as $n \to \infty$. Because A has bounded inverse, this implies that $x_n \to A^{-1}f$, the unique solution of $Ax_n = f$. This completes the proof. □

Remark 9. (1) According to the estimates (6–8) of Martynjuk [2], we have

$$\begin{aligned}&\|Kx_{n+1} - Kx^*\| \\ &\quad \leq \frac{1 + \epsilon_1(1 - c) + \alpha\epsilon_1^2(1 - c + \alpha)}{1 + \epsilon_1}\|Kx_n - Kx^*\| \\ &\quad = \theta\|Kx_n - Kx^*\|,\end{aligned} \tag{45}$$

where

$$\begin{aligned}\theta &= \frac{1 + \epsilon_1(1 - c) + \alpha\epsilon_1^2(1 - c + \alpha)}{1 + \epsilon_1} \\ &= 1 - \frac{\epsilon_1}{1 + \epsilon_1}(c - \alpha(1 - c + \alpha)\epsilon_1) \\ &= 1 - \frac{\epsilon_1}{1 + \epsilon_1}\eta,\end{aligned} \tag{46}$$

for $\eta = c - \alpha(1 - c + \alpha)\epsilon_1$ or $\epsilon_1 = (c - \eta)/\alpha(1 - c + \alpha)$, $\eta \in (0, c)$. Thus,

$$\begin{aligned}\theta &= 1 - \frac{c - \eta}{\alpha(1 - c + \alpha) + c - \eta}\eta \\ &= 1 - \frac{c^2}{4\alpha(1 - c + \alpha) + 2c}.\end{aligned} \tag{47}$$

Table 1

n	1	2	3	4	5	6	7	8
x_n	0.0922	0.0851	0.07859	0.07528	0.06947	0.06411	0.05916	0.05459

Table 2

n	1	2	3	4	5	6	7	8
x_n	0.0893	0.0798	0.07136	0.06376	0.05698	0.05092	0.04550	0.04066

Table 3

n	1	2	3	4	5	6	7	8
x_n	0.0098	0.0096	0.00949	0.00933	0.00917	0.00901	0.00885	0.00870

Table 4

n	1	2	3	4	5	6	7	8
x_n	0.0098	0.0096	0.00941	0.00923	0.00905	0.00887	0.00869	0.00852

(2) For $\alpha > c/2$, we observe that

$$\begin{aligned}\rho &= 1 - \frac{c^2}{4\alpha(1-c/2)+2c} \\ &= \theta - \frac{4\alpha c^2}{(4\alpha(1-c/2)+2c)(4\alpha(1-c+\alpha)+2c)}\left(\alpha - \frac{c}{2}\right).\end{aligned} \tag{48}$$

Thus, the relation between Martynjuk [2] and our parameter of convergence, that is, between θ and ρ, respectively, is the following:

$$\rho < \theta. \tag{49}$$

Despite the fact that our scheme is implicit, inequality (49) shows that the results of Osilike and Udomene [7] are improved in the sense that our scheme converges faster.

Example 10. Suppose $E = \mathbb{R}$, $D(A) = \mathbb{R}_+$, $Ax = x$, $Kx = 2I$ ($x^* = 0$ is the solution of $Ax = f$); then for the explicit iterative scheme due to Osilike and Udomene [7] we have

$$Kx_{n+1} = Kx_n - \epsilon_1 Ax_n, \tag{50}$$

which implies that

$$2x_{n+1} = 2x_n - \epsilon_1 x_n, \tag{51}$$

and hence

$$x_{n+1} = \left(1 - \frac{\epsilon_1}{2}\right)x_n. \tag{52}$$

Also for the implicit iterative scheme we have that

$$Kx_n = Kx_{n-1} - \epsilon Ax_n, \tag{53}$$

which implies that

$$x_n = \frac{1}{1+\epsilon/2}x_{n-1}. \tag{54}$$

It can be easily seen that for $c \le 1/2$ and $\alpha \ge 1/2$, (4) and (6) are satisfied. Suppose $c = 1/4$ and $\alpha = 3/5$; then $\eta = 0.125$, $\epsilon = (c-\eta)/\alpha(1-\eta) = 0.23810$, $\epsilon_1 = (c-\eta)/\alpha(1-c+\alpha) = 0.15432$, $\rho = 0.97596$, and $\theta = 0.983288$ and so $\rho < \theta$. Take $x_0 = 0.1$; then from (52) we have Table 1 and for (54) we get Table 2.

Example 11. Let us take $E = \mathbb{R}$, $D(A) = \mathbb{R}_+$, $Ax = (1/4)x$, $Kx = 2x$ ($x^* = 0$ is the solution of $Ax = f$); then for the explicit iterative scheme due to Osilike and Udomene [7] we have

$$Kx_{n+1} = Kx_n - \epsilon_1 Ax_n, \tag{55}$$

which implies that

$$2x_{n+1} = 2x_n - \frac{\epsilon_1}{4}x_n, \tag{56}$$

and hence

$$x_{n+1} = \left(1 - \frac{\epsilon_1}{8}\right)x_n. \tag{57}$$

Also for the implicit iterative scheme we have that

$$Kx_n = Kx_{n-1} - \epsilon Ax_n, \tag{58}$$

which implies that

$$x_n = \frac{1}{1+\epsilon/8}x_{n-1}. \tag{59}$$

It can be easily seen that for $c \le 1/8$ and $\alpha \ge 1/8$, (4) and (6) are satisfied. Suppose $c = 0.0625$ and $\alpha = 0.2$; then $\eta = 0.03125$, $\epsilon = (c-\eta)/\alpha(1-\eta) = 0.16129$, $\epsilon_1 = (c-\eta)/\alpha(1-c+\alpha) = 0.13736$, $\rho = 0.99566$, and $\theta = 0.99623$ and so $\rho < \theta$. Take $x_0 = 0.01$; then from (57) we have Table 3 and for (59) we get Table 4.

Even though our scheme is implicit we observe that it converges strongly to the solution of the Kpd operator equation $Ax = f$ with the error estimate which is faster in comparison to the explicit error estimate obtained by Osilike and Udomene [7].

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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