Research Article

Existence of Nonradial Solutions for Hénon Type Biharmonic Equation Involving Critical Sobolev Exponents

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We prove the existence of nonradial solutions under some conditions for a semilinear biharmonic Dirichlet problem involving critical Sobolev exponents.

1. Introduction

In this paper we consider the following Hénon type biharmonic problem:

$$\begin{aligned} \Delta^2 u &= |x|^\alpha |u|^{2^*-2} u \quad \text{in } \Omega, \\ u &= \frac{\partial u}{\partial \mathbf{n}} = 0 \quad \text{on } \partial\Omega, \end{aligned} \quad (1)$$

where $\alpha \geq 0$, $2^* = 2N/(N-4)$, Ω is the unit ball of $\mathbb{R}^N$, $N \geq 5$, and $\mathbf{n}$ denotes the unit outward normal at the boundary $\partial\Omega$.

We consider first the case where $\alpha = 0$, namely, the equation

$$\begin{aligned} \Delta^2 u &= |u|^{2^*-2} u \quad \text{in } \Omega, \\ u &= \frac{\partial u}{\partial \mathbf{n}} = 0 \quad \text{on } \partial\Omega. \end{aligned} \quad (2)$$

It is well known that (2) admits no nontrivial radial solution (see [1], Theorem 3.11, or [2], Theorem 4). The nonexistence of any nontrivial solution to (2) seems to be still unknown; only more restricted results are available. In order to obtain existence results for (2), one should either add subcritical perturbations or modify the topology or the geometry of the domain. For subcritical perturbations, we refer to [2, 3] and references therein. Domains with nontrivial topology are studied in [2, 4]. They demonstrated how domains with topology often carry solutions that cannot be present

otherwise. The corresponding second order elliptic problem has been investigated by Bahri and Coron in [5].

Berchio et al. [6], among other things, considered the minimization problem

$$\inf_{u \in H_{0,\text{rad}}^2(\Omega) \setminus \{0\}} \frac{\int_{\Omega} |\Delta u|^2 dx}{\left(\int_{\Omega} |x|^\alpha |u|^{2^*} dx \right)^{2/2^*}}, \quad (3)$$

where $H_{0,\text{rad}}^2(\Omega)$ denotes the subspace of radial functions in $H_0^2(\Omega)$. Actually, they treated general polyharmonic problem. They proved the infimum in (3) is attained. The minimizers of (3), after rescaling, are a solution of (1). It is natural to ask whether (1) has a nontrivial nonradial solution. We will answer this problem partially here.

Our main result is as follows.

Theorem 1. *Let $N \geq 8$ and let Ω be the unit ball in $\mathbb{R}^N$. Then, for every $\alpha > 0$ large enough, problem (1) admits at least one nonradial solution.*

The corresponding second order elliptic problem, namely, the Hénon equation,

$$\begin{aligned} -\Delta u &= |x|^\alpha |u|^{p-1}, \quad u > 0 \text{ in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega, \end{aligned} \quad (4)$$

has been studied by many authors, where $p > 1$. Ni [7], among other things, proved the existence of radial positive

solutions of (4) for all $p \in (1, (N+2+2\alpha)/(N-2))$. For the case $p \in (1, (N+2)/(N-2))$, there are many works concerning the limiting behavior of the ground state solutions of (4); see, for example, [8–11] and the references therein. In particular, Smets et al. [11] proved that, for every $p \in (1, (N+2)/(N-2))$, no minimizer of

$$\frac{\int_{\Omega} |\nabla u|^2 dx}{\left(\int_{\Omega} |x|^\alpha |u|^p dx\right)^{2/p}} \quad (5)$$

is radial provided α is large enough. Serra [12] studied the case $p = (N+2)/(N-2)$ and proved the existence of nonradial positive solutions of (4) for α large. Theorem 1 can be regarded as an extension of Serra's result to biharmonic problem.

In order to outline the proof of Theorem 1, we introduce some notations. We write $\mathbb{R}^N = \mathbb{R}^2 \times \mathbb{R}^{N-2} \simeq \mathbb{C} \times \mathbb{R}^{N-2}$ and $x = (z, y)$. For a given integer k , let G_k be the group $\mathbb{Z}_k \times \mathbf{O}(N-2)$. We consider the action of G_k on $H_0^2(\Omega)$ given by

$$g(u)(x) = g(u)(z, y) = u(e^{j(2\pi i/k)} z, Ry), \quad (6)$$

where $j \in \{0, 1, \dots, k-1\}$ and $R \in \mathbf{O}(N-2)$.

Define

$$\begin{aligned} H_k &= \{u \in H_0^2(\Omega) : u(e^{2\pi i/k} z, Ry) \\ &= u(z, y), \forall R \in \mathbf{O}(N-2)\}. \end{aligned} \quad (7)$$

It is easy to see that functions in H_k are radial in y . Since both the numerator and the denominator of the functional Q_α are invariant under the action of G_k , the functional Q_α is invariant. So the critical points of Q_α restricted to H_k are critical points of Q_α . After scaling, these correspond to weak solution of (1), which are in fact classical solutions by standard elliptic theory (see [13, 14]).

Set

$$Q_\alpha(u) = \frac{\int_{\Omega} |\Delta u|^2 dx}{\left(\int_{\Omega} |x|^\alpha |u|^{2^*} dx\right)^{2/2^*}}, \quad (8)$$

$$\Sigma_k = \inf_{u \in H_k \setminus \{0\}} Q_\alpha(u). \quad (9)$$

Notice that since $|x| \leq 1$ in Ω , we have $\Sigma_k \geq S$, the best Sobolev constant for the embedding $H_0^2 \hookrightarrow L^{2^*}$; that is,

$$S = \inf_{u \in H_0^2(\Omega) \setminus \{0\}} \frac{\int_{\Omega} |\Delta u|^2 dx}{\left(\int_{\Omega} |u|^{2^*} dx\right)^{2/2^*}}. \quad (10)$$

We now briefly outline the proof of Theorem 1. Firstly, we show that, for every $\alpha > 0$, there exists $k_\alpha > 0$ such that $\Sigma_k < k^{4/N} S$ for every integer $k \geq k_\alpha$. Next we prove that if $\Sigma_k < k^{4/N} S$, then Σ_k is achieved. Finally, we obtain a bound from below for $\inf_{u \in H_{0,\text{rad}}^2(\Omega) \setminus \{0\}} Q_\alpha(u)$ and then prove

$$\Sigma_k < \inf_{u \in H_{0,\text{rad}}^2(\Omega) \setminus \{0\}} Q_\alpha(u) \quad (11)$$

for all α large enough, where $H_{0,\text{rad}}^2(\Omega)$ denotes the space of radial functions in $H_0^2(\Omega)$. Therefore, our solution cannot be radial.

The paper is organized as follows. In Section 2, we establish some estimates we will need and investigate the compactness properties of Palais-Smale sequences for Q_α . In Section 3, we prove Theorem 1. Throughout this paper, the constant C will denote various generic constants.

2. Asymptotic Estimates and Analysis of Palais-Smale Sequences

In this section, we first establish the estimate to prove that $\Sigma < k^{4/N} S$ for suitable values of k and α .

In what follows, when we need to, we may define the trivial extension of functions u in $H_0^2(\Omega)$ by zero; namely, $u(x) = 0$ for $x \in \mathbb{R}^N \setminus \Omega$. For any fixed $\lambda > 0, q \in \mathbb{R}^N$, the rescaling $\mathcal{T} = \mathcal{T}(\lambda, q) : D^{2,2}(\mathbb{R}^N) \rightarrow D^{2,2}(\mathbb{R}^N)$ is defined by

$$\mathcal{T}v(x) = \lambda^{-(N-4)/2} v\left(\frac{x}{\lambda} + q\right). \quad (12)$$

Notice that $\mathcal{T}^{-1} = \mathcal{T}(1/\lambda, -\lambda q)$; that is,

$$\mathcal{T}^{-1}v(x) = \lambda^{(N-4)/2} v(\lambda(x - q)). \quad (13)$$

We choose $r \in (0, 1)$ and for every $k \in \mathbb{N}$ we define k points $x^{(j)}$ in $\mathbb{R}^N \simeq \mathbb{C} \times \mathbb{R}^{N-2}$ as

$$x^{(j)} = ((1-r)e^{j(2\pi i/k)}, 0), \quad j = 0, 1, \dots, k-1. \quad (14)$$

Notice that the points $x^{(j)}$ are all in Ω . Define

$$\begin{aligned} U_{\lambda, x^{(j)}}(x) &= \mathcal{T}(\lambda, x^{(j)})^{-1} U(x) \\ &= \frac{C_N \lambda^{(N-4)/2}}{(1 + \lambda^2 |x - x^{(j)}|^2)^{(N-4)/2}}, \end{aligned} \quad (15)$$

where $C_N = [(N-4)(N-2)N(N+2)]^{(N-4)/8}$. We recall that $U_{\lambda, x^{(j)}}$ are the unique positive solutions, radial about $x^{(j)}$, of the equation $\Delta^2 U = U^{2^*-1}$ in $D^{2,2}(\mathbb{R}^N)$, and $\int_{\mathbb{R}^N} |\Delta U_{\lambda, x^{(j)}}|^2 dx = \int_{\mathbb{R}^N} U_{\lambda, x^{(j)}}^{2^*} dx = S^{N/4}$ for all λ and $x^{(j)}$. To fix ideas, we anticipate that we will let $r \rightarrow 0, n \rightarrow \infty$, and $\lambda \rightarrow \infty$, with appropriate relations between r, n , and λ . The functions $U_{\lambda, x^{(j)}}$ are not in $H_0^2(\Omega)$, so we will use instead their projections $PU_{\lambda, x^{(j)}}$ on $H_0^2(\Omega)$ defined by

$$\begin{aligned} \Delta^2 PU_{\lambda, x^{(j)}} &= \Delta^2 U_{\lambda, x^{(j)}} \quad \text{in } \Omega, \\ PU_{\lambda, x^{(j)}} &= \frac{\partial PU_{\lambda, x^{(j)}}}{\partial \mathbf{n}} = 0 \quad \text{on } \partial\Omega. \end{aligned} \quad (16)$$

By Lemma 2.27 in [15], we have $PU_{\lambda, x^{(j)}}(x) > 0$ in Ω . If we set $\varphi_{\lambda, x^{(j)}} = U_{\lambda, x^{(j)}} - PU_{\lambda, x^{(j)}}$, then

$$\begin{aligned} \Delta^2 \varphi_{\lambda, x^{(j)}} &= 0 \quad \text{in } \Omega, \\ \varphi_{\lambda, x^{(j)}} &= U_{\lambda, x^{(j)}} \quad \text{on } \partial\Omega, \end{aligned} \quad (17)$$

$$\frac{\partial \varphi_{\lambda, x^{(j)}}}{\partial \mathbf{n}} = \frac{\partial U_{\lambda, x^{(j)}}}{\partial \mathbf{n}} \quad \text{on } \partial\Omega.$$

To avoid heavy notation from now on, we will write simply U_j for $U_{\lambda, x^{(j)}}$, PU_j for $PU_{\lambda, x^{(j)}}$, and φ_j for $\varphi_{\lambda, x^{(j)}}$.

We set

$$\begin{aligned} d_{jl} &= x^{(l)} - x^{(j)}, \\ d &= \frac{1}{2} \min_{j \neq l} |d_{jl}| = \frac{1}{2} |x^{(1)} - x^{(0)}|, \end{aligned} \quad (18)$$

and we assume that $2d \leq r$ and $\lambda d \geq 1$ for all λ under consideration. Note also that due to the definition of $x^{(j)}$, we have

$$d = 2(1-r) \sin \frac{\pi}{k} \sim \frac{C}{k} \quad (19)$$

for all r small.

Lemma 2.

$$\begin{aligned} \int_{\Omega} U_j^{2^*-1} U_l dx &= \begin{cases} S^{N/4} + O((\lambda r)^{-N}), & j = l, \\ \frac{C_N a_0}{(\lambda |d_{jl}|)^{N-4}} + O\left(\frac{1}{(\lambda |d_{jl}|)^{N-2}}\right) \\ \quad + O\left(\frac{1}{(\lambda r)^N}\right), & j \neq l, \end{cases} \end{aligned} \quad (20)$$

where $a_0 = \int_{\mathbb{R}^N} U^{2^*-1} dx$.

Proof.

Case 1. Consider $j = l$.

Direct computations yield that

$$\begin{aligned} S^{N/4} &\geq \int_{\Omega} U_j^{2^*} dx \\ &\geq \int_{B_r(x^{(j)})} U_j^{2^*} dx \\ &= S^{N/4} - \int_{\mathbb{R}^N \setminus B_r(x^{(j)})} U_j^{2^*}, \\ \int_{\mathbb{R}^N \setminus B_r(x^{(j)})} &= C_N^{2^*} \int_{\mathbb{R}^N \setminus B_{\lambda r}(0)} \frac{dx}{(1+|x|^2)^{(N-4)/2}} \\ &\geq C_N^{2^*} \omega_N \int_{\lambda r}^{+\infty} \frac{\rho^{N-1} d\rho}{(1+\rho^2)^N} \\ &\leq C(\lambda r)^{-N}, \end{aligned} \quad (21)$$

where ω denotes surface area of unit sphere in $\mathbb{R}^N$. Combining (21), we prove the first case of Lemma 2.

Case 2. Consider $j \neq l$.

Writing

$$\int_{\Omega} U_j^{2^*-1} U_l dx = \int_{\mathbb{R}^N} U_j^{2^*-1} U_l dx - \int_{\mathbb{R}^N \setminus \Omega} U_j^{2^*-1} U_l dx, \quad (22)$$

with the same type of calculation as in the proof of the first case, we see that

$$\begin{aligned} \int_{\mathbb{R}^N \setminus \Omega} U_j^{2^*-1} U_l dx &\leq \left(\int_{\mathbb{R}^N \setminus \Omega} U_j^{2^*} \right)^{(2^*-1)/2^*} \left(\int_{\mathbb{R}^N \setminus \Omega} U_l^{2^*} dx \right)^{1/2^*} \\ &\leq \frac{C}{(\lambda r)^N}. \end{aligned} \quad (23)$$

To estimate the integral over $\mathbb{R}^N$ in (22), we follow exactly the calculation in [5] (see page 279-280). It is easy to see that

$$\begin{aligned} \int_{\mathbb{R}^N} U_j^{2^*-1} U_l dx &= C_N^{2^*} \int_{\mathbb{R}^N} \frac{dx}{(1+|x|^2)^{(N+4)/2} (1+|x-\lambda d_{jl}|)^{(N-4)/2}}. \end{aligned} \quad (24)$$

We have also

$$1 + |x - \lambda d_{jl}|^2 = (1 + \lambda^2 |d_{jl}|^2) \left(1 + \frac{|x|^2 - 2\lambda x \cdot d_{jl}}{1 + \lambda^2 |d_{jl}|^2} \right). \quad (25)$$

Hence

$$\begin{aligned} &(1 + |x - \lambda d_{jl}|^2)^{(4-N)/2} \\ &= (1 + \lambda^2 |d_{jl}|^2)^{(4-N)/2} \\ &\quad \times \left[1 + \frac{(N-4)\lambda x \cdot d_{jl}}{1 + \lambda^2 |d_{jl}|^2} + O\left(\frac{|x|^2}{1 + \lambda^2 |d_{jl}|^2}\right) \right] \end{aligned} \quad (26)$$

for $|x| \leq (1/4)\lambda |d_{jl}|$.

Direct calculations yield

$$\begin{aligned} \int_{|x| \leq (1/4)\lambda |d_{jl}|} \frac{dx}{(1+|x|^2)^{(N+4)/2} (1+|x-\lambda d_{jl}|^2)^{(N-4)/2}} \\ = \frac{1}{(1 + \lambda^2 |d_{jl}|^2)^{(N-4)/2}} \end{aligned}$$

$$\begin{aligned} & \times \left[\int_{|x| \leq (1/4)\lambda|d_{jl}|} \frac{dx}{(1+|x|^2)^{(N+4)/2}} \right. \\ & \quad \left. + \frac{1}{1+\lambda^2|d_{jl}|^2} O \left(\int_{|x| \leq (1/4)\lambda|d_{jl}|} \frac{|x|^2 dx}{(1+|x|^2)^{(N+4)/2}} \right) \right], \end{aligned} \quad (27)$$

$$\int_{|x| \leq (1/4)\lambda|d_{jl}|} \frac{|x|^2 dx}{(1+|x|^2)^{(N+4)/2}} = b_0 + O\left(\frac{1}{\lambda^2|d_{jl}|^2}\right), \quad (28)$$

where $b_0 = \int_{\mathbb{R}^N} |x|^2 dx / (1+|x|^2)^{(N+4)/2}$, and

$$\int_{|x| \leq (1/4)\lambda|d_{jl}|} \frac{dx}{(1+|x|^2)^{(N+4)/2}} = \frac{a_0}{C_N^{2^*-1}} + O\left(\frac{1}{\lambda^4|d_{jl}|^4}\right). \quad (29)$$

From (27)–(29), we obtain

$$\begin{aligned} & \int_{|x| \leq (1/4)\lambda|d_{jl}|} \frac{dx}{(1+|x|^2)^{(N+4)/2} (1+|x-\lambda d_{jl}|^2)^{(N-4)/2}} \\ & = \frac{a_0}{C_N^{2^*-1}} \frac{1}{\lambda^{N-4}|d_{jl}|^{N-4}} + O\left(\frac{1}{\lambda^{N-2}|d_{jl}|^{N-2}}\right). \end{aligned} \quad (30)$$

Let

$$\begin{aligned} \Gamma_1 &= \left\{ x \in \mathbb{R}^N : |x - \lambda d_{jl}| \leq \frac{\lambda|d_{jl}|}{4} \right\}, \\ \Gamma_2 &= \left\{ x \in \mathbb{R}^N : |x| \leq \frac{\lambda|d_{jl}|}{4} \right\}. \end{aligned} \quad (31)$$

We have

$$\begin{aligned} & \int_{\mathbb{R}^N \setminus (\Gamma_1 \cup \Gamma_2)} \frac{dx}{(1+|x|^2)^{(N+4)/2} (1+|x-\lambda d_{jl}|^2)^{(N-4)/2}} \\ & \leq \frac{C}{\lambda^{N-4}|d_{jl}|^{N-4}} \int_{(1/4)\lambda|d_{jl}|}^{+\infty} \frac{\rho^{N-1}}{(1+\rho^2)^{(N+4)/2}} d\rho \\ & = O\left(\frac{1}{\lambda^N|d_{jl}|^N}\right), \end{aligned} \quad (32)$$

$$\begin{aligned} & \int_{\Gamma_1} \frac{dx}{(1+|x|^2)^{(N+4)/2} (1+|x-\lambda d_{jl}|^2)^{(N-4)/2}} \\ & = \int_0^{(1/4)\lambda|d_{jl}|} d\rho \\ & \quad \times \int_{|x-\lambda d_{jl}|=\rho} \frac{dS}{(1+|x|^2)^{(N+4)/2} (1+\rho^2)^{(N-4)/2}} \\ & \leq \frac{C}{\lambda^{N+4}|d_{jl}|^{N+4}} \int_0^{(1/4)\lambda|d_{jl}|} \frac{\rho^{N-1}}{(1+\rho^2)^{(N-4)/2}} d\rho \\ & = O\left(\frac{1}{\lambda^N|d_{jl}|^N}\right). \end{aligned} \quad (33)$$

From (30)–(33), it follows that

$$\begin{aligned} & \int_{\mathbb{R}^N} \frac{dx}{(1+|x|^2)^{(N+4)/2} (1+|x-\lambda d_{jl}|^2)^{(N-4)/2}} \\ & = \frac{a_0}{C_N^{2^*-1} \lambda^{N-4} |d_{jl}|^{N-4}} + O\left(\frac{1}{\lambda^{N-2} |d_{jl}|^{N-2}}\right). \end{aligned} \quad (34)$$

Hence

$$\int_{\mathbb{R}^N} U_j^{2^*-1} U_l dx = \frac{C_N a_0}{\lambda^{N-4} |d_{jl}|^{N-4}} + O\left(\frac{1}{\lambda^{N-2} |d_{jl}|^{N-2}}\right). \quad (35)$$

Combining (23) and (35), we prove

$$\begin{aligned} \int_{\Omega} U_j^{2^*-1} U_l dx &= \frac{C_N a_0}{(\lambda|d_{jl}|)^{N-4}} \\ &+ O\left(\frac{1}{(\lambda|d_{jl}|)^{N-2}}\right) + O\left(\frac{1}{(\lambda r)^N}\right). \end{aligned} \quad (36)$$

□

We denote by G Green's function of Δ^2 ; that is,

$$\begin{aligned} \Delta^2 G(x, \cdot) &= 2(N-4)(N-2)\omega_N \delta_x \quad \text{in } \Omega, \\ G(x, \cdot) &= \frac{\partial G(x, \cdot)}{\partial \mathbf{n}_y} = 0 \quad \text{on } \partial\Omega, \end{aligned} \quad (37)$$

where δ_x denotes the Dirac mass at x , and $\mathbf{n}_y$ is the outer unit normal at $y \in \partial\Omega$. We also denote by H the regular part of G ; that is,

$$\begin{aligned} H(x, y) &= |x-y|^{4-N} - G(x, y) \\ &\text{for } (x, y) \in \Omega \times \Omega. \end{aligned} \quad (38)$$

By Remark 1 in [16], we have

$$H(x, y) = \left| |x|y - \frac{x}{|x|} \right|^{4-N} + \frac{N-4}{2} (1 - |x|^2)(1 - |y|^2) \left| |x|y - \frac{x}{|x|} \right|^{2-N}. \quad (39)$$

Lemma 3. Consider

$$\begin{aligned} & \int_{\Omega} U_j^{2^*-1} \varphi_l dx \\ &= \begin{cases} \frac{C_N a_0}{\lambda^{N-4}} H(x^{(j)}, x^{(j)}) + O(\lambda^{2-N} r^{1-N}), & j = l \\ \frac{C_N a_0}{\lambda^{N-4}} H(x^{(j)}, x^{(l)}) + O((\lambda r)^{-N}), & j \neq l, \end{cases} \end{aligned} \quad (40)$$

where $a_0 = \int_{\mathbb{R}^N} U^{2^*-1} dx$.

Proof.

Case 1. Consider $j = l$.

Set

$$\Gamma(x) = \varphi_j(x) - C_N \lambda^{(4-N)/2} H(x^{(j)}, x). \quad (41)$$

By the definition of φ_j and $H(x^{(j)}, x)$, we get

$$\begin{aligned} \Delta^2 \Gamma(x) &= 0 \quad \text{in } \Omega, \\ \Gamma(x) &= U_j - C_N \lambda^{(4-N)/2} |x - x^{(j)}|^{4-N} \quad \text{on } \partial\Omega, \\ \frac{\partial \Gamma(x)}{\partial \mathbf{n}} &= \frac{\partial U_j}{\partial \mathbf{n}} - C_N \lambda^{(4-N)/2} \frac{\partial |x - x^{(j)}|^{4-N}}{\partial \mathbf{n}} \quad \text{on } \partial\Omega. \end{aligned} \quad (42)$$

For each $x \in \partial\Omega$, we have

$$\begin{aligned} \Gamma(x) &= C_N \lambda^{(4-N)/2} |x - x^{(j)}|^{4-N} \\ &\times \left[\left(1 - \frac{1}{1 + \lambda^2 |x - x^{(j)}|^2} \right)^{(N-4)/2} - 1 \right]. \end{aligned} \quad (43)$$

Thus,

$$\begin{aligned} |\Gamma(x)| &\leq C_N \lambda^{(4-N)/2} r^{4-N} \left[1 - \left(1 - \frac{1}{1 + \lambda^2 r^2} \right)^{(N-4)/2} \right] \\ &\leq C \lambda^{-N/2} r^{2-N}; \end{aligned} \quad (44)$$

consequently,

$$\Gamma(x) = O(\lambda^{-N/2} r^{2-N}), \quad \text{for } x \in \partial\Omega. \quad (45)$$

For each $x \in \partial\Omega$, we have

$$\begin{aligned} \frac{\partial \Gamma}{\partial \mathbf{n}}(x) &= C_N (4-N) (1 - x \cdot x^{(j)}) \lambda^{(4-N)/2} \\ &\times |x - x^{(j)}|^{2-N} \left[\left(1 - \frac{1}{1 + \lambda^2 |x - x^{(j)}|^2} \right)^{(N-2)/2} - 1 \right]. \end{aligned} \quad (46)$$

Since $1 - x \cdot x^{(j)} \leq 1 - |x| \cdot |x^{(j)}| = r$, we have

$$\frac{\partial \Gamma}{\partial \mathbf{n}}(x) = O(\lambda^{-N/2} r^{1-N}), \quad \text{for } x \in \partial\Omega. \quad (47)$$

By [15] (page 155), we have the following explicit formula:

$$\begin{aligned} \Gamma(x) &= \int_{\partial\Omega} K(x, y) (U_j(y) - C_N \lambda^{(4-N)/2} |y - x^{(j)}|^{4-N}) dS(y) \\ &+ C_N (N-4) \int_{\partial\Omega} L(x, y) (|y|^2 - y \cdot x^{(j)}) \\ &\times \left[\lambda^{N/2} (1 + \lambda^2 |y - x^{(j)}|^2)^{(2-N)/2} \right. \\ &\quad \left. - \lambda^{(4-N)/2} |y - x^{(j)}|^{2-N} \right] dS(y), \end{aligned} \quad (48)$$

where

$$\begin{aligned} K(x, y) &= \frac{1}{2\omega_N} \frac{(1 - |x|^2)^2}{|x - y|^{N+2}} [2 + (N-4)x \cdot y - (N-2)|x|^2], \\ L(x, y) &= \frac{1}{2\omega_N} \frac{(1 - |x|^2)^2}{|x - y|^N}, \end{aligned} \quad (49)$$

with $x \in \Omega$, $y \in \partial\Omega$. Using (45) and (47), we have, for all $x \in \Omega$,

$$\begin{aligned} |\Gamma(x)| &\leq C \left[\lambda^{-N/2} r^{2-N} \int_{\partial\Omega} |K(x, y)| dS(y) \right. \\ &\quad \left. + \lambda^{-N/2} r^{1-N} \int_{\partial\Omega} L(x, y) dS(y) \right] \\ &\leq C \lambda^{-N/2} r^{1-N} \int_{\partial\Omega} L(x, y) dS(y) \\ &\leq C \lambda^{-N/2} r^{1-N}. \end{aligned} \quad (50)$$

Thus,

$$\Gamma(x) = O(\lambda^{-N/2} r^{1-N}), \quad \text{for } x \in \Omega. \quad (51)$$

We split the term to be estimated as

$$\begin{aligned} & \int_{\Omega} U_j^{2^*-1} \varphi_j dx \\ &= C_N \lambda^{(4-N)/2} \left(\int_{B_{r/2}(x^{(j)})} + \int_{\Omega \setminus B_{r/2}(x^{(j)})} \right) \\ & \quad \times U_j^{2^*-1}(x) H(x^{(j)}, x) dx \\ & \quad + \int_{\Omega} U_j^{2^*-1}(x) \Gamma(x) dx, \end{aligned} \quad (52)$$

and then for the last integral we have

$$\begin{aligned} & \left| \int_{\Omega} U_j^{2^*-1}(x) \Gamma(x) dx \right| \\ & \leq O(\lambda^{-N/2} r^{1-N}) \int_{\mathbb{R}^N} U_j^{2^*-1} dx \\ & \leq O(\lambda^{-N/2} r^{1-N}) C \lambda^{(4-N)/2} = O(\lambda^{2-N} r^{1-N}). \end{aligned} \quad (53)$$

Concerning the integral over $\Omega \setminus B_{r/2}(x^{(j)})$, we first notice that, by (39),

$$\begin{aligned} H(x^{(j)}, x) & \leq r^{4-N} + \frac{N-4}{2} r^{3-N} (2-r) (1-|x|^2) \\ & \leq 2(N-4) r^{3-N}. \end{aligned} \quad (54)$$

Therefore,

$$\begin{aligned} & C_N \lambda^{(4-N)/2} \int_{\Omega \setminus B_{r/2}(x^{(j)})} U_j^{2^*-1}(x) H(x^{(j)}, x) dx \\ & \leq C_N \lambda^{(4-N)/2} 2(N-4) r^{3-N} \int_{\Omega \setminus B_{r/2}(x^{(j)})} U_j^{2^*-1}(x) dx \\ & = O(\lambda^{-N} r^{1-N}). \end{aligned} \quad (55)$$

As in [5, 12] we expand $H(x^{(j)}, \cdot)$ up to the fifth order near $x^{(j)}$, writing

$$H(x^{(j)}, x) = H(x^{(j)}, x^{(j)}) + \sum_{k=1}^5 H_k + R, \quad (56)$$

where H_k denotes the k th order term (e.g., $H_1 = \nabla H(x^{(j)}, x^{(j)})(x - x^{(j)})$). Since $|R| \leq \sup_{B_{r/2}(x^{(j)})} \|\nabla^6 H(x^{(j)}, \cdot)\| \cdot |x - x^{(j)}|^6$, using the explicit form of H , it is not difficult to check that

$$|R| \leq \sup_{B_{r/2}(x^{(j)})} \|\nabla^6 H(x^{(j)}, \cdot)\| \leq \frac{C}{r^{N+2}} \quad (57)$$

so that

$$\begin{aligned} & \left| \int_{B_{r/2}(x^{(j)})} U_j^{2^*-1} R \right| \\ & \leq \frac{C}{r^{N+2}} \int_{B_{r/2}(x^{(j)})} U_j^{2^*-1} |x - x^{(j)}|^6 dx \\ & \leq \frac{C}{\lambda^{(N+4)/2} r^N}. \end{aligned} \quad (58)$$

Note that $\Delta^2 H(x^{(j)}, x) = 0$. Using the symmetry of U_j and the usual scaling arguments, we have

$$\begin{aligned} & C_N \lambda^{(4-N)/2} \int_{B_{r/2}(x^{(j)})} U_j^{2^*-1} H(x^{(j)}, x) dx \\ & \leq C_N a_0 \lambda^{4-N} H(x^{(j)}, x^{(j)}) + O(r^{-N} \lambda^{-N}). \end{aligned} \quad (59)$$

By (53)–(59), we obtain

$$\int_{\Omega} U_j^{2^*-1} \varphi_j dx = \frac{C_N a_0}{\lambda^{N-4}} H(x^{(j)}, x^{(j)}) + O(\lambda^{2-N} r^{1-N}). \quad (60)$$

The proof of Lemma 3 is completed.

Case 2. Consider $j \neq l$.

Using the same argument similar to the ones in the proof of the first case, we get the desired result. $\square$

Lemma 4. (i) Consider $\int_{B_j} U_j^{2^*} dx = S^{N/4} + O(1/(\lambda d)^N)$.

(ii) Consider $\int_{B_j} U_j^{2^*-1} U_l dx \geq C_N a_0 / (\lambda |d_{jl}|)^{N-4} + O(1/(\lambda |d_{jl}|)^{N-2}) + O(1/\lambda^N d^4 |d_{jl}|^{N-4})$.

Proof. (i) The proof makes use of the same estimate as the one in the proof of Lemma 2, with r replaced by d this time.

(ii) We first write

$$\int_{B_j} U_j^{2^*-1} U_l dx = \int_{\mathbb{R}^N} U_j^{2^*-1} U_l dx - \int_{\mathbb{R}^N \setminus B_j} U_j^{2^*-1} U_l dx \quad (61)$$

and notice that the first integral in the right hand side in (61) has been estimated in (35). Next, we treat the second integral. We will make use of notation and formulas already established in the proof of Lemma 2 to get estimate (35).

Since $d/2 \leq |d_{jl}|/4$ by definition, we have the decomposition

$$\begin{aligned} \mathbb{R}^N \setminus B_{\lambda d/2}(0) &= \Gamma_1 \cup (\Gamma_2 \setminus B_{\lambda d/2}(0)) \\ &\quad \cup (\mathbb{R}^N \setminus (\Gamma_1 \cup \Gamma_2)). \end{aligned} \quad (62)$$

Consequently,

$$\begin{aligned} & \int_{\mathbb{R}^N \setminus B_j} U_j^{2^*-1} U_l dx \\ & \leq \int_{\mathbb{R}^N \setminus B_{d/2}(x^{(j)})} U_j^{2^*-1} U_l dx \\ & = C_N^{2^*} \left(\int_{\Gamma_1} + \int_{\Gamma_2 \setminus B_{\lambda d/2}(0)} + \int_{\mathbb{R}^N \setminus (\Gamma_1 \cup \Gamma_2)} \right) \\ & \quad \times \frac{dx}{(1+|x|^2)^{(N+4)/2} (1+|x - \lambda d_{jl}|^2)^{(N-4)/2}}. \end{aligned} \quad (63)$$

Now we have to evaluate three integrals in the right hand side in (63). The first integral and the third integral have been estimated in (33) and (32), respectively. Finally, we deal with the second integral over $\Gamma_2 \setminus B_{\lambda d/2}(0)$.

Similar to (27), we have

$$\begin{aligned} & \int_{\Gamma_2 \setminus B_{\lambda d/2}(0)} \frac{dx}{(1+|x|^2)^{(N+4)/2} (1+|x-\lambda d_{jl}|^2)^{(N-4)/2}} \\ & \leq \frac{1}{(\lambda |d_{jl}|)^{N-4}} \int_{\Gamma_2 \setminus B_{\lambda d/2}(0)} \frac{dx}{(1+|x|^2)^{(N+4)/2}} \\ & \quad + \frac{1}{(\lambda |d_{jl}|)^{N-2}} \int_{\Gamma_2 \setminus B_{\lambda d/2}(0)} \frac{|x|^2 dx}{(1+|x|^2)^{(N+4)/2}}. \end{aligned} \quad (64)$$

Now with elementary computations we have

$$\begin{aligned} & \int_{\Gamma_2 \setminus B_{\lambda d/2}(0)} \frac{dx}{(1+|x|^2)^{(N+4)/2}} \\ & \leq \int_{\mathbb{R}^N \setminus B_{\lambda d/2}(0)} \frac{dx}{(1+|x|^2)^{(N+4)/2}} = O((\lambda d)^{-4}), \\ & \int_{\Gamma_2 \setminus B_{\lambda d/2}(0)} \frac{|x|^2 dx}{(1+|x|^2)^{(N+4)/2}} \\ & \leq \int_{\Gamma_2} \frac{|x|^2 dx}{(1+|x|^2)^{(N+4)/2}} = \int_{\mathbb{R}^N} \frac{|x|^2 dx}{(1+|x|^2)^{(N+4)/2}} \\ & \quad + O((\lambda |d_{jl}|)^{-2}). \end{aligned} \quad (65)$$

Inserting these in (64), we obtain

$$\begin{aligned} & \int_{\Gamma_2 \setminus B_{\lambda d/2}(0)} \frac{dx}{(1+|x|^2)^{(N+4)/2} (1+|x-\lambda d_{jl}|^2)^{(N-4)/2}} \\ & \leq O\left(\frac{1}{\lambda^N d^4 |d_{jl}|^{N-4}}\right) + O\left(\frac{1}{(\lambda |d_{jl}|)^{N-2}}\right). \end{aligned} \quad (66)$$

Substituting (33), (32), and (66) into (63), we obtain

$$\begin{aligned} & \int_{\mathbb{R}^N \setminus B_j} U_j^{2^*-1} U_l dx \\ & \leq O\left(\frac{1}{\lambda^N d^4 |d_{jl}|^{N-4}}\right) + O\left(\frac{1}{(\lambda |d_{jl}|)^{N-2}}\right). \end{aligned} \quad (67)$$

By (61), (35), and the inequality above, we get the desired estimate. $\square$

Lemma 5. Consider

$$\begin{aligned} & \int_{B_j} U_j^{2^*-1} \varphi_l dx \\ & \leq \begin{cases} \frac{C_N a_0}{\lambda^{N-4}} H(x^{(j)}, x^{(j)}) + O\left(\frac{1}{\lambda^{N-1} r^N}\right), & j = l, \\ \frac{C_N a_0}{\lambda^{N-4}} H(x^{(j)}, x^{(l)}) + O\left(\frac{1}{\lambda^{N-1} d^N}\right) & j \neq l, \end{cases} \end{aligned} \quad (68)$$

Proof.

Case 1. Consider $j = l$.

Set

$$\Gamma(x) = \varphi_j(x) - C_N \lambda^{(N-4)/2} H(x^{(j)}, x). \quad (69)$$

For $x \in \Omega$, $y \in \partial\Omega$, we have

$$\begin{aligned} & 2 + (N-4)x \cdot y - (N-2)|x|^2 \\ & \leq 2 + (N-4)|x| - (N-2)|x|^2 < 0. \end{aligned} \quad (70)$$

By definition of $K(x, y)$, we get

$$K(x, y) < 0 \quad \text{for } x \in \Omega, \quad y \in \partial\Omega. \quad (71)$$

By (48), we obtain

$$\Gamma(x) < 0 \quad \text{for } x \in \Omega. \quad (72)$$

Therefore, we can write

$$\begin{aligned} & \int_{B_j} U_j^{2^*-1} \varphi_j dx \\ & = \frac{C_N}{\lambda^{N-4}} \int_{B_j} U_j^{2^*-1} H(x^{(j)}, x) dx + \int_{B_j} U_j^{2^*-1} \Gamma dx \\ & \leq \frac{C_N}{\lambda^{N-4}} \int_{B_j} U_j^{2^*-1} H(x^{(j)}, x) dx. \end{aligned} \quad (73)$$

Since $d \leq (1/2)r$, the last term can be estimated as in (59); namely,

$$\begin{aligned} & \frac{C_N}{\lambda^{(N-4)/2}} \int_{B_j} U_j^{2^*-1} H(x^{(j)}, x) dx \\ & \leq \frac{C_N}{\lambda^{(N-4)/2}} \int_{B_{r/2}(x^{(j)})} U_j^{2^*-1} H(x^{(j)}, x) dx \\ & \leq \frac{C_N a_0}{\lambda^{N-4}} H(x^{(j)}, x^{(j)}) + O(r^{-N} \lambda^{1-N}), \end{aligned} \quad (74)$$

which gives the required estimate.

Case 2. Consider $j \neq l$.

The computation can be adapted from the ones in the proof of Case 1.

Define

$$\tilde{u}(x) = \sum_{j=0}^{k-1} P U_j = \sum_{j=0}^{k-1} (U_j - \varphi_j). \quad (75)$$

Due to the definition of the points $x^{(j)}$, we have $\tilde{u} \in H_k$. Notice that $\tilde{u}$ depends on λ, k , and r through the choice of the points $x^{(j)}$. $\square$

Lemma 6. As $\lambda d \rightarrow \infty$ (i.e., $\lambda \rightarrow 0$), one has

$$\begin{aligned} & \int_{\Omega} |\Delta \tilde{u}|^2 dx \\ &= kS^{N/4} - \frac{C_N a_0}{\lambda^{N-4}} \\ & \times \left\{ \sum_{j=0}^{k-1} H(x^{(j)}, x^{(j)}) - \sum_{\substack{j,l=0 \\ j \neq l}}^{k-1} G(x^{(j)}, x^{(j)}) \right\} \\ &+ k^{N-1} O(\lambda^{2-N}) + k^2 O((\lambda r)^{-N}), \end{aligned} \quad (76)$$

$$\begin{aligned} & \int_{\Omega} |x|^{\alpha} \tilde{u}^{2^*} dx \\ & \geq (1-2r)^{\alpha} \\ & \times \left[kS^{N/4} - \frac{2^* C_N a_0}{\lambda^{N-4}} \right. \\ & \times \left\{ \sum_{j=0}^{k-1} H(x^{(j)}, x^{(j)}) - \sum_{\substack{j,l=0 \\ j \neq l}}^{k-1} G(x^{(j)}, x^{(j)}) \right\} \\ & \left. + k^{N-1} O\left(\frac{1}{\lambda^{N-2}}\right) + k^2 O\left(\frac{1}{\lambda^{N-1} r^N}\right) \right]. \end{aligned} \quad (77)$$

Proof. By definition of u (see (75)) and PU_j ,

$$\begin{aligned} & \int_{\Omega} |\Delta \tilde{u}|^2 dx \\ &= \sum_{j,l=0}^{k-1} \left(\int_{\Omega} U_j^{2^*} dx - \int_{\Omega} U_j^{2^*-1} \varphi_j dx \right) \\ &+ \sum_{\substack{j,l=0 \\ j \neq l}}^{k-1} \left(\int_{\Omega} U_j^{2^*-1} U_l dx - \int_{\Omega} U_j^{2^*-1} \varphi_l dx \right). \end{aligned} \quad (78)$$

By Lemmas 2 and 3, we have

$$\begin{aligned} & \sum_{j=0}^{k-1} \left(\int_{\Omega} U_j^{2^*} dx - \int_{\Omega} U_j^{2^*-1} \varphi_j dx \right) \\ &= kS^{N/4} - \frac{C_N a_0}{\lambda^{N-4}} \sum_{j=0}^{k-1} H(x^{(j)}, x^{(j)}) \\ &+ kO((\lambda r)^{-N}), \sum_{\substack{j,l=0 \\ j \neq l}}^{k-1} \int_{\Omega} U_j^{2^*-1} \varphi_l dx \\ &= \frac{C_N a_0}{\lambda^{N-4}} \sum_{\substack{j,l=0 \\ j \neq l}}^{k-1} H(x^{(j)}, x^{(l)}) \\ &+ (k^2 - k) O((\lambda r)^{-N}). \end{aligned} \quad (79)$$

By the symmetry of the points $x^{(j)}$, we have

$$\begin{aligned} & \sum_{\substack{j,l=0 \\ j \neq l}}^{k-1} \frac{1}{|d_{jl}|^{N-2}} = k \sum_{j=1}^{k-1} \frac{1}{(2-2r)^{N-2} \sin^{N-2}(\pi j/k)} \\ & \sim \frac{2k}{(2-2r)^{N-2}} \sum_{j=1}^{[(k-1)/2]} \frac{C}{(j/k)^{N-2}} \\ & \sim Ck^{N-1}, \end{aligned} \quad (80)$$

since the series of j^{1-N} is convergent. Recall that r will be taken small so that we can always assume $r \leq 1/2$. By (80), we obtain from Lemma 2

$$\begin{aligned} & \sum_{\substack{j,l=0 \\ j \neq l}}^{k-1} \int_{\Omega} U_j^{2^*-1} U_l dx \\ &= \frac{C_N a_0}{\lambda^{N-4}} \sum_{\substack{j,l=0 \\ j \neq l}}^{k-1} \frac{1}{|x^{(j)} - x^{(l)}|^{N-4}} \\ &+ k^{N-1} O(\lambda^{2-N}) + (k^2 - k) O((\lambda r)^{-N}). \end{aligned} \quad (81)$$

Substituting (79) and (81) into (78) and recalling the definition of G , we prove (76).

Having completed the estimate of the numerator of Q_{α} , we now go on to estimate the denominator, namely, $\int_{\Omega} |x|^{\alpha} \tilde{u}^{2^*} dx$. Recall that we denote $d = (1/2) \min_{j \neq l} |x^{(j)} - x^{(l)}|$ and that we assume $2d \leq r$. We now set $B_j = B_d(x^{(j)})$ for $j = 0, 1, \dots, k-1$. Then the B_j 's are positive disjoint and they are all contained in $\Omega \setminus \bar{B}_{1-2r}(0)$. Hence

$$\begin{aligned} & \int_{\Omega} |x|^{\alpha} \tilde{u}^{2^*} dx \geq (1-2r)^{\alpha} \int_{\Omega \setminus \bar{B}_{1-2r}(0)} \tilde{u}^{2^*} dx \\ & \geq (1-2r)^{\alpha} \sum_{j=0}^{k-1} \int_{B_j} \tilde{u}^{2^*} dx. \end{aligned} \quad (82)$$

It is easy to see that

$$\begin{aligned} & \tilde{u}^{2^*} = \left(\sum_{l=0}^{k-1} PU_l \right)^{2^*} \\ &= \left[U_j + \sum_{\substack{l=0 \\ l \neq j}}^{k-1} (U_l - \varphi_l) - \varphi_j \right]^{2^*} \\ & \geq U_j^{2^*} + 2^* U_j^{2^*-1} \left[\sum_{\substack{l=0 \\ l \neq j}}^{k-1} (U_l - \varphi_l) - \varphi_j \right]^{2^*}, \end{aligned} \quad (83)$$

where we have used the inequality

$$(a+b)^p \geq a^p + pa^{p-1}b, \quad (84)$$

for $p \geq 1, a \geq 0, a + b \geq 0$. We obtain therefore

$$\begin{aligned} & \sum_{j=0}^{k-1} \int_{B_j} \tilde{u}^{2^*} dx \\ & \geq \sum_{j=0}^{k-1} \left(\int_{B_j} U_j^{2^*} dx - 2^* \int_{B_j} U_j^{2^*-1} \varphi_j dx \right) \\ & \quad + 2^* \sum_{\substack{j,l=0 \\ j \neq l}}^{k-1} \left(\int_{B_j} U_j^{2^*-1} U_l dx - \int_{B_j} U_j^{2^*-1} \varphi_l dx \right), \end{aligned} \quad (85)$$

and we estimate the four integrals separately as above.

By the first part of Lemmas 4 and 5, and recalling that $r > d \sim C/k$ and

$$\begin{aligned} & \sum_{j=0}^{k-1} \left(\int_{B_j} U_j^{2^*} dx - 2^* \int_{B_j} U_j^{2^*-1} \varphi_j dx \right) \\ & \geq k S^{N/4} - \frac{2^* C_N a_0}{\lambda^{N-4}} \sum_{j=0}^{k-1} H(x^{(j)}, x^{(j)}) \\ & \quad + k^N O\left(\frac{1}{\lambda^{N-1}}\right). \end{aligned} \quad (86)$$

The remainders generated by the second part of Lemma 4 can be dealt with as in (80). We obtain

$$\begin{aligned} & \sum_{\substack{j,l=0 \\ j \neq l}}^{k-1} \frac{1}{|d_{jl}|^{N-2}} \sim C k^{N-1}, \\ & \sum_{\substack{j,l=0 \\ j \neq l}}^{k-1} \frac{1}{|d_{jl}|^{N-4}} \sim C k^{N-3}. \end{aligned} \quad (87)$$

Therefore,

$$\begin{aligned} 2^* \sum_{\substack{j,l=0 \\ j \neq l}}^{k-1} \int_{B_j} U_j^{2^*-1} U_l dx & \geq \frac{2^* C_N a_0}{\lambda^{N-4}} \sum_{\substack{j,l=0 \\ j \neq l}}^{k-1} \frac{1}{|x^{(j)} - x^{(l)}|^{N-4}} \\ & \quad + k^{N-1} O\left(\frac{1}{\lambda^{N-2}}\right) + k^{N+1} O\left(\frac{1}{\lambda^N}\right) \\ & = \frac{2^* C_N a_0}{\lambda^{N-4}} \sum_{\substack{j,l=0 \\ j \neq l}}^{k-1} \frac{1}{|x^{(j)} - x^{(l)}|^{N-4}} \\ & \quad + k^{N-1} O\left(\frac{1}{\lambda^{N-2}}\right), \end{aligned} \quad (88)$$

since $k/\lambda \rightarrow 0$.

Finally, from Lemma 5,

$$\begin{aligned} 2^* \sum_{\substack{j,l=0 \\ j \neq l}}^{k-1} \int_{B_j} U_j^{2^*-1} \varphi_l dx & \leq \frac{2^* C_N a_0}{\lambda^{N-4}} \sum_{\substack{j,l=0 \\ j \neq l}}^{k-1} H(x^{(j)}, x^{(l)}) \\ & \quad + (k^2 - k) O\left(\frac{1}{\lambda^{N-1} r^N}\right). \end{aligned} \quad (89)$$

Substituting (86), (88), and (89) into (85) and recalling the definition of G , we obtain the required estimate. $\square$

Proposition 7. Let $N \geq 6$. For every $\alpha > 0$, there exists $k_\alpha > 0$ such that, for every integer $k \geq k_\alpha$,

$$\Sigma_k < k^{4/N} S. \quad (90)$$

Proof. The function $\tilde{u}$ constructed in (75) depends on k, r , and λ , and for each k it belongs to H_k . We show that, for appropriate choice of these parameters, there results $Q_\alpha(\tilde{u}) < k^{4/N} S$. For simplicity, we set

$$A := \sum_{j=0}^{k-1} H(x^{(j)}, x^{(j)}) - \sum_{\substack{j,l=0 \\ j \neq l}}^{k-1} G(x^{(j)}, x^{(l)}) \quad (91)$$

and we begin with an estimate of A , noticing that we can write it as

$$A = \sum_{j,l=0}^{k-1} H(x^{(j)}, x^{(l)}) - \sum_{\substack{j,l=0 \\ j \neq l}}^{k-1} \frac{1}{|x^{(j)} - x^{(l)}|^{N-4}}. \quad (92)$$

By definition of H and since $|x^{(j)} - x^{(l)}|/|x^{(l)}|^2 \geq r$ for all j, l , we have

$$\begin{aligned} & \sum_{j,l=0}^{k-1} H(x^{(j)}, x^{(l)}) \\ & \leq \sum_{j,l=0}^{k-1} \left(\frac{1}{|x^{(j)}|^{N-4} r^{N-4}} + \frac{1}{|x^{(j)}|^{N-2} r^{N-2}} \right) \\ & \leq \frac{C_1 k^2}{r^{N-2}}. \end{aligned} \quad (93)$$

Moreover, as in (87),

$$\sum_{\substack{j,l=0 \\ j \neq l}}^{k-1} \frac{1}{|x^{(j)} - x^{(l)}|^{N-4}} \sim C_2 k^{N-3}, \quad (94)$$

so we obtain

$$A \leq \frac{C_1 k^2}{r^{N-2}} - C_2 k^{N-3}. \quad (95)$$

Note that $(1 - 2r)^{2\alpha/2^*} \geq 1 - 3\alpha r$, for all $N \geq 6, \alpha > 0$ and r small enough; we see that, from Lemma 6,

$$\begin{aligned} & \left(\int_{\Omega} |x|^\alpha \tilde{u}^{2^*} dx \right)^{2/2^*} \\ & \geq (1 - 3\alpha r) k^{2/2^*} \\ & \quad \times \left[S^{N/4} - \frac{2^* C_N a_0}{k \lambda^{N-4}} A \right. \\ & \quad \left. + k^{N-2} O\left(\frac{1}{\lambda^{N-2}}\right) + k O\left(\frac{1}{\lambda^{N-1} r^N}\right) \right]^{2/2^*}. \end{aligned} \quad (96)$$

Choose $r = k^{-3(N-5)/4(N-2)}$ and $\lambda = k^{1+\varepsilon}$ with $\varepsilon > 0$ and small. It is easy to see that all the quantities depending on k in the square brackets tend to zero as $\lambda/k \rightarrow \infty$; therefore, we obtain

$$\begin{aligned} Q_\alpha(\bar{u}) &\leq \left(kS^{N/4} - \frac{C_N a_0}{\lambda^{N-4}} A + k^{N-1} O\left(\frac{1}{\lambda^{N-2}}\right) + k^2 O\left(\frac{1}{(\lambda r)^N}\right) \right) \\ &\quad \times \left((1 - 3\alpha r) k^{2/2^*} \right. \\ &\quad \times \left[S^{(N-4)/4} - \frac{2C_N a_0}{kS\lambda^{N-4}} A \right. \\ &\quad \left. \left. + k^{N-2} O\left(\frac{1}{\lambda^{N-2}}\right) + k O\left(\frac{1}{\lambda^{N-1} r^N}\right) \right] \right)^{-1}. \end{aligned} \quad (97)$$

We must check that, for suitable values of the parameters, the right hand side is strictly less than $k^{4/N} S$. Direct computations show that it is enough to prove

$$\begin{aligned} R := 3\alpha r S^{N/4} + (1 - 6\alpha r) \frac{C_N a_0}{k\lambda^{N-4}} A + k^{N-2} O\left(\frac{1}{\lambda^{N-2}}\right) \\ - (1 - 3\alpha r) S k^{N-2} O\left(\frac{1}{\lambda^{N-2}}\right) \\ + k^2 \left(\frac{1}{(\lambda r)^N} \right) - (1 - 3\alpha r) S k O\left(\frac{1}{\lambda^{N-1} r^N}\right) < 0. \end{aligned} \quad (98)$$

We take k so large that

$$A \leq -\frac{C_2}{2} k^{N-3}, \quad (99)$$

and this is possible because

$$A \leq \frac{C_1 k^2}{r^{N-2}} - C_2 k^{N-3} = C_1 k^{2+(3/4)(N-5)} - C_2 k^{N-3} \quad (100)$$

and $2 + (3/4)(N-5) < N-3$ for all $N \geq 6$, as one immediately checks. Furthermore, noticing that $3N(N-5)/4(N-2) < N-2$, we see that

$$\begin{aligned} \frac{k^2}{(\lambda r)^N} &= \frac{k^{2+3N(N-5)/4(N-2)}}{\lambda^N} \leq \frac{k^N}{\lambda^N} \leq \frac{k^{N-2}}{\lambda^{N-2}}, \\ \frac{k}{\lambda^{N-1} r^N} &= \frac{k^{1+3N(N-5)/4(N-2)}}{\lambda^{N-1}} \leq \frac{k^{N-1}}{\lambda^{N-1}} \leq \frac{k^{N-2}}{\lambda^{N-2}}, \end{aligned} \quad (101)$$

since $k/\lambda \rightarrow 0$. Therefore, the third and the last big O is unnecessary in the expression of R . We are thus led to

$$\begin{aligned} R &\leq \frac{3\alpha S^{N/4}}{k^{3(N-5)/4(N-2)}} - \left(1 - \frac{6\alpha}{k^{3(N-5)/4(N-2)}} \right) \\ &\quad \cdot \frac{C_N a_0 C_2}{2} \cdot \frac{k^{N-4}}{\lambda^{N-4}} + k^{N-2} O\left(\frac{1}{\lambda^{N-2}}\right) \\ &= \frac{3\alpha S^{N/4}}{k^{3(N-5)/4(N-2)}} - \left(1 - \frac{6\alpha}{k^{3(N-5)/4(N-2)}} \right) \\ &\quad \cdot \frac{C_N a_0 C_2}{2} \cdot \frac{1}{k^{\varepsilon(N-4)}} + O\left(\frac{1}{k^{\varepsilon(N-2)}}\right). \end{aligned} \quad (102)$$

Since α is fixed, we have $R < 0$ for k large (depending on α) if we take ε small enough (essentially $\varepsilon < 3(N-5)/4(N-2)(N-4)$). $\square$

Next we show that if $\Sigma_k < k^{4/N} S$ then Σ_k is achieved. So we are led to analyze what happens if a minimizing sequence in H_k tends weakly to zero in $H_0^2(\Omega)$. Let $u_n \in H_k$ be a minimizing sequence for problem (9) such that $u_n \rightharpoonup 0$ weakly in $H_0^2(\Omega)$. Without loss of generality, we can assume that $Q'_\alpha(u_n) \rightarrow 0$ in H_k by Ekeland's variational principle.

Since Q_α is invariant under the action of G_k , we also have $Q'_\alpha(u_n) \rightarrow 0$ in $H_0^2(\Omega)$. By homogeneity of $Q_\alpha(u)$, we normalize u_n to obtain a sequence (still denoted by u_n) such that as $n \rightarrow \infty$,

$$\begin{aligned} Q_\alpha(u_n) &\rightarrow \Sigma_k, \quad Q'_\alpha(u_n) \rightarrow 0 \quad \text{in } H_0^2(\Omega), \\ u_n &\rightharpoonup 0 \quad \text{in } H_0^2(\Omega), \end{aligned} \quad (103)$$

$$\int_\Omega |x|^\alpha |u_n|^{2^*} dx = \Sigma_k^{N/4},$$

$$\int_\Omega |\Delta u_n|^2 dx = \Sigma_k^{N/4} + o(1).$$

The corresponding energy functional of problem (1) is defined by

$$\begin{aligned} J_\alpha(u) &= \frac{1}{2} \int_\Omega |\Delta u|^2 dx - \frac{1}{2} \int_\Omega |x|^\alpha |u|^{2^*} dx, \\ u &\in H_0^2(\Omega). \end{aligned} \quad (104)$$

By (103), direct computation shows that

$$\begin{aligned} J_\alpha(u_n) &\rightarrow \frac{2}{N} \Sigma_k^{N/4}, \quad J'_\alpha(u_n) \rightarrow 0 \\ &\text{as } n \rightarrow \infty. \end{aligned} \quad (105)$$

Thus, $\{u_n\}$ is a Palais-Smale sequence for the functional J_α at level $(2/N)\Sigma_k^{N/4}$.

Since for all n

$$\begin{aligned} \int_{\mathbb{R}^N} |u_n|^{2^*} dx \\ = \int_\Omega |u_n|^{2^*} dx \geq \int_\Omega |x|^\alpha |u_n|^{2^*} dx = \Sigma_k^{N/4} > 0, \end{aligned} \quad (106)$$

by Lemma 10.1 in [17], there exists a sequence of rescaling $\mathcal{T}_n = \mathcal{T}(\lambda_n, q_n)$ such that

$$\mathcal{T}_n u_n \rightharpoonup u \quad \text{weakly in } D^{2,2}(\mathbb{R}^N), \quad u \neq 0. \quad (107)$$

Since $\text{supp } u_n \subset \Omega$, we get $\lambda_n \rightarrow \infty$ as $n \rightarrow \infty$ and $q_n \in \bar{\Omega}$. We can also assume that $q_n \rightarrow q \in \bar{\Omega}$.

Lemma 8. *Let $\mathcal{T}_n = \mathcal{T}(\lambda_n, q_n)$ be the above sequence of rescaling satisfying $\lambda \rightarrow \infty$, $q_n \rightarrow q \in \bar{\Omega}$, and $\mathcal{T}_n u_n \rightharpoonup u \neq 0$ weakly in $D^{2,2}(\mathbb{R}^N)$. Then $\lambda_n \text{dist}(q_n, \partial\Omega) \rightarrow \infty$, $q \neq 0$, and u satisfies*

$$\Delta^2 u = |q|^\alpha |u|^{2^*-2} u \quad \text{in } \mathbb{R}^N. \quad (108)$$

Proof. We first prove $\lambda_n \text{dist}(q_n, \partial\Omega) \rightarrow \infty$ as $n \rightarrow \infty$. Assume to the contrary that $\lambda_n \text{dist}(q_n, \partial\Omega) \leq C < \infty$ for all n . Considering $\Omega_n = \{x \in \mathbb{R}^N : x/\lambda_n + q_n \in \Omega\}$, we may regard $\mathcal{T}_n u_n \in H_0^2(\Omega_n) \subset D^{2,2}(\mathbb{R}^N)$. After a rotation of coordinates, we may assume that the sequence $\{\Omega_n\}$ exhausts the half space $\mathbb{R}_+^N$. For any $\psi \in C_0^\infty(\mathbb{R}_+^N)$, we have $\psi \in C_0^\infty(\Omega_n)$ for all large n . Thus, the support of $\mathcal{T}_n^{-1}\psi$ is contained in Ω for all large n .

Set

$$\begin{aligned}\phi(x) &= \begin{cases} |x|^\alpha & x \in \bar{\Omega}, \\ 0, & x \notin \bar{\Omega}, \end{cases} \\ \phi_n(x) &= \phi\left(\frac{x}{\lambda_n} + q_n\right).\end{aligned}\quad (109)$$

The fact that $\{u_n\}$ is a Palais-Smale sequence for J_α implies that

$$\begin{aligned}o(1) &= \langle J'_\alpha(u_n), \mathcal{T}_n^{-1}\psi \rangle \\ &= \int_{\mathbb{R}^N} \Delta u \Delta \psi \, dx - \int_{\mathbb{R}^N} \phi(q) u^{2^*-1} \psi \, dx + o(1).\end{aligned}\quad (110)$$

We have used the fact that $\mathcal{T}_n u_n \rightarrow u$ in $D^{2,2}(\mathbb{R}^N)$ and $\phi_n(x) \rightarrow \phi(q)$. Since this happens for all $\psi \in C_0^\infty(\mathbb{R}_+^N)$, we see that u satisfies

$$\begin{aligned}\Delta^2 u &= |q|^\alpha u^{2^*-1}, \quad u \in D^{2,2}(\mathbb{R}^N) \text{ in } \Pi, \\ u &= \frac{\partial u}{\partial x_N} = 0 \quad \text{on } \partial\Pi,\end{aligned}\quad (111)$$

where $\Pi = \{(x', x_N) \in \mathbb{R}^N : x_N \geq -L\}$ for some constant $L \geq 0$. By Lemma 4 in [2] (also see [4]), (111) has a unique solution $u \equiv 0$, which contradicts $u \not\equiv 0$. Therefore, we must have $\lambda_n \text{dist}(q_n, \partial\Omega) \rightarrow \infty$. In this case, we can repeat the above argument and take test function $\psi \in C_0^\infty(\mathbb{R}^N)$. The above computations imply that u is a nontrivial solution of $\Delta^2 u = |q|^\alpha |u|^{2^*-2} u$ in $\mathbb{R}^N$, which also shows that $q \neq 0$ since $u \in D^{2,2}(\mathbb{R}^N)$. $\square$

Remark 9. It is proved in [18] that the radial function

$$U(x) = C_N (1 + |x|^2)^{-(N-4)/2} \quad (112)$$

solves $\Delta^2 U = U^{2^*-1}$ in $\mathbb{R}^N$, where $C_N = [(N-4)(N-2)N(N+2)]^{(N-4)/8}$. So the function u is nothing else than a multiple of U . Precisely, $u = |q|^{\alpha(4-N)/8} U$.

Let $\chi \in C_0^\infty(\mathbb{R}^N)$ be a cut-off function satisfying $0 \leq \chi(x) \leq 1$ for all $x \in \mathbb{R}^N$, $\chi(x) = 1$ for $|x| \leq 1$, $\chi(x) = 0$ for $|x| \geq 2$, $|\nabla \chi| \leq C$, and $|\Delta \chi| \leq C$.

Since $\lambda_n \text{dist}(q_n, \partial\Omega) \rightarrow \infty$, we may take a sequence $\{\bar{\lambda}_n\}$ such that $\bar{\lambda}_n > 0$, $\bar{\lambda}_n \rightarrow \infty$, $\lambda_n/\bar{\lambda}_n \rightarrow \infty$ and $\bar{\lambda}_n \text{dist}(q_n, \partial\Omega) \rightarrow \infty$ as $n \rightarrow \infty$. Set $\chi_n(x) = \chi((\bar{\lambda}_n/\lambda_n)x)$. In this way, $\text{supp } \mathcal{T}_n^{-1}\chi_n \subset \Omega$ for all n large and $u_n - \mathcal{T}_n^{-1}(\chi_n u) \in H_0^2(\Omega)$.

Note that

$$\begin{aligned}& \int_{\mathbb{R}^N} |\Delta(\chi_n u - u)|^2 \, dx \\ & \leq C \left(\int_{\mathbb{R}^N \setminus B_{\lambda_n/\bar{\lambda}_n}(0)} |\Delta u|^2 \, dx \right. \\ & \quad + \left(\frac{\bar{\lambda}_n}{\lambda_n} \right)^2 \int_{B_{2\lambda_n/\bar{\lambda}_n}(0) \setminus B_{\lambda_n/\bar{\lambda}_n}(0)} |\nabla u|^2 \, dx \\ & \quad \left. + \left(\frac{\bar{\lambda}_n}{\lambda_n} \right)^4 \int_{B_{2\lambda_n/\bar{\lambda}_n}(0) \setminus B_{\lambda_n/\bar{\lambda}_n}(0)} |u|^2 \, dx \right).\end{aligned}\quad (113)$$

Since $\Delta u \in L^2(\mathbb{R}^N)$, the first term tends to 0 as $n \rightarrow \infty$. By Hölder inequality, the second term and the third term in inequality above,

$$\begin{aligned}& \left(\frac{\bar{\lambda}_n}{\lambda_n} \right)^2 \int_{B_{2\lambda_n/\bar{\lambda}_n}(0) \setminus B_{\lambda_n/\bar{\lambda}_n}(0)} |\nabla u|^2 \, dx \\ & \leq C \left(\frac{\bar{\lambda}_n}{\lambda_n} \right)^2 \\ & \quad \times \left(\int_{B_{2\lambda_n/\bar{\lambda}_n}(0) \setminus B_{\lambda_n/\bar{\lambda}_n}(0)} |\nabla u|^{2N/(N-2)} \, dx \right)^{(N-2)/N} \rightarrow 0, \\ & \left(\frac{\bar{\lambda}_n}{\lambda_n} \right)^4 \int_{B_{2\lambda_n/\bar{\lambda}_n}(0) \setminus B_{\lambda_n/\bar{\lambda}_n}(0)} |u|^2 \, dx \\ & \leq C \left(\frac{\bar{\lambda}_n}{\lambda_n} \right)^4 \left(\int_{B_{2\lambda_n/\bar{\lambda}_n}(0) \setminus B_{\lambda_n/\bar{\lambda}_n}(0)} |u|^{2^*} \, dx \right)^{2/2^*} \rightarrow 0\end{aligned}\quad (114)$$

as $n \rightarrow \infty$. Thus, $\chi_n u \rightarrow u$ strongly in $D^{2,2}(\mathbb{R}^N)$.

Lemma 10. Let $\mathcal{T}_n u_n$ be the sequence constructed above. Then, as $n \rightarrow \infty$, one has

- (i) $J_\alpha(u_n - \mathcal{T}_n^{-1}(\chi_n u)) = J_\alpha(u_n) - J_{\alpha,q}(u) + o(1)$,
- (ii) $J'_\alpha(u_n - \mathcal{T}_n^{-1}(\chi_n u)) = J'_\alpha(u_n) + o(1)$ in $H_0^2(\Omega)$,

where

$$J_{\alpha,q}(u) = \frac{1}{2} \int_{\mathbb{R}^N} |\Delta u|^2 \, dx - \frac{1}{2^*} |q|^\alpha \int_{\mathbb{R}^N} |u|^{2^*} \, dx. \quad (115)$$

Moreover, the sequence $\{u_n - \mathcal{T}_n^{-1}(\chi_n u)\}$ is a Palais-Smale sequence for J_α at level $(2/N)\Sigma_k^{N/4} - J_{\alpha,q}(u)$; namely, it satisfies

- (i) $J_\alpha(u_n - \mathcal{T}_n^{-1}(\chi_n u)) = (2/N)\Sigma_k^{N/4} - J_{\alpha,q}(u) + o(1)$;
- (ii) $J'_\alpha(u_n - \mathcal{T}_n^{-1}(\chi_n u)) = o(1)$ in $H_0^2(\Omega)$.

Proof. (i) Set $w_n = u_n - \mathcal{T}_n^{-1}(\chi_n u)$; then $w_n \in H_0^2(\Omega)$. Since $\mathcal{T}_n u_n \rightarrow u$ and $\chi_n u \rightarrow u$ in $D^{2,2}(\mathbb{R}^N)$, we have

$$\int_{\Omega} |\Delta w_n|^2 \, dx = \int_{\mathbb{R}^N} |\Delta u_n|^2 \, dx - \int_{\mathbb{R}^N} |\Delta u|^2 \, dx + o(1). \quad (116)$$

Set $\phi_n(x) = \phi(x/\lambda_n + q_n)$. Changing variables as in the first part, we have

$$\begin{aligned} \int_{\Omega} |x|^\alpha |w_n|^{2^*} dx &= \int_{\mathbb{R}^N} \phi(x) |w_n|^{2^*} dx \\ &= \int_{\mathbb{R}^N} \phi_n(x) |\mathcal{T}_n u_n - \chi_n u|^{2^*} dx. \end{aligned} \quad (117)$$

Since $\chi_n u \rightarrow u$ in $D^{2,2}(\mathbb{R}^N)$, we get by the Brézis-Lieb lemma [19]:

$$\begin{aligned} \int_{\mathbb{R}^N} \phi_n(x) |\mathcal{T}_n u_n - \chi_n u|^{2^*} dx &= \int_{\mathbb{R}^N} \phi_n(x) |\mathcal{T}_n u_n - u|^{2^*} dx + o(1) \\ &= \int_{\mathbb{R}^N} \phi_n(x) |\mathcal{T}_n u_n|^{2^*} dx \\ &\quad - \int_{\mathbb{R}^N} \phi_n(x) |u|^{2^*} dx + o(1). \end{aligned} \quad (118)$$

By changing variables, we obtain

$$\begin{aligned} \int_{\mathbb{R}^N} \phi_n(x) |\mathcal{T}_n u_n|^{2^*} dx &= \int_{\mathbb{R}^N} \phi(x) |u_n|^{2^*} dx \\ &= \int_{\Omega} |x|^\alpha |u_n|^{2^*} dx, \\ \int_{\mathbb{R}^N} \phi_n(x) |u|^{2^*} dx &= \int_{\mathbb{R}^N} \phi(q) |u|^{2^*} dx + o(1) \\ &= |q|^\alpha \int_{\mathbb{R}^N} |u|^{2^*} dx + o(1). \end{aligned} \quad (119)$$

Inserting these into (118), we get

$$\begin{aligned} \int_{\mathbb{R}^N} \phi_n(x) |\mathcal{T}_n u_n - \chi_n u|^{2^*} dx &= \int_{\Omega} |x|^\alpha |u_n|^{2^*} dx - |q|^\alpha \int_{\Omega} |u|^{2^*} dx + o(1), \end{aligned} \quad (120)$$

which, combined with (116), yields (i).

(ii) Without loss of generality, we assume that $u_n - \mathcal{T}_n^{-1}(\chi_n u)$ is nonnegative; otherwise, one replaces its (2^*-1) th power by $|u_n - \mathcal{T}_n^{-1}(\chi_n u)|^{2^*-2}(u_n - \mathcal{T}_n^{-1}(\chi_n u))$. Taking $\psi \in C_0^\infty(\Omega)$,

$$\begin{aligned} \langle J'_\alpha(u_n - \mathcal{T}_n^{-1}(\chi_n u)), \psi \rangle &= \int_{\mathbb{R}^N} \Delta(u_n - \mathcal{T}_n^{-1}(\chi_n u)) \Delta\psi dx \\ &\quad - \int_{\mathbb{R}^N} \phi(x) (u_n - \mathcal{T}_n^{-1}(\chi_n u))^{2^*-1} \psi dx. \end{aligned} \quad (121)$$

Since $\chi_n u \rightarrow u$ in $D^{2,2}(\mathbb{R}^N)$, we have

$$\begin{aligned} \int_{\mathbb{R}^N} \Delta(u_n - \mathcal{T}_n^{-1}(\chi_n u)) \Delta\psi dx &= \int_{\mathbb{R}^N} \Delta u_n \Delta\psi dx - \int_{\mathbb{R}^N} \Delta u \Delta(\mathcal{T}_n \psi) dx \\ &\quad + o(1) \int_{\mathbb{R}^N} |\Delta\psi|^2 dx, \end{aligned} \quad (122)$$

$$\begin{aligned} \int_{\mathbb{R}^N} \phi(x) (u_n - \mathcal{T}_n^{-1}(\chi_n u))^{2^*-1} \psi dx &= \int_{\mathbb{R}^N} \phi_n(x) (\mathcal{T}_n u_n - u)^{2^*-1} \mathcal{T}_n \psi dx \\ &\quad + o(1) \int_{\mathbb{R}^N} |\Delta\psi|^2 dx. \end{aligned} \quad (123)$$

By the Brézis-Lieb lemma, we have

$$\begin{aligned} \int_{\mathbb{R}^N} \phi_n(x) (\mathcal{T}_n u_n - u)^{2^*-1} \mathcal{T}_n \psi dx &= \int_{\mathbb{R}^N} \phi_n(x) (\mathcal{T}_n u_n)^{2^*-1} \mathcal{T}_n \psi dx \\ &\quad - \int_{\mathbb{R}^N} \phi_n |u|^{2^*-1} \mathcal{T}_n \psi dx + o(1) \int_{\mathbb{R}^N} |\Delta\psi|^2 dx. \end{aligned} \quad (124)$$

By (123), (124), and $\phi_n \rightarrow \phi(q)$, we obtain

$$\begin{aligned} \int_{\mathbb{R}^N} \phi(x) (u_n - \mathcal{T}_n^{-1}(\chi_n u))^{2^*-1} \psi dx &= \int_{\mathbb{R}^N} \phi(x) |u_n|^{2^*-1} \psi dx \\ &\quad - |q|^\alpha \int_{\mathbb{R}^N} |u|^{2^*-1} \mathcal{T}_n \psi dx + o(1) \int_{\mathbb{R}^N} |\Delta\psi|^2 dx. \end{aligned} \quad (125)$$

Combing (122) and (125), we have

$$\begin{aligned} \langle J'_\alpha(u_n - \mathcal{T}_n^{-1}(\chi_n u)), \psi \rangle &= \langle J'_\alpha(u_n), \psi \rangle - \langle J'_{\alpha,q}(u), \mathcal{T}_n \psi \rangle \\ &\quad + o(1) \int_{\mathbb{R}^N} |\Delta\psi|^2 dx \\ &= \langle J'_\alpha(u_n), \psi \rangle + o(1) \int_{\mathbb{R}^N} |\Delta\psi|^2 dx, \end{aligned} \quad (126)$$

since u is a critical point of $J_{\alpha,q}$ by Lemma 8. Thus, we prove (ii).

An immediate consequence of (i) and (ii) is the the sequence $\{u_n - \mathcal{T}_n^{-1}(\chi_n u)\}$ which is a Palais-Smale sequence for J_α at level $(2/N)\Sigma_k^{N/4} - J_{\alpha,q}(u)$. $\square$

We are now ready to describe the behavior of Palais-Smale sequence of J_α .

Lemma 11. *Let $\{u_n\}$ be a Palais-Smale sequence for J_α at level $(2/N)\Sigma^{N/4}$ and $u_n \rightarrow 0$ in $H_0^2(\Omega)$. Then there is a positive m*

(depending only on Σ_k) such that, for every $j = 1, 2, \dots, m$, there exist sequences $\{\lambda_{jn}\} \subset \mathbb{R}^+$ and $\{q_{jn}\} \subset \bar{\Omega}$, with $\lambda_{jn} \rightarrow \infty$ and $q_{jn} \rightarrow q_j \in \bar{\Omega} \setminus \{0\}$ as $n \rightarrow \infty$, and there exists a nontrivial critical point $v_j \in D^{2,2}(\mathbb{R}^N)$ of J_{α,q_j} such that (up to subsequence)

$$u_n = \sum_{j=1}^m \mathcal{T}_{jn}^{-1}(v_j) + o(1) \quad \text{in } D^{2,2}(\mathbb{R}^N), \quad (127)$$

$$J_\alpha(u_n) = \sum_{j=1}^m J_{\alpha,q_j}(v_j) + o(1),$$

where $\mathcal{T}_{jn} = \mathcal{T}(\lambda_{jn}, q_{jn})$.

Proof. It is clear that there exists a sequence of positive numbers $\lambda_n \rightarrow \infty$, a sequence q_{1n} of points of $\bar{\Omega}$ with $q_{1n} \rightarrow q_1 \in \bar{\Omega} \setminus \{0\}$, and a nontrivial critical point v_1 of J_{α,q_1} such that, setting $\mathcal{T}_{1n} = \mathcal{T}(\lambda_{1n}, q_{1n})$, the sequence

$$w_{1n} := u_n - \mathcal{T}_{1n}^{-1}(\chi_{1n} v_1) \quad (128)$$

is a Palais-Smale sequence for J_α at level $(2/N)\Sigma^{N/4} - J_\alpha(v_1)$.

We now iterate this scheme. If $w_{1n} \rightarrow 0$ strongly in $L^{2^*}(\mathbb{R}^N)$, then the fact that it is a Palais-Smale sequence implies that $w_{1n} \rightarrow 0$ strongly in $D^{2,2}(\mathbb{R}^N)$. Since also $\chi_{1n} v_1 \rightarrow v_1$ strongly in $D^{2,2}(\mathbb{R}^N)$, we can write

$$u_n = \mathcal{T}_{1n}^{-1}(v_1) + o(1) \quad \text{in } D^{2,2}(\mathbb{R}^N), \quad (129)$$

and the lemma is proved with $m = 1$. Otherwise, $w_{1n} \rightarrow 0$ weakly in $L^{2^*}(\mathbb{R}^N)$ but not strongly. In this case, starting with Lemma 10.1 in [17], we can work on w_{1n} as we did for u_n . So we can find sequences $\lambda_{2n} \rightarrow \infty$, $q_{2n} \rightarrow q_2 \in \bar{\Omega} \setminus \{0\}$ and a nontrivial critical point v_2 of J_{α,q_2} such that the sequence

$$w_{2n} := w_{1n} - \mathcal{T}_{2n}^{-1}(\chi_{2n} v_2) \quad (130)$$

is a Palais-Smale sequence for J_α at level $(2/N)\Sigma_k^{N/4} - J_{\alpha,q_1}(v_1) - J_{\alpha,q_2}(v_2)$. Once again, if $w_{2n} \rightarrow 0$ strongly in $L^{2^*}(\mathbb{R}^N)$, then we obtain

$$u_n = \mathcal{T}_{1n}^{-1}(v_1) + w_{1n} = \mathcal{T}_{1n}^{-1}(v_1) + \mathcal{T}_{2n}^{-1}(v_2) + o(1) \quad (131)$$

$$\text{in } D^{2,2}(\mathbb{R}^N),$$

and the lemma is proved with $m = 2$. Otherwise, $w_{2n} \rightarrow 0$ weakly in $L^{2^*}(\mathbb{R}^N)$ but not strongly, and we iterate the above argument. This procedure will end after a finite number of steps. Actually, notice that, by Remark 9, for all j ,

$$J_{\alpha,q_j} = J_{\alpha,q_j} \left(|q_j|^{\alpha(4-N)/8} U \right) \quad (132)$$

$$= |q_j|^{\alpha(4-N)/4} \frac{2}{N} S^{N/4} \geq \frac{2}{N} S^{N/4},$$

by definition of U , so that, after at most $m := [(\Sigma_k/S)^{N/4}]$ steps, the remainder will be a Palais-Smale sequence at level zero; namely, it will be $o(1)$ in $D^{2,2}(\mathbb{R}^N)$, obtaining the requested representation for u_n and $J_\alpha(u_n)$. $\square$

Remark 12. Checking the process of the proof of Lemma 11, it is easy to see that if one does not suppress the cut-off functions χ_{jn} , one can obtain the following representation of u_n :

$$u_n = \sum_{j=1}^m \mathcal{T}_{jn}^{-1}(\chi_{jn} v_j) + o(1) \quad (133)$$

$$= \sum_{j=1}^m \lambda_{jn}^{(N-4)/2} (\chi_{jn} v_j) (\lambda_{jn}(\cdot - q_{jn})) + o(1)$$

$$\text{in } H_0^2(\Omega).$$

Lemma 13. Let $\{u_n\} \subset H_k$ be a minimizing sequence for problem (9) and $u_n \rightharpoonup u$ weakly in $H_0^2(\Omega)$. If $u \neq 0$, then u is a minimum point and $u_n \rightarrow u$ strongly in $H_0^2(\Omega)$.

Proof. Notice first that $u \in H_k$ since H_k is weakly closed in $H_0^2(\Omega)$. Weak convergence and the Brézis-Lieb Lemma yield

$$\lim_{n \rightarrow \infty} \left(\int_\Omega |\Delta u_n|^2 dx - \int_\Omega |\Delta u_n - \Delta u|^2 dx \right) = \int_\Omega |\Delta u|^2 dx,$$

$$\lim_{n \rightarrow \infty} \left(\int_\Omega |x|^\alpha |u_n|^{2^*} dx - \int_\Omega |x|^\alpha |u_n - u|^{2^*} dx \right)$$

$$= \int_\Omega |x|^\alpha |u|^{2^*} dx. \quad (134)$$

Therefore,

$$Q_\alpha(u) = \frac{\int_\Omega |\Delta u_n|^2 dx - \int_\Omega |\Delta u_n - \Delta u|^2 dx + o(1)}{\left(\int_\Omega |x|^\alpha |u_n|^{2^*} dx - \int_\Omega |x|^\alpha |u_n - u|^{2^*} dx + o(1) \right)^{2/2^*}}. \quad (135)$$

But $\{u_n\}$ is a minimizing sequence so that

$$\int_\Omega |\Delta u_n|^2 dx = \Sigma_k \left(\int_\Omega |x|^\alpha |u_n|^{2^*} dx \right)^{2/2^*} + o(1). \quad (136)$$

Since $u_n - u \in H_k$, we have

$$\int_\Omega |\Delta(u_n - u)|^2 dx \geq \Sigma_k \left(\int_\Omega |x|^\alpha |u_n - u|^{2^*} dx \right)^{2/2^*} + o(1). \quad (137)$$

Inserting (136) and (137) into (135), we obtain

$$\begin{aligned}
Q_\alpha(u) &\leq \Sigma_k \left(\int_\Omega |\Delta u_n|^2 dx - \int_\Omega |\Delta u_n - \Delta u|^2 dx + o(1) \right) \\
&\quad \times \left(\left[\left(\int_\Omega |\Delta u_n|^2 dx \right)^{2^*/2} \right. \right. \\
&\quad \left. \left. - \left(\int_\Omega |\Delta(u_n - u)|^2 dx \right)^{2^*/2} + o(1) \right]^{2/2^*} \right)^{-1}.
\end{aligned} \tag{138}$$

Since $\int_\Omega |\Delta u_n - \Delta u|^2 dx < \int_\Omega |\Delta u_n|^2 dx$ for n large enough, it is easy to see that $\int_\Omega |\Delta u_n - \Delta u|^2 dx \rightarrow 0$ as $n \rightarrow \infty$. Otherwise, the right hand side of (138) will be strictly less than Σ_k , which contradicts the definition of Σ_k . Thus, we have $u_n \rightarrow u$ strongly in $H_0^2(\Omega)$. $\square$

The following proposition is the main result of this section.

Proposition 14. *If $\Sigma_k < k^{4/N}S$, then Σ_k is achieved.*

Proof. Let $\{u_n\} \subset H_k$ be a (bounded) minimizing sequence for Q_α over H_k . Then there exists a subsequence (still denoted by u_n) such that $u_n \rightharpoonup u$ in $H_0^2(\Omega)$. If $u \neq 0$, by Lemma 13, u is a minimum point in H_k , and the proof is completed. If, on the contrary, $u_n \rightarrow 0$ in $H_0^2(\Omega)$, then, as we did above, without loss of generality we can assume that $\{u_n\}$ is a Palais-Smale sequence for J_α at level $(2/N)\Sigma_k^{N/4}$. The behavior of such sequence is described in Lemma 11. In particular,

$$\begin{aligned}
J_\alpha(u_n) &= \sum_{j=1}^m J_{\alpha, q_j}(v_j) + o(1) \\
&= \sum_{j=1}^m |q_j|^{\alpha(4-N)/4} \frac{2}{N} S^{N/4} + o(1) \\
&\geq \frac{2m}{N} S^{N/4} + o(1).
\end{aligned} \tag{139}$$

Since v_j is a multiple of the radial function U , the cut-off function χ_{j_n} is also radial, and $\mathbf{O}(N-2)$ is a continuous group, we obtain that, for every $j = 1, \dots, m$, $\lambda_{j_n} \text{dist}(q_{j_n}, \mathbb{R}^2 \times \{0\}) \rightarrow 0$ as $n \rightarrow \infty$. Otherwise, (133) would be incompatible with the symmetry properties of u_n . Thus, we can replace q_{j_n} in (133) with its projection on $\mathbb{R}^2 \times \{0\}$ so that we can assume that (133) holds in H_k . This means that m must be a multiple of k ; say, $m = lk$, for some integer $l \geq 1$. But, from (139),

$$\begin{aligned}
\frac{2}{N} \Sigma_k^{N/4} &= J_\alpha(u_n) + o(1) \\
&\geq lk \frac{2}{N} S^{N/4} + o(1) \geq \frac{2k}{N} S^{N/4} + o(1);
\end{aligned} \tag{140}$$

namely, $\Sigma_k \geq k^{4/N}S$, which is impossible. $\square$

3. Proof of Theorem 1

This last section is devoted to the proof of Theorem 1.

Lemma 15. *For any $2 \leq p \leq 2^*$, there exists $C > 0$ depending on N and p such that*

$$\inf_{u \in H_{0, \text{rad}}^2(\Omega) \setminus \{0\}} Q_{\alpha, p}(u) \geq C \left(\frac{\alpha + N}{N} \right)^{1+2/p}$$

as $\alpha \rightarrow +\infty$,

where $H_{0, \text{rad}}^2(\Omega)$ denotes the space of radial functions in $H_0^2(\Omega)$ and

$$Q_{\alpha, p}(u) = \frac{\int_\Omega |\Delta u|^2 dx}{\left(\int_\Omega |x|^\alpha |u|^p dx \right)^{2/p}}. \tag{142}$$

Proof. Let $u \in H_{0, \text{rad}}^2(\Omega)$ and define $v(|x|) = u(|x|^\beta)$, where $\beta = N/(\alpha + N)$. Then

$$\int_\Omega |x|^\alpha |u|^p dx = \beta \int_\Omega |v(x)|^p dx. \tag{143}$$

It is easy to see that

$$\Delta v = \beta \left[\beta u''(\rho^\beta) \rho^{2(\beta-1)} + (N + \beta - 2) u'(\rho^\beta) \rho^{\beta-2} \right]. \tag{144}$$

We obtain

$$\begin{aligned}
&\beta^{-1} \int_\Omega |\Delta v|^2 |x|^{(\beta-1)(N-4)} dx \\
&= N \omega_N \int_0^1 \left[\beta u''(\rho) + (N + \beta - 2) \frac{u'(\rho)}{\rho} \right]^2 \rho^{N-1} d\rho.
\end{aligned} \tag{145}$$

Set

$$F(\beta) = N \omega_N \int_0^1 \left[\beta u''(\rho) + (N + \beta - 2) \frac{u'(\rho)}{\rho} \right]^2 \rho^{N-1} d\rho. \tag{146}$$

Direct computations yield $F''(\beta) > 0$, which shows $F(\beta)$ is convex. Hence

$$F(\beta) \leq \max \{F(0), F(1)\}; \tag{147}$$

that is

$$F(\beta) \leq \max \left\{ \int_\Omega \frac{|\nabla u|^2}{|x|^2} dx, \int_\Omega |\Delta u|^2 dx \right\}. \tag{148}$$

By Hardy's inequality, we have

$$\int_\Omega \left| \nabla \frac{\partial u}{\partial x_j} \right|^2 dx \geq \frac{(N-2)^2}{4} \int_\Omega \frac{|\partial u / \partial x_j|^2}{|x|^2} dx, \tag{149}$$

$$j = 1, 2, \dots, N;$$

consequently,

$$\sum_{j=1}^N \int_{\Omega} \left| \nabla \frac{\partial u}{\partial x_j} \right|^2 dx \geq \frac{(N-2)^2}{4} \int_{\Omega} \frac{|\nabla u|^2}{|x|^2} dx. \quad (150)$$

Therefore, by (148) and (150), there exists $C > 0$ depending only on N such that

$$\int_{\Omega} |\Delta u|^2 dx \geq C\beta^{-1} \int_{\Omega} |\Delta v|^2 |x|^{(\beta-1)(N-4)} dx. \quad (151)$$

By (143), we have

$$\begin{aligned} & \frac{\int_{\Omega} |\Delta u|^2 dx}{\left(\int_{\Omega} |x|^{\alpha} |u|^p dx \right)^{2/p}} \\ & \geq C\beta^{-1} \frac{\int_{\Omega} |\Delta v|^2 |x|^{(\beta-1)(N-4)} dx}{\beta^{2/p} \left(\int_{\Omega} |v(x)|^p dx \right)^{2/p}} \\ & = \frac{C}{\beta^{1+2/p}} \frac{\int_{\Omega} |\Delta v|^2 |x|^{(\beta-1)(N-4)} dx}{\left(\int_{\Omega} |v(x)|^p dx \right)^{2/p}}. \end{aligned} \quad (152)$$

For every $\beta \in [0, 1]$,

$$c_{\beta} = \inf_{v \in H_{0,\text{rad}}^2(\Omega) \setminus \{0\}} \frac{\int_{\Omega} |\Delta v|^2 |x|^{(\beta-1)(N-4)} dx}{\left(\int_{\Omega} |v(x)|^p dx \right)^{2/p}} \quad (153)$$

is achieved by standard arguments. Since c_{β} is nondecreasing on $[0, 1]$, by (152), we obtain (141). $\square$

We are now ready for the main result of the paper.

Proof of Theorem 1. For every $\alpha > 0$, problem (1) has a solution in some H_k . Indeed, given $\alpha > 0$, there exists $k_{\alpha} > 0$ such that, for $k \geq K_{\alpha}$,

$$\Sigma_k < k^{4/N} S. \quad (154)$$

By Proposition 14, Σ_k is achieved by a function $u \in H_k$. By invariance, u is a critical point of Q_{α} on $H_0^2(\Omega)$ which, after scaling, gives rise to a weak solution of (1). By [13], u is a classical solution. We have to show that, at least for α large, u is not radial.

By Lemma 15,

$$\inf_{u \in H_{0,\text{rad}}^2(\Omega) \setminus \{0\}} Q_{\alpha}(u) \geq C\alpha^{(2N-4)/N}, \quad (155)$$

where the constant C depends only on N . We now show that the level Σ_k of the solution we find is strictly below this threshold for α large. To this aim, we must evaluate how large k_{α} of Proposition 7 has to be in terms of α . If we choose k of the order of $\alpha^{4(N-2)/3(N-5)+1/(N-5)}$, we see that (102) essentially becomes

$$\begin{aligned} R & \leq \frac{3S^{N/4}}{\alpha^{3/4(N-2)}} - \left(1 - \frac{6}{\alpha^{3/4(N-2)}} \right) \\ & \quad \cdot \frac{C_N a_0 C_2}{2} \alpha^{-(4(N-2)/3(N-5)+1/(N-5))\epsilon(N-4)} \\ & \quad + O\left(\alpha^{-(4(N-2)/3(N-5)+1/(N-5))\epsilon(N-2)}\right). \end{aligned} \quad (156)$$

Therefore, R will be negative for all α big enough when ϵ is sufficiently small, so we find a solution to (1) in the corresponding H_k . Therefore, we find a solution at level

$$\begin{aligned} \Sigma_k & < k^{4/N} S \leq C\alpha^{4(4N-3)/3N(N-5)} \\ & \leq C\alpha^{(2N-4)/N} \leq \inf_{u \in H_{0,\text{rad}}^2(\Omega) \setminus \{0\}} Q_{\alpha}(u) \end{aligned} \quad (157)$$

for all α large enough since $4(4N-3)/3N(N-5) \leq (2N-4)/N$ for all $N \geq 8$. Therefore, our solution cannot be radial. $\square$

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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