Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2014, Article ID 691632, 9 pages
http://dx.doi.org/10.1155/2014/691632

*Research Article*

# New Difference Sequence Spaces Defined by Musielak-Orlicz Function

**M. Mursaleen,[1] Sunil K. Sharma,[2] S. A. Mohiuddine,[3] and A. Kılıçman[4]**

[1] *Department of Mathematics, Aligarh Muslim University, Aligarh 202002, India*
[2] *Department of Mathematics, Model Institute of Engineering & Technology, Kot Bhalwal, Jammu and Kashmir 181122, India*
[3] *Department of Mathematics, Faculty of Science, King Abdulaziz University, P.O. Box 80203, Jeddah 21589, Saudi Arabia*
[4] *Department of Mathematics, University Putra Malaysia (UPM), 43400 Serdang, Selangor, Malaysia*

Correspondence should be addressed to A. Kılıçman; akilic@upm.edu.my

Received 17 March 2014; Revised 11 July 2014; Accepted 11 July 2014; Published 22 July 2014

Academic Editor: Feyzi Başar



We introduce new sequence spaces by using Musielak-Orlicz function and a generalized $B_\wedge^\mu$-difference operator on $n$-normed space. Some topological properties and inclusion relations are also examined.

## 1. Introduction and Preliminaries

The notion of the difference sequence space was introduced by Kızmaz [1]. It was further generalized by Et and Çolak [2] as follows: $Z(\Delta^\mu) = \{x = (x_k) \in \omega : (\Delta^\mu x_k) \in z\}$ for $z = \ell_\infty, c$, and $c_0$, where $\mu$ is a nonnegative integer and

$$\Delta^\mu x_k = \Delta^{\mu-1} x_k - \Delta^{\mu-1} x_{k+1}, \quad \Delta^0 x_k = x_k \quad \forall k \in \mathbb{N} \tag{1}$$

or equivalent to the following binomial representation:

$$\Delta^\mu x_k = \sum_{\nu=0}^{\mu} (-1)^\nu \binom{\mu}{\nu} x_{k+\nu}. \tag{2}$$

These sequence spaces were generalized by Et and Basarir [3] taking $z = \ell_\infty(p), c(p)$, and $c_0(p)$.

Dutta [4] introduced the following difference sequence spaces using a new difference operator:

$$Z\left(\Delta_{(\eta)}\right) = \left\{x = (x_k) \in \omega : \Delta_{(\eta)} x \in z\right\} \quad \text{for } z = \ell_\infty, c, c_0, \tag{3}$$

where $\Delta_{(\eta)} x = (\Delta_{(\eta)} x_k) = (x_k - x_{k-\eta})$ for all $k, \eta \in \mathbb{N}$.

In [5], Dutta introduced the sequence spaces $\overline{c}(\|\cdot,\cdot\|, \Delta_{(\eta)}^\mu, p)$, $\overline{c_0}(\|\cdot,\cdot\|, \Delta_{(\eta)}^\mu, p)$, $\ell_\infty(\|\cdot,\cdot\|, \Delta_{(\eta)}^\mu, p)$, $m(\|\cdot,\cdot\|, \Delta_{(\eta)}^\mu, p)$, and $m_0(\|\cdot,\cdot\|, \Delta_{(\eta)}^\mu, p)$, where $\eta, \mu \in \mathbb{N}$ and $\Delta_{(\eta)}^\mu x_k = (\Delta_{(\eta)}^\mu x_k) = (\Delta_{(\eta)}^{\mu-1} x_k - \Delta_{(\eta)}^{\mu-1} x_{k-\eta})$ and $\Delta_{(\eta)}^0 x_k = x_k$ for all $k, \eta \in \mathbb{N}$, which is equivalent to the following binomial representation:

$$\Delta_{(\eta)}^\mu x_k = \sum_{\nu=0}^{\mu} (-1)^\nu \binom{\mu}{\nu} x_{k-\eta\nu}. \tag{4}$$

The difference sequence spaces have been studied by authors [6–14] and references therein. Başar and Altay [15] introduced the generalized difference matrix $B = (b_{mk})$ for all $k, m \in \mathbb{N}$, which is a generalization of $\Delta_{(1)}$-difference operator by

$$b_{mk} = \begin{cases} r, & k = m \\ s, & k = m-1 \\ 0, & (k > m) \text{ or } (0 \le k < m-1). \end{cases} \tag{5}$$

Başarir and Kayikçi [16] defined the matrix $B^\mu(b_{mk}^\mu)$ which reduced the difference matrix $\Delta_{(1)}^\mu$ in case $r = 1$, $s = -1$. The generalized $B^\mu$-difference operator is equivalent to the following binomial representation:

$$B^\mu x = B^\mu\left(x_k\right) = \sum_{\nu=0}^{\mu} \binom{\mu}{\nu} r^{\mu-\nu} s^\nu x_{k-\nu}. \tag{6}$$

Let $\wedge = (\wedge_k)$ be a sequence of nonzero scalars. Then, for a sequence space $E$, the multiplier sequence space $E_\wedge$, associated with the multiplier sequence $\wedge$, is defined as

$$E_\wedge = \{x = (x_k) \in \omega : (\wedge_k x_k) \in E\}. \tag{7}$$

An Orlicz function $M$ is a function, $M : [0,\infty) \to [0,\infty)$, which is continuous, nondecreasing, and convex with $M(0) = 0$, $M(x) > 0$ for $x > 0$, and $M(x) \to \infty$ as $x \to \infty$.

We say that an Orlicz function $M$ satsfies the $\Delta_2$-condition if there exists $K > 2$ and $x_0 \geq 0$ such that $M(2x) \leq KM(x)$ for all $x \geq x_0$. The $\Delta_2$-condition is equivalent to $M(Lx) \leq KLM(x)$ for all $x > x_0 > 0$ and for $L, K > 1$.

Lindenstrauss and Tzafriri [17] used the idea of Orlicz function to define the following sequence space:

$$\ell_M = \left\{x \in \omega : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) < \infty\right\} \tag{8}$$

which is called an Orlicz sequence space. The space $\ell_M$ is a Banach space with the norm

$$\|x\| = \inf\left\{\rho > 0 : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) \leq 1\right\}. \tag{9}$$

It is shown in [17] that every Orlicz sequence space $\ell_M$ contains a subspace isomorphic to $\ell_p$ $(p \geq 1)$.

A sequence $\mathscr{M} = (M_k)$ of Orlicz function is called a Musielak-Orlicz function; see [18, 19]. A sequence $\mathscr{N} = (N_k)$ defined by

$$N_k(v) = \sup\{|v|\, u - M_k(u) : u \geq 0\}, \quad k = 1, 2, \ldots, \tag{10}$$

is called the complimentary function of a Musielak-Orlicz function $\mathscr{M}$. For a given Musielak-Orlicz function $\mathscr{M}$, the Musielak-Orlicz sequence space $t_{\mathscr{M}}$ and its subspace $h_{\mathscr{M}}$ are defined as follows:

$$\begin{gathered} t_{\mathscr{M}} = \{x \in \omega : I_M(cx) < \infty \text{ for some } c > 0\}, \\ h_{\mathscr{M}} = \{x \in \omega : I_M(cx) < \infty \ \forall c > 0\}, \end{gathered} \tag{11}$$

where $I_{\mathscr{M}}$ is a convex modular defined by

$$I_{\mathscr{M}}(x) = \sum_{k=1}^{\infty} M_k(x_k), \quad x = (x_k) \in t_M. \tag{12}$$

We consider $t_{\mathscr{M}}$ equipped with the Luxemburg norm

$$\|x\| = \inf\left\{k > 0 : I_M\left(\frac{x}{k}\right) \leq 1\right\} \tag{13}$$

or equipped with the Orlicz norm

$$\|x\|^0 = \inf\left\{\frac{1}{k}\left(1 + I_M(kx)\right) : k > 0\right\}. \tag{14}$$

By a lacunary sequence $\theta = (i_r)$, $r = 0, 1, 2, \ldots$, where $i_0 = 0$, we will mean an increasing sequence of nonnegative integers $h_r = (i_r - r_{r-1}) \to \infty$ $(r \to \infty)$. The intervals determined by $\theta$ are denoted by $I_r = (i_{r-1}, i_r]$ and the ratio $i_r/i_{r-1}$ will be denoted by $q_r$. The space of lacunary strongly convergent sequences $N_\theta$ was defined by Freedman et al. [20] as follows:

$$N_\theta = \left\{x = (x_k) : \lim_{r\to\infty} \frac{1}{h_r} \sum_{k\in I_r} |x_k - L| = 0, \text{for some } L\right\}. \tag{15}$$

The concept of 2-normed spaces was initially developed by Gähler [21] in the mid of 1960's, while that of $n$-normed spaces one can see in Misiak [22]. Since then, many others have studied this concept and obtained various results; see Gunawan [23, 24] and Gunawan and Mashadi [25]. For more details about sequence spaces see [26–33] and references therein. Let $n \in \mathbb{N}$ and $X$ be linear space over the field $\mathbb{K}$, where $\mathbb{K}$ is the field of real or complex numbers of dimension $d$, where $d \geq n \geq 2$.

A real valued function $\|\cdot, \ldots, \cdot\|$ on $X^n$ satisfying the following four conditions:

(1) $\|(x_1, x_2, \ldots, x_n)\| = 0$ if and only if $x_1, x_2, x_3, \ldots, x_n$ are linearly dependent in $X$;

(2) $\|(x_1, x_2, \ldots, x_n)\|$ is invariant under permutation;

(3) $\|(\alpha x_1, x_2, \ldots, x_n)\| = |\alpha| \|(x_1, x_2, \ldots, x_n)\|$ for any $\alpha \in \mathbb{K}$;

(4) $\|(x + x', x_2, \ldots, x_n)\| \leq \|(x, x_2, \ldots, x_n)\| + \|(x', x_2, \ldots, x_n)\|$

is called an $n$-norm on $X$ and the pair $(X, \|\cdot, \ldots, \cdot\|)$ is called an $n$-normed space over the field $\mathbb{K}$. For example, we may take $X = \mathbb{R}^n$ being equipped with the Euclidean $n$-norm $\|(x_1, x_2, \ldots, x_n)\|_E$ = the volume of the $n$-dimensional parallelepiped spanned by the vectors $x_1, x_2, \ldots, x_n$ which may be given explicitly by the formula

$$\|(x_1, x_2, \ldots, x_n)\|_E = \left|\det\left(x_{ij}\right)\right|, \tag{16}$$

where $x_i = (x_1, x_2, x_3, \ldots, x_n) \in \mathbb{R}^n$ for each $i = 1, 2, 3, \ldots, n$ and $\|\cdot\|_E$ denotes the Euclidean norm. Let $(X, \|\cdot, \ldots, \cdot\|)$ be an $n$-normed space of dimension $d \geq n \geq 2$ and $\{a_1, a_2, \ldots, a_n\}$ linearly independent set in $X$. Then the following function $\|(\cdot, \ldots, \cdot)\|_\infty$ on $X^{n-1}$ defined by

$$\begin{aligned} &\|(x_1, x_2, \ldots, x_n)\|_\infty \\ &\quad = \max\{\|(x_1, x_2, \ldots, x_{n-1}, a_i)\| : i = 1, 2, \ldots, n\} \end{aligned} \tag{17}$$

defines an $(n-1)$ norm on $X$ with respect to $\{(a_1, a_2, \ldots, a_n)\}$.

A sequence $(x_k)$ in an $n$-normed space $(X, \|\cdot, \ldots, \cdot\|)$ is said to converge to some $L \in X$ if

$$\begin{gathered} \lim_{k\to\infty} \|(x_k - L, z_1, \ldots, z_{n-1})\| = 0, \\ \text{for every } z_1, \ldots, z_{n-1} \in X. \end{gathered} \tag{18}$$

A sequence $(x_k)$ in a normed space $(X, \|\cdot, \ldots, \cdot\|)$ is said to be Cauchy if

$$\begin{gathered} \lim_{\substack{k\to\infty \\ p\to\infty}} \left\|\left(x_k - x_p, z_1, \ldots, z_{n-1}\right)\right\| = 0, \\ \text{for every } z_1, \ldots, z_{n-1} \in X. \end{gathered} \tag{19}$$

If every Cauchy sequence in $X$ converges to some $L \in X$ then $X$ is said to be complete with respect to the $n$-norm. Any complete $n$-normed space is said to be $n$-Banach space.

Let $(X, \|\cdot, \ldots, \cdot\|)$ be an $n$-normed space and let $s(\omega - x)$ denote the space of $X$-valued sequences. Let $p = (p_k)$ be any bounded sequence of positive real numbers and $\mathscr{M} = (M_k)$ a Musielak-Orlicz function. We define the following sequence spaces in this paper:

$$\begin{aligned}
&w_0^\theta(\mathscr{M}, B_\wedge^\mu, p, \|\cdot, \ldots, \cdot\|) \\
&= \left\{ x = (x_k) \in s(w - x) : \lim_{r\to\infty} \frac{1}{h_r} \right. \\
&\qquad \left. \times \sum_{k\in I_r} M_k\left(\left\|\left(\frac{B_\wedge^\mu x_k}{\rho}, z_1, \ldots, z_{n-1}\right)\right\|\right)^{p_k} = 0,\ \rho > 0 \right\}, \\
&w^\theta(\mathscr{M}, B_\wedge^\mu, p, \|\cdot, \ldots, \cdot\|) \\
&= \left\{ x = (x_k) \in s(w - x) : \lim_{r\to\infty} \frac{1}{h_r} \right. \\
&\qquad \times \sum_{k\in I_r} M_k\left(\left\|\left(\frac{B_\wedge^\mu x_k - L}{\rho}, z_1, \ldots, z_{n-1}\right)\right\|\right)^{p_k} = 0, \\
&\qquad \left. \text{for some } L, \rho > 0 \right\}, \\
&w_\infty^\theta(\mathscr{M}, B_\wedge^\mu, p, \|\cdot, \ldots, \cdot\|) \\
&= \left\{ x = (x_k) \in s(w - x) : \lim_{r\to\infty} \frac{1}{h_r} \right. \\
&\qquad \left. \times \sum_{k\in I_r} M_k\left(\left\|\left(\frac{B_\wedge^\mu x_k}{\rho}, z_1, \ldots, z_{n-1}\right)\right\|\right)^{p_k} < \infty,\ \rho > 0 \right\};
\end{aligned} \tag{20}$$

when $\mathscr{M}(x) = x$, we get

$$\begin{aligned}
&w_0^\theta(B_\wedge^\mu, p, \|\cdot, \ldots, \cdot\|) \\
&= \left\{ x = (x_k) \in s(w - x) : \lim_{r\to\infty} \frac{1}{h_r} \right. \\
&\qquad \left. \times \sum_{k\in I_r} \left(\left\|\left(\frac{B_\wedge^\mu x_k}{\rho}, z_1, \ldots, z_{n-1}\right)\right\|\right)^{p_k} = 0,\ \rho > 0 \right\}, \\
&w^\theta(B_\wedge^\mu, p, \|\cdot, \ldots, \cdot\|) \\
&= \left\{ x = (x_k) \in s(w - x) : \lim_{r\to\infty} \frac{1}{h_r} \right. \\
&\qquad \times \sum_{k\in I_r} \left(\left\|\left(\frac{B_\wedge^\mu x_k - L}{\rho}, z_1, \ldots, z_{n-1}\right)\right\|\right)^{p_k} = 0, \\
&\qquad \left. \text{for some } L,\ \rho > 0 \right\}, \\
&w_\infty^\theta(B_\wedge^\mu, p, \|\cdot, \ldots, \cdot\|) \\
&= \left\{ x = (x_k) \in s(w - x) : \lim_{r\to\infty} \frac{1}{h_r} \right. \\
&\qquad \left. \times \sum_{k\in I_r} \left(\left\|\left(\frac{B_\wedge^\mu x_k}{\rho}, z_1, \ldots, z_{n-1}\right)\right\|\right)^{p_k} < \infty,\ \rho > 0 \right\};
\end{aligned} \tag{21}$$

when $p_k = 1$, for all $k$, we get

$$\begin{aligned}
&w_0^\theta(\mathscr{M}, B_\wedge^\mu, \|\cdot, \ldots, \cdot\|) \\
&= \left\{ x = (x_k) \in w(s - x) : \lim_{r\to\infty} \frac{1}{h_r} \right. \\
&\qquad \left. \times \sum_{k\in I_r} M_k\left(\left\|\left(\frac{B_\wedge^\mu x_k}{\rho}, z_1, \ldots, z_{n-1}\right)\right\|\right) = 0,\ \rho > 0 \right\}, \\
&w^\theta(\mathscr{M}, B_\wedge^\mu, \|\cdot, \ldots, \cdot\|) \\
&= \left\{ x = (x_k) \in w(s - x) : \lim_{r\to\infty} \frac{1}{h_r} \right. \\
&\qquad \times \sum_{k\in I_r} M_k\left(\left\|\left(\frac{B_\wedge^\mu x_k - L}{\rho}, z_1, \ldots, z_{n-1}\right)\right\|\right) = 0, \\
&\qquad \left. \text{for some } L,\ \rho > 0 \right\}, \\
&w_\infty^\theta(\mathscr{M}, B_\wedge^\mu, \|\cdot, \ldots, \cdot\|) \\
&= \left\{ x = (x_k) \in w(s - x) : \lim_{r\to\infty} \frac{1}{h_r} \right. \\
&\qquad \left. \times \sum_{k\in I_r} M_k\left(\left\|\left(\frac{B_\wedge^\mu x_k}{\rho}, z_1, \ldots, z_{n-1}\right)\right\|\right) < \infty,\ \rho > 0 \right\}.
\end{aligned} \tag{22}$$

The following inequality will be used throughout the paper. If $0 \le p_k \le \sup p_k = H$, $k = \max(1, 2^{H-1})$, then

$$|a_k + b_k|^{p_k} \le K\left\{|a_k|^{p_k} + |b_k|^{p_k}\right\} \tag{23}$$

for all $k$ and $a_k, b_k \in \mathbb{C}$. Also $|a|^{p_k} \le \max(1, |a|^H)$ for all $a \in \mathbb{C}$.

## 2. Main Results

**Theorem 1.** *Let $\mathscr{M} = (M_k)$ be a Musielak-Orlicz function and $p = (p_k)$ a bounded sequence of positive real numbers; the spaces $w_0^\theta(\mathscr{M}, B_\wedge^\mu, p, \|\cdot, \ldots, \cdot\|)$, $w^\theta(\mathscr{M}, B_\wedge^\mu, p, \|\cdot, \ldots, \cdot\|)$, and $w_\infty^\theta(\mathscr{M}, B_\wedge^\mu, p, \|\cdot, \ldots, \cdot\|)$ are linear over the field of complex numbers $\mathbb{C}$.*

*Proof.* Let $x = (x_k)$, $y = (y_k) \in w_0^\theta(\mathscr{M}, B_\wedge^\mu, p, \|\cdot, \ldots, \cdot\|)$, and $\alpha, \beta \in \mathbb{C}$. Then there exist positive real numbers $\rho_1$ and $\rho_2$ such that

$$\begin{aligned} \lim_{r\to\infty} \frac{1}{h_r} \sum_{k\in I_r} M_k\left(\left\|\left(\frac{B_\wedge^\mu x_k}{\rho_1}, z_1, \ldots, z_{n-1}\right)\right\|\right)^{p_k} &= 0, \\ \lim_{r\to\infty} \frac{1}{h_r} \sum_{k\in I_r} M_k\left(\left\|\left(\frac{B_\wedge^\mu y_k}{\rho_2}, z_1, \ldots, z_{n-1}\right)\right\|\right)^{p_k} &= 0. \end{aligned} \tag{24}$$

Define $\rho_3 = \max(2|\alpha|\rho_1, 2|\beta|\rho_2)$. Since $\|\cdot, \ldots, \cdot\|$ is an $n$-norm on $X$ and $M_k's$ are nondecreasing and convex functions so by using inequality (23) we have

$$\begin{aligned} &\lim_{r\to\infty} \frac{1}{h_r} \sum_{k\in I_r} M_k\left(\left\|\left(\frac{B_\wedge^\mu(\alpha x_k + \beta y_k)}{\rho_3}, z_1, \ldots, z_{n-1}\right)\right\|\right)^{p_k} \\ &\quad \le \lim_{r\to\infty} \frac{1}{h_r} \sum_{k\in I_r} M_k\left(\left\|\left(\frac{B_\wedge^\mu \alpha x_k}{\rho_3}, z_1, \ldots, z_{n-1}\right)\right\| \right. \\ &\qquad\qquad \left. + \left\|\left(\frac{B_\wedge^\mu \beta y_k}{\rho_3}, z_1, \ldots, z_{n-1}\right)\right\|\right)^{p_k} \\ &\quad \le K \lim_{r\to\infty} \frac{1}{h_r} \sum_{k\in I_r} \frac{1}{2^{p_k}} M_k\left(\left\|\left(\frac{B_\wedge^\mu x_k}{\rho_1}, z_1, \ldots, z_{n-1}\right)\right\|\right)^{p_k} \\ &\qquad + K \lim_{r\to\infty} \frac{1}{h_r} \sum_{k\in I_r} \frac{1}{2^{p_k}} M_k\left(\left\|\left(\frac{B_\wedge^\mu y_k}{\rho_2}, z_1, \ldots, z_{n-1}\right)\right\|\right)^{p_k} \\ &\quad = 0. \end{aligned} \tag{25}$$

Thus, we have $\alpha x + \beta y \in w_0^\theta(\mathscr{M}, B_\wedge^\mu, p, \|\cdot, \ldots, \cdot\|)$. Hence $w_0^\theta(\mathscr{M}, B_\wedge^\mu, p, \|\cdot, \ldots, \cdot\|)$ is a linear space. Similarly, we can prove that $w^\theta(\mathscr{M}, B_\wedge^\mu, p, \|\cdot, \ldots, \cdot\|)$ and $w_\infty^\theta(\mathscr{M}, B_\wedge^\mu, p, \|\cdot, \ldots, \cdot\|)$ are linear spaces. This completes the proof of the theorem. □

**Theorem 2.** *Let $\mathscr{M} = (M_k)$ be a Musielak-Orlicz function and $p = (p_k)$ a bounded sequence of positive real numbers; the space $w_0^\theta(\mathscr{M}, B_\wedge^\mu, p, \|\cdot, \ldots, \cdot\|)$ is a topological linear space paranormed by*

$$\begin{aligned} g(x) &= \inf\left\{\rho^{p_r/M} : \left(\frac{1}{h_r} \sum_{k\in I_r} M_k\left(\left\|\left(\frac{B_\wedge^\mu x_k}{\rho}, z_1, \ldots, z_{n-1}\right)\right\|\right)^{p_k}\right)^{1/M} \le 1\right\}, \end{aligned} \tag{26}$$

*where $M = \max(1, \sup_k p_k < \infty)$.*

*Proof.* Clearly $g(x) \ge 0$ for $x = (x_k) \in w_0^\theta(\mathscr{M}, B_\wedge^\mu, p, \|\cdot, \ldots, \cdot\|)$. Since $M_k(0) = 0$, we get $g(0) = 0$. Again, if $g(x) = 0$, then

$$\begin{aligned} g(x) &= \inf\left\{\rho^{p_r/M} : \left(\frac{1}{h_r} \sum_{k\in I_r} M_k\left(\left\|\left(\frac{B_\wedge^\mu x_k}{\rho}, z_1, \ldots, z_{n-1}\right)\right\|\right)^{p_k}\right)^{1/M} \le 1\right\} = 0. \end{aligned} \tag{27}$$

This implies that, for a given $\varepsilon > 0$, there exist some $\rho_\varepsilon$ $(0 < \rho_\varepsilon < \varepsilon)$ such that

$$\left(\frac{1}{h_r} \sum_{k\in I_r} M_k\left(\left\|\left(\frac{B_\wedge^\mu x_k}{\rho_\varepsilon}, z_1, \ldots, z_{n-1}\right)\right\|\right)^{p_k}\right)^{1/M} \le 1. \tag{28}$$

Thus

$$\begin{aligned} &\left(\frac{1}{h_r} \sum_{k\in I_r} M_k\left(\left\|\left(\frac{B_\wedge^\mu x_k}{\varepsilon}, z_1, \ldots, z_{n-1}\right)\right\|\right)^{p_k}\right)^{1/M} \\ &\quad \le \left(\frac{1}{h_r} \sum_{k\in I_r} M_k\left(\left\|\left(\frac{B_\wedge^\mu x_k}{\rho_\varepsilon}, z_1, \ldots, z_{n-1}\right)\right\|\right)^{p_k}\right)^{1/M} \\ &\quad \le 1 \end{aligned} \tag{29}$$

for each $r$, and suppose that $x_k \ne 0$ for each $k \in \mathbb{N}$. This implies that $B_\wedge^\mu x_k \ne 0$ for each $k \in \mathbb{N}$. Let $\varepsilon \to 0$, then

$$\left(\left(\left\|\left(\frac{B_\wedge^\mu x_k}{\rho_\varepsilon}, z_1, \ldots, z_{n-1}\right)\right\|\right)^{p_k}\right)^{1/M} \longrightarrow \infty; \tag{30}$$

which is a contradiction. Therefore, $B_{\wedge}^{\mu}x_k = 0$ for each $k$ and thus $x_k = 0$ for each $k \in \mathbb{N}$. Let $\rho_1 > 0$ and $\rho_2 > 0$ be such that

$$\left(\frac{1}{h_r}\sum_{k\in I_r} M_k\left(\left\|\left(\frac{B_{\wedge}^{\mu}x_k}{\rho_1}, z_1, \ldots, z_{n-1}\right)\right\|\right)^{p_k}\right)^{1/M} \leq 1,$$
$$\left(\frac{1}{h_r}\sum_{k\in I_r} M_k\left(\left\|\left(\frac{B_{\wedge}^{\mu}y_k}{\rho_2}, z_1, \ldots, z_{n-1}\right)\right\|\right)^{p_k}\right)^{1/M} \leq 1 \quad \text{for each } r. \tag{31}$$

Let $\rho = \rho_1 + \rho_2$; then by using Minkowski's inequality, we have

$$\begin{aligned}
&\left(\frac{1}{h_r}\sum_{k\in I_r} M_k\left(\left\|\left(\frac{B_{\wedge}^{\mu}(x_k + y_k)}{\rho}, z_1, \ldots, z_{n-1}\right)\right\|\right)^{p_k}\right)^{1/M} \\
&\leq \left(\frac{1}{h_r}\sum_{k\in I_r} M_k\left(\left\|\left(\frac{B_{\wedge}^{\mu}x_k + B_{\wedge}^{\mu}y_k}{\rho_1 + \rho_2}, z_1, \ldots, z_{n-1}\right)\right\|\right)^{p_k}\right)^{1/M} \\
&\leq \left(\frac{1}{h_r}\sum_{k\in I_r} M_k \times \left(\left(\frac{\rho_1}{\rho_1 + \rho_2}\right)\left\|\left(\frac{B_{\wedge}^{\mu}x_k}{\rho_1}, z_1, \ldots, z_{n-1}\right)\right\| + \left(\frac{\rho_2}{\rho_1 + \rho_2}\right)\left\|\left(\frac{B_{\wedge}^{\mu}y_k}{\rho_2}, z_1, \ldots, z_{n-1}\right)\right\|\right)^{p_k}\right)^{1/M} \\
&\leq \left(\frac{\rho_1}{\rho_1 + \rho_2}\right) \times \left(\frac{1}{h_r}\sum_{k\in I_r} M_k\left(\left\|\left(\frac{B_{\wedge}^{\mu}x_k}{\rho_1}, z_1, \ldots, z_{n-1}\right)\right\|\right)^{p_k}\right)^{1/M} + \left(\frac{\rho_2}{\rho_1 + \rho_2}\right) \times \left(\frac{1}{h_r}\sum_{k\in I_r} M_k\left(\left\|\left(\frac{B_{\wedge}^{\mu}y_k}{\rho_2}, z_1, \ldots, z_{n-1}\right)\right\|\right)^{p_k}\right)^{1/M} \leq 1.
\end{aligned} \tag{32}$$

Since $\rho's$ are nonnegative, we have

$$\begin{aligned}
g(x+y) &= \inf\left\{\rho^{p_r/M} : \left(\frac{1}{h_r}\sum_{k\in I_r} M_k \times \left(\left\|\left(\frac{B_{\wedge}^{\mu}x_k}{\rho}, z_1, \ldots, z_{n-1}\right)\right\|\right)^{p_k}\right)^{1/M} \leq 1\right\} \\
&\leq \inf\left\{\rho_1{}^{p_r/M} : \left(\frac{1}{h_r}\sum_{k\in I_r} M_k \times \left(\left\|\left(\frac{B_{\wedge}^{\mu}x_k}{\rho_1}, z_1, \ldots, z_{n-1}\right)\right\|\right)^{p_k}\right)^{1/M} \leq 1\right\} \\
&\quad + \inf\left\{\rho_2{}^{p_r/M} : \left(\frac{1}{h_r}\sum_{k\in I_r} M_k \times \left(\left\|\left(\frac{B_{\wedge}^{\mu}y_k}{\rho_2}, z_1, \ldots, z_{n-1}\right)\right\|\right)^{p_k}\right)^{1/M} \leq 1\right\}.
\end{aligned} \tag{33}$$

Therefore, $g(x + y) \leq g(x) + g(y)$.

Finally, we prove that the scalar multiplication is continuous. Let $\nu$ be any complex number. By definition,

$$g(\nu x) = \inf\left\{\rho^{p_r/M} : \left(\frac{1}{h_r}\sum_{k\in I_r} M_k \times \left(\left\|\left(\frac{\nu B_{\wedge}^{\mu}x_k}{\rho}, z_1, \ldots, z_{n-1}\right)\right\|\right)^{p_k}\right)^{1/M} \leq 1\right\}. \tag{34}$$

Then

$$g(\nu x) = \inf\left\{(|\nu|\,t)^{p_r/M} : \left(\frac{1}{h_r}\sum_{k\in I_r} M_k \times \left(\left\|\left(\frac{B_\wedge^\mu x_k}{t}, z_1, \ldots, z_{n-1}\right)\right\|\right)^{p_k}\right)^{1/M} \le 1\right\}, \tag{35}$$

where $t = \rho/|\nu|$. Since $|\nu|^{p_r} \le \max(1, |\nu| \sup p_k)$, we have

$$g(\nu x) = \max(1, |\nu| \sup p_k) \inf \times \left\{(t)^{p_r/M} : \left(\frac{1}{h_r}\sum_{k\in I_r} M_k \times \left(\left\|\left(\frac{B_\wedge^\mu x_k}{t}, z_1, \ldots, z_{n-1}\right)\right\|\right)^{p_k}\right)^{1/M} \le 1\right\}. \tag{36}$$

So, the fact that scalar multiplication is continuous follows from the above inequality. This completes the proof of theorem. □

**Theorem 3.** *Let $\mathscr{M} = (M_k)$ be a Musielak-Orlicz function. If $\sup_k (M_k(x))^{p_k} < \infty$ for all fixed $x > 0$, then $w^\theta(\mathscr{M}, B_\wedge^\mu, p, \|\cdot, \ldots, \cdot\|) \subset w_\infty^\theta(\mathscr{M}, B_\wedge^\mu, p, \|\cdot, \ldots, \cdot\|)$.*

*Proof.* Let $x = (x_k) \in w^\theta(\mathscr{M}, B_\wedge^\mu, p, \|\cdot, \ldots, \cdot\|)$. Then there exists some positive number $\rho_1$ such that

$$\lim_{r\to\infty}\frac{1}{h_r}\sum_{k\in I_r} M_k\left(\left\|\left(\frac{B_\wedge^\mu x_k - L}{\rho_1}, z_1, \ldots, z_{n-1}\right)\right\|\right)^{p_k} = 0. \tag{37}$$

Define $\rho = 2\rho_1$. Since $M_k$ is nondecreasing and convex and by using inequality (23), we have

$$\begin{aligned}
&\lim_{r\to\infty}\frac{1}{h_r}\sum_{k\in I_r} M_k\left(\left\|\left(\frac{B_\wedge^\mu x_k}{\rho}, z_1, \ldots, z_{n-1}\right)\right\|\right)^{p_k} \\
&= \lim_{r\to\infty}\frac{1}{h_r}\sum_{k\in I_r} M_k\left(\left\|\left(\frac{B_\wedge^\mu x_k - L + L}{\rho}, z_1, \ldots, z_{n-1}\right)\right\|\right)^{p_k} \\
&\le K\left\{\lim_{r\to\infty}\frac{1}{h_r}\sum_{k\in I_r} M_k \frac{1}{2^{p_k}}\left(\left\|\left(\frac{B_\wedge^\mu x_k - L}{\rho_1}, z_1, \ldots, z_{n-1}\right)\right\|\right)^{p_k} + \lim_{r\to\infty}\frac{1}{h_r}\sum_{k\in I_r} M_k \frac{1}{2^{p_k}}\left(\left\|\left(\frac{L}{\rho_1}, z_1, \ldots, z_{n-1}\right)\right\|\right)^{p_k}\right\} \\
&< K\left\{\lim_{r\to\infty}\frac{1}{h_r}\sum_{k\in I_r} M_k\left(\left\|\left(\frac{B_\wedge^\mu x_k - L}{\rho_1}, z_1, \ldots, z_{n-1}\right)\right\|\right)^{p_k} + \lim_{r\to\infty}\frac{1}{h_r}\sum_{k\in I_r} M_k\left(\left\|\left(\frac{L}{\rho_1}, z_1, \ldots, z_{n-1}\right)\right\|\right)^{p_k}\right\}.
\end{aligned} \tag{38}$$

Hence $x = (x_k) \in w_\infty^\theta(\mathscr{M}, B_\wedge^\mu, p, \|\cdot, \ldots, \cdot\|)$. This completes the proof of the theorem. □

**Theorem 4.** *Let $\mathscr{M} = (M_k)$ be a Musielak-Orlicz function and $0 < h = \inf p_k$. Then*

$$w_\infty^\theta(\mathscr{M}, B_\wedge^\mu, p, \|\cdot, \ldots, \cdot\|) \subset w_0^\theta(\mathscr{M}, B_\wedge^\mu, p, \|\cdot, \ldots, \cdot\|) \tag{39}$$

*if and only if*

$$\lim_{r\to\infty}\frac{1}{h_r}\sum_{k\in I_r} M_k(t)^{p_k} = \infty, \quad \text{for some } t > 0. \tag{40}$$

*Proof.* Let $w_\infty^\theta(\mathscr{M}, B_\wedge^\mu, p, \|\cdot, \ldots, \cdot\|) \subset w_0^\theta(\mathscr{M}, B_\wedge^\mu, p, \|\cdot, \ldots, \cdot\|)$. Suppose (40) does not hold. Therefore there are a subinterval $I_{r(m)}$ of the set of intervals $I_r$ and a number $n_0$, where $n_0 = \|(B_\wedge^\mu x_k/\rho, z_1, \ldots, z_{n-1})\|$ for all $k$, such that

$$\frac{1}{h_r(m)}\sum_{k\in I_{r(m)}} M(n_0)^{p_k} \le N < \infty, \quad m = 1, 2, 3, \ldots. \tag{41}$$

Let us define $x = (x_k)$ as follows:

$$B_\wedge^\mu x_k = \begin{cases} \rho n_0, & k \in I_{r(m)} \\ 0, & k \notin I_{r(m)}. \end{cases} \tag{42}$$

Thus by (41), $x = (x_k) \in w_\infty^\theta(\mathscr{M}, B_\wedge^\mu, p, \|\cdot, \ldots, \cdot\|)$. But $x = (x_k) \notin w_0^\theta(\mathscr{M}, B_\wedge^\mu, p, \|\cdot, \ldots, \cdot\|)$. Hence (40) must hold.

Conversely, suppose that (40) holds and let $x = (x_k) \in w_\infty^\theta(\mathscr{M}, B_\wedge^\mu, p, \|\cdot, \ldots, \cdot\|)$. Then,

$$\frac{1}{h_r}\sum_{k\in I_r} M_k\left(\left\|\left(\frac{B_\wedge^\mu x_k}{\rho}, z_1, \ldots, z_{n-1}\right)\right\|\right)^{p_k} \le N < \infty. \tag{43}$$

Suppose that $x = (x_k) \notin w_0^\theta(\mathscr{M}, B_\wedge^\mu, p, \|\cdot, \ldots, \cdot\|)$. Then for some number $\varepsilon$, $1 > \varepsilon > 0$, there is a number $N_0$ such that, for a subinterval $I_{r(m)}$ of the set of intervals $I_r$,

$$\left\|\left(\frac{B_\wedge^\mu x_k}{\rho}, z_1, \ldots, z_{n-1}\right)\right\| > \varepsilon \quad \text{for } N \ge N_0. \tag{44}$$

We have $M_k(\|(B_\wedge^\mu x_k/\rho, z_1, \ldots, z_{n-1})\|) \geq M(\varepsilon)^{p_k}$, which contradicts (40) by using (43). Hence we get

$$w_\infty^\theta(\mathscr{M}, B_\wedge^\mu, p, \|\cdot, \ldots, \cdot\|) \subset w_0^\theta(\mathscr{M}, B_\wedge^\mu, p, \|\cdot, \ldots, \cdot\|). \quad (45)$$

This completes the proof. □

**Theorem 5.** *Let $0 < h = \inf p_k \leq \sup p_k = H < \infty$. For any Musielak-Orlicz function $\mathscr{M} = (M_k)$ which satisfies $\Delta_2$-condition, one has*

(i) $w_0^\theta(B_\wedge^\mu, p, \|\cdot, \ldots, \cdot\|) \subset w_0^\theta(\mathscr{M}, B_\wedge^\mu, p, \|\cdot, \ldots, \cdot\|)$

(ii) $w^\theta(B_\wedge^\mu, p, \|\cdot, \ldots, \cdot\|) \subset w^\theta(\mathscr{M}, B_\wedge^\mu, p, \|\cdot, \ldots, \cdot\|)$

(iii) $w_\infty^\theta(B_\wedge^\mu, p, \|\cdot, \ldots, \cdot\|) \subset w_\infty^\theta(\mathscr{M}, B_\wedge^\mu, p, \|\cdot, \ldots, \cdot\|)$.

*Proof.* (i) Let $x = (x_k) \in w_0^\theta(B_\wedge^\mu, p, \|\cdot, \ldots, \cdot\|)$. Then, we have

$$\frac{1}{h_r}\sum_{k\in I_r}\left(\left\|\left(\frac{B_\wedge^\mu x_k}{\rho}, z_1, \ldots, z_{n-1}\right)\right\|\right)^{p_k} \longrightarrow 0 \quad \text{as } r \longrightarrow \infty. \quad (46)$$

Let $\varepsilon > 0$, and choose $\delta$ with $0 < \delta < 1$ such that $M_k < \varepsilon$ for $0 \leq t \leq \delta$. We can write

$$\begin{aligned}
&\frac{1}{h_r}\sum_{k\in I_r}\left(\left\|\left(\frac{B_\wedge^\mu x_k}{\rho}, z_1, \ldots, z_{n-1}\right)\right\|\right)^{p_k} \\
&= \frac{1}{h_r}\sum_{\substack{k\in I_r \\ \|(B_\wedge^\mu x_k/\rho, z_1, \ldots, z_{n-1})\|\leq\delta}}\left(\left\|\left(\frac{B_\wedge^\mu x_k}{\rho}, z_1, \ldots, z_{n-1}\right)\right\|\right)^{p_k} \\
&\quad + \frac{1}{h_r}\sum_{\substack{k\in I_r \\ \|(B_\wedge^\mu x_k/\rho, z_1, \ldots, z_{n-1})\|>\delta}}\left(\left\|\left(\frac{B_\wedge^\mu x_k}{\rho}, z_1, \ldots, z_{n-1}\right)\right\|\right)^{p_k}.
\end{aligned} \quad (47)$$

For the first summation above, we can write

$$\begin{aligned}
&\frac{1}{h_r}\sum_{\substack{k\in I_r \\ \|(B_\wedge^\mu x_k/\rho, z_1, \ldots, z_{n-1})\|\leq\delta}}\left(\left\|\left(\frac{B_\wedge^\mu x_k}{\rho}, z_1, \ldots, z_{n-1}\right)\right\|\right)^{p_k} \\
&\quad < \max\left(\varepsilon, \varepsilon^h\right).
\end{aligned} \quad (48)$$

By using continuity of $M_k$, for the second summation we can write

$$\left\|\left(\frac{B_\wedge^\mu x_k}{\rho}, z_1, \ldots, z_{n-1}\right)\right\| < 1 + \frac{(\|(B_\wedge^\mu x_k/\rho, z_1, \ldots, z_{n-1})\|)}{\delta}. \quad (49)$$

Since each $M_k$ is nondecreasing and convex and satisfies $\Delta_2$-condition, it follows that

$$\begin{aligned}
&\frac{1}{h_r}\sum_{k\in I_r} M_k\left(\left\|\left(\frac{B_\wedge^\mu x_k}{\rho}, z_1, \ldots, z_{n-1}\right)\right\|\right)^{p_k} \\
&\leq \max\left(\varepsilon, \varepsilon^h\right) \\
&\quad + \max\left\{1, \left[\frac{2M_k(\|(B_\wedge^\mu x_k/\rho, z_1, \ldots, z_{n-1})\|)}{\delta}\right]^h\right\} \\
&\quad \times \frac{1}{h_r}\sum_{k\in I_r}\left(\left\|\left(\frac{B_\wedge^\mu x_k}{\rho}, z_1, \ldots, z_{n-1}\right)\right\|\right)^{p_k}.
\end{aligned} \quad (50)$$

Taking limit as $\varepsilon \to 0$ and $r \to \infty$, it follows that $x = (x_k) \in w_0^\theta(\mathscr{M}, B_\wedge^\mu, p, \|\cdot, \ldots, \cdot\|)$. Hence $w_0^\theta(B_\wedge^\mu, p, \|\cdot, \ldots, \cdot\|) \subset w_0^\theta(\mathscr{M}, B_\wedge^\mu, p, \|\cdot, \ldots, \cdot\|)$. Similarly, we can prove (ii) and (iii). This completes the proof of the theorem. □

**Theorem 6.** *Let $\mathscr{M} = (M_k)$ be a Musielak-Orlicz function. Then the following statements are equivalent:*

(i) $w_\infty^\theta(B_\wedge^\mu, p, \|\cdot, \ldots, \cdot\|) \subset w_\infty^\theta(\mathscr{M}, B_\wedge^\mu, p, \|\cdot, \ldots, \cdot\|)$;

(ii) $w_0^\theta(B_\wedge^\mu, p, \|\cdot, \ldots, \cdot\|) \subset w_\infty^\theta(\mathscr{M}, B_\wedge^\mu, p, \|\cdot, \ldots, \cdot\|)$;

(iii) $\sup_r(1/h_r)\sum_{k\in I_r} M_k(t)^{p_k} < \infty$ *for all* $t > 0$, *where* $t = \|B_\wedge^\mu x_k/\rho, z_1, \ldots, z_{n-1}\|$.

*Proof.* (i) ⇒ (ii) Suppose (i) holds. In order to prove (ii) we have to show that

$$w^\theta(B_\wedge^\mu, p, \|\cdot, \ldots, \cdot\|) \subset w_\infty^\theta(\mathscr{M}, B_\wedge^\mu, p, \|\cdot, \ldots, \cdot\|). \quad (51)$$

Let $x = (x_k) \in w_0^\theta(B_\wedge^\mu, p, \|\cdot, \ldots, \cdot\|)$. Then for a given $\varepsilon > 0$ there exists $s > s_\varepsilon$ such that

$$\frac{1}{h_r}\sum_{k\in I_r}\left(\left\|\left(\frac{B_\wedge^\mu x_k}{\rho}, z_1, \ldots, z_{n-1}\right)\right\|\right)^{p_k} < \varepsilon. \quad (52)$$

Hence there exists $K > 0$ such that

$$\sup_r \frac{1}{h_r}\sum_{k\in I_r}\left(\left\|\left(\frac{B_\wedge^\mu x_k}{\rho}, z_1, \ldots, z_{n-1}\right)\right\|\right)^{p_k} < K. \quad (53)$$

This shows that $x = (x_k) \in w_\infty^\theta(\mathscr{M}, B_\wedge^\mu, p, \|\cdot, \ldots, \cdot\|)$.

(ii) ⇒ (iii) Suppose (ii) holds and (iii) fails to hold. Then for some $t > 0$,

$$\sup_r \frac{1}{h_r}\sum_{k\in I_r} M_k(\varepsilon)^{p_k} = \infty, \quad (54)$$

and, therefore, we can find a subinterval $I_{r(m)}$ of the set of intervals $I_r$ such that

$$\frac{1}{h_r(m)}\sum_{k\in I_{r(m)}} M_k\left(\frac{1}{m}\right)^{p_k} \geq m, \quad m = 1, 2, 3, \ldots. \quad (55)$$

Let us define $x = (x_k)$ as follows:

$$B_\wedge^\mu x_k = \begin{cases} \dfrac{\rho}{m}, & k \in I_{r(m)} \\ 0, & k \notin I_{r(m)}. \end{cases} \tag{56}$$

Thus $x = (x_k) \in w_0^\theta(B_\wedge^\mu, p, \|\cdot, \ldots, \cdot\|)$. But by (55), $x = (x_k) \notin w_\infty^\theta(\mathscr{M}, B_\wedge^\mu, p, \|\cdot, \ldots, \cdot\|)$ which contradicts (ii). Hence (iii) must hold.

(iii) $\Rightarrow$ (i) Let (iii) hold. Suppose that $x = (x_k) \notin w_\infty^\theta(\mathscr{M}, B_\wedge^\mu, p, \|\cdot, \ldots, \cdot\|)$. Then for $x = (x_k) \in w_\infty^\theta(B_\wedge^\mu, p, \|\cdot, \ldots, \cdot\|)$.

$$\sup_r \frac{1}{h_r} \sum_{k \in I_r} M_k\left(\left\|\left(\frac{B_\wedge^\mu x_k}{\rho}, z_1, \ldots, z_{n-1}\right)\right\|\right)^{p_k} = \infty. \tag{57}$$

Let $t = \|B_\wedge^\mu x_k/\rho, z_1, \ldots, z_{n-1}\|$ for each $k$, and then by (57) $\sup_r (1/h_r) \sum_{k \in I_r} M_k(t)^{p_k} = \infty$, which contradicts (iii). Hence (i) must hold. This completes the proof of the theorem. □

**Theorem 7.** *Let $\mathscr{M} = (M_k)$ be a Musielak-Orlicz function. Then the following statements are equivalent:*

(i) $w_0^\theta(\mathscr{M}, B_\wedge^\mu, p, \|\cdot, \ldots, \cdot\|) \subset w_0^\theta(B_\wedge^\mu, p, \|\cdot, \ldots, \cdot\|)$;

(ii) $w_0^\theta(\mathscr{M}, B_\wedge^\mu, p, \|\cdot, \ldots, \cdot\|) \subset w_\infty^\theta(B_\wedge^\mu, p, \|\cdot, \ldots, \cdot\|)$;

(iii) $\inf_r (1/h_r) \sum_{k \in I_r} M_k(t)^{p_k} > 0$ *for all* $t > 0$.

*Proof.* (i) $\Rightarrow$ (ii) is obvious

(ii) $\Rightarrow$ (iii) Let (ii) hold and let (iii) fail to hold. Then

$$\inf_r \frac{1}{h_r} \sum_{k \in I_r} M_k(t)^{p_k} = 0 \quad \text{for some } t > 0, \tag{58}$$

and we can find a subinterval $I_{r(m)}$ of the set of intervals $I_r$ such that

$$\frac{1}{h_r(m)} \sum_{k \in I_{r(m)}} M_k(m)^{p_k} < \frac{1}{m}, \quad m = 1, 2, 3, \ldots. \tag{59}$$

Let us define $x = (x_k)$ as follows:

$$B_\wedge^\mu x_k = \begin{cases} \rho m, & k \in I_{r(m)} \\ 0, & k \notin I_{r(m)}. \end{cases} \tag{60}$$

Thus by (iii), $x = (x_k) \in w_0^\theta(\mathscr{M}, B_\wedge^\mu, p, \|\cdot, \ldots, \cdot\|)$. But $x = (x_k) \notin w_\infty^\theta(B_\wedge^\mu, p, \|\cdot, \ldots, \cdot\|)$ which contradict (ii). Hence (iii) must hold.

(iii) $\Rightarrow$ (i) Let (iii) hold. Suppose that $x = (x_k) \in w_0^\theta(\mathscr{M}, B_\wedge^\mu, p, \|\cdot, \ldots, \cdot\|)$. Therefore,

$$\frac{1}{h_r} \sum_{k \in I_r} M_k\left(\left\|\left(\frac{B_\wedge^\mu x_k}{\rho}, z_1, \ldots, z_{n-1}\right)\right\|\right)^{p_k} \longrightarrow 0 \quad \text{as } r \longrightarrow \infty. \tag{61}$$

Again suppose $x = (x_k) \notin w_0^\theta(B_\wedge^\mu, p, \|\cdot, \ldots, \cdot\|)$ for some number $\varepsilon > 0$ and a subinterval $I_{r(m)}$ of the set of intervals $I_r$, we have

$$\left\|\left(\frac{B_\wedge^\mu x_k}{\rho}, z_1, \ldots, z_{n-1}\right)\right\| \geq \varepsilon \quad \forall k. \tag{62}$$

Then, from properties of the Orlicz function, we can write

$$M_k\left(\left\|\left(\frac{B_\wedge^\mu x_k}{\rho}, z_1, \ldots, z_{n-1}\right)\right\|\right)^{p_k} \geq M_k(\varepsilon)^{p_k}. \tag{63}$$

Consequently, by (61), we have $\lim_{r \to \infty} (1/h_r) \sum_{k \in I_r} M_k(\varepsilon)^{p_k} = 0$, which contradicts (iii). Hence (i) must hold. This completes the proof of the theorem. □

**Theorem 8.** *(i) If $0 < \inf p_k \leq p_k \leq 1$ for all $k$, then $w^\theta(\mathscr{M}, B_\wedge^\mu, \|\cdot, \ldots, \cdot\|) \subseteq w^\theta(\mathscr{M}, B_\wedge^\mu, p, \|\cdot, \ldots, \cdot\|)$.*

*(ii) If $1 \leq p_k \leq \sup p_k = H < \infty$, then $w^\theta(\mathscr{M}, B_\wedge^\mu, p, \|\cdot, \ldots, \cdot\|) \subseteq w^\theta(\mathscr{M}, B_\wedge^\mu, \|\cdot, \ldots, \cdot\|)$.*

*Proof.* (i) Let $x \in w^\theta(\mathscr{M}, B_\wedge^\mu, \|\cdot, \ldots, \cdot\|)$. Since $0 < \inf p_k \leq 1$, we get

$$\begin{aligned} &\frac{1}{h_r} \sum_{k \in I_r} M_k\left(\left\|\left(\frac{B_\wedge^\mu x_k - L}{\rho}, z_1, \ldots, z_{n-1}\right)\right\|\right) \\ &\quad \leq \frac{1}{h_r} \sum_{k \in I_r} M_k\left(\left\|\left(\frac{B_\wedge^\mu x_k - L}{\rho}, z_1, \ldots, z_{n-1}\right)\right\|\right)^{p_k}, \end{aligned} \tag{64}$$

and hence $x \in w^\theta(\mathscr{M}, B_\wedge^\mu, p, \|\cdot, \ldots, \cdot\|)$.

(ii) $1 \leq p_k \leq \sup p_k = H < \infty$ and $x = (x_k) \in w^\theta(\mathscr{M}, B_\wedge^\mu, p, \|\cdot, \ldots, \cdot\|)$. Then for each $0 < \varepsilon < 1$ there exists a positive integer $s_0$ such that

$$\begin{aligned} &\frac{1}{h_r} \sum_{k \in I_r} M_k\left(\left\|\left(\frac{B_\wedge^\mu x_k - L}{\rho}, z_1, \ldots, z_{n-1}\right)\right\|\right)^{p_k} \\ &\quad \leq \varepsilon < 1 \quad \forall r > s_0. \end{aligned} \tag{65}$$

This implies that

$$\begin{aligned} &\frac{1}{h_r} \sum_{k \in I_r} M_k\left(\left\|\left(\frac{B_\wedge^\mu x_k - L}{\rho}, z_1, \ldots, z_{n-1}\right)\right\|\right)^{p_k} \\ &\quad \leq \frac{1}{h_r} \sum_{k \in I_r} M_k\left(\left\|\left(\frac{B_\wedge^\mu x_k - L}{\rho}, z_1, \ldots, z_{n-1}\right)\right\|\right). \end{aligned} \tag{66}$$

Therefore $x = (x_k) \in w^\theta(\mathscr{M}, B_\wedge^\mu, \|\cdot, \ldots, \cdot\|)$. This completes the proof of the theorem. □

## Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

## Acknowledgments

The authors gratefully acknowledge that this research was partially supported by the University Putra Malaysia under the ERGS Grant Scheme having Project no. ERGS 1-2013/5527179. The authors are grateful also to the anonymous referees for a careful checking of the details and for helpful comments that improved the paper.

## References


[1] H. Kızmaz, "On certain sequence spaces," *Canadian Mathematical Bulletin*, vol. 24, no. 2, pp. 169–176, 1981.

[2] M. Et and R. Çolak, "On some generalized difference sequence spaces," *Soochow Journal of Mathematics*, vol. 21, no. 4, pp. 377–386, 1995.

[3] M. Et and M. Basarir, "On some new generalized difference sequence spaces," *Periodica Mathematica Hungarica*, vol. 35, no. 3, pp. 169–175, 1997.

[4] H. Dutta, "On some differece sequences spaces," *Pacific Journal of Science and Technology*, vol. 10, pp. 243–247, 2009.

[5] H. Dutta, "Some statistically convergent difference sequence spaces defined over real 2-normed linear spaces," *Applied Sciences*, vol. 12, pp. 37–47, 2010.

[6] M. Işik, "On statistical convergence of generalized difference sequences," *Soochow Journal of Mathematics*, vol. 30, no. 2, pp. 197–205, 2004.

[7] M. Mursaleen, "Generalized spaces of difference sequences," *Journal of Mathematical Analysis and Applications*, vol. 203, no. 3, pp. 738–745, 1996.

[8] M. Mursaleen, V. Karakaya, H. Polat, and N. Simşek, "Measure of noncompactness of matrix operators on some difference sequence spaces of weighted means," *Computers & Mathematics with Applications*, vol. 62, no. 2, pp. 814–820, 2011.

[9] M. Mursaleen and A. K. Noman, "On some new difference sequence spaces of non-absolute type," *Mathematical and Computer Modelling*, vol. 52, no. 3-4, pp. 603–617, 2010.

[10] M. Mursaleen and A. K. Noman, "Compactness of matrix operators on some new difference sequence spaces," *Linear Algebra and Its Applications*, vol. 436, no. 1, pp. 41–52, 2012.

[11] S. A. Mohiuddine and A. Alotaibi, "Some spaces of double sequences obtained through invariant mean and related concepts," *Abstract and Applied Analysis*, vol. 2013, Article ID 507950, 11 pages, 2013.

[12] S. A. Mohiuddine, K. Raj, and A. Alotaibi, "Some paranormed double difference sequence spaces for Orlicz functions and bounded-regular matrices," *Abstract and Applied Analysis*, vol. 2014, Article ID 419064, 10 pages, 2014.

[13] K. Raj, S. K. Sharma, and A. K. Sharma, "Some difference sequence spaces in $n$-normed spaces defined by Musielak-Orlicz function," *Armenian Journal of Mathematics*, vol. 3, pp. 127–141, 2010.

[14] T. Bilgin, "Some new difference sequences spaces defined by an Orlicz function," *Filomat*, no. 17, pp. 1–8, 2003.

[15] F. Başar and B. Altay, "On the space of sequences of $p$-bounded variation and related matrix mappings," *Ukrainian Mathematical Journal*, vol. 55, no. 1, pp. 108–118, 2003.

[16] M. Başarir and M. Kayikçi, "On the generalized $B^m$-Riesz difference sequence space and $\beta$-property," *Journal of Inequalities and Applications*, vol. 2009, Article ID 385029, 18 pages, 2009.

[17] J. Lindenstrauss and L. Tzafriri, "On Orlicz sequence spaces," *Israel Journal of Mathematics*, vol. 10, pp. 379–390, 1971.

[18] L. Maligranda, *Orlicz Spaces and Interpolation*, Seminars in Mathematics, Polish Academy of Science, Warsaw, Poland, 1989.

[19] J. Musielak, *Orlicz Spaces and Modular Spaces*, vol. 1034 of *Lecture Notes in Mathematics*, Springer, Berlin, Germany, 1983.

[20] A. R. Freedman, J. J. Sember, and M. Raphael, "Some Cesaro-type summability spaces," *Proceedings of the London Mathematical Society*, vol. 37, no. 3, pp. 508–520, 1978.

[21] S. Gähler, "2-metrische Räume und ihre topologische Struktur," *Mathematische Nachrichten*, vol. 26, pp. 115–148, 1963.

[22] A. Misiak, "$n$-inner product spaces," *Mathematische Nachrichten*, vol. 140, pp. 299–319, 1989.

[23] H. Gunawan, "On $n$-inner product, $n$-norms and the Cauchy Schwartz inequality," *Scientiae Mathematicae Japonicae*, vol. 5, pp. 47–54, 2001.

[24] H. Gunawan, "The space of $p$-summable sequences and its natural $n$-norm," *Bulletin of the Australian Mathematical Society*, vol. 64, no. 1, pp. 137–147, 2001.

[25] H. Gunawan and M. Mashadi, "On $n$-normed spaces," *International Journal of Mathematics and Mathematical Sciences*, vol. 27, no. 10, pp. 631–639, 2001.

[26] K. Raj, A. K. Sharma, and S. K. Sharma, "A sequence space defined by Musielak-Orlicz function," *International Journal of Pure and Applied Mathematics*, vol. 67, no. 4, pp. 475–484, 2011.

[27] K. Raj and S. K. Sharma, "Some sequence spaces in 2-normed spaces defined by Musielak-Orlicz function," *Acta Universitatis Sapientiae. Mathematica*, vol. 3, no. 1, pp. 97–109, 2011.

[28] M. Mursaleen, S. K. Sharma, and A. Kılıçman, "Sequence spaces defined by Musielak-Orlicz function over $n$-normed spaces," *Abstract and Applied Analysis*, vol. 2013, Article ID 364743, 10 pages, 2013.

[29] K. Raj and S. K. Sharma, "Some multiplier sequence spaces defined by a Musielak-Orlicz function in $n$-normed spaces," *New Zealand Journal of Mathematics*, vol. 42, pp. 45–56, 2012.

[30] M. Et, Y. Altin, B. Choudhary, and B. C. Tripathy, "On some classes of sequences defined by sequences of Orlicz functions," *Mathematical Inequalities & Applications*, vol. 9, no. 2, pp. 335–342, 2006.

[31] M. Gungor and M. Et, "$\Delta^r$-strongly almost summable sequences defined by Orlicz functions," *Indian Journal of Pure and Applied Mathematics*, vol. 34, no. 8, pp. 1141–1151, 2003.

[32] M. Mursaleen, R. Çolak, and M. Et, "Some geometric inequalities in a new Banach sequence space," *Journal of Inequalities and Applications*, vol. 2007, Article ID 86757, 6 pages, 2007.

[33] B. C. Tripathy and H. Dutta, "On some new paranormed difference sequence spaces defined by Orlicz functions," *Kyungpook Mathematical Journal*, vol. 50, no. 1, pp. 59–69, 2010.