Research Article

New Difference Sequence Spaces Defined by Musielak-Orlicz Function

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We introduce new sequence spaces by using Musielak-Orlicz function and a generalized $B^\mu$-difference operator on $n$-normed space. Some topological properties and inclusion relations are also examined.

1. Introduction and Preliminaries

The notion of the difference sequence space was introduced by Kizmaz [1]. It was further generalized by Et and Çolak [2] as follows: $Z(\Delta^\mu) = \{x = (x_k) \in \omega : (\Delta^\mu x_k) \in z\}$ for $z = \ell_{\infty}$, $\ell_1$, and $\ell_0$, where $\mu$ is a nonnegative integer and

$$\Delta^\mu x_k = \Delta^{\mu-1} x_k - \Delta^{\mu-1} x_{k+1}, \quad \Delta^0 x_k = x_k \quad \forall k \in \mathbb{N} \quad (1)$$

or equivalent to the following binomial representation:

$$\Delta^\mu x_k = \sum_{v=0}^{\mu} (-1)^v \binom{\mu}{v} x_{k+v} \quad (2)$$

These sequence spaces were generalized by Et and Basarir [3] taking $z = \ell_{\infty}(p)$, $c(p)$, and $c_0(p)$.

Dutta [4] introduced the following difference sequence spaces using a new difference operator:

$$Z(\Delta^\eta) = \{x = (x_k) \in \omega : \Delta^\eta x_k \in z\} \quad \text{for } z = \ell_{\infty}, c, c_0. \quad (3)$$

where $\Delta^\eta x_k = (\Delta^\eta x_k) = (x_k - x_{k-\eta})$ for all $k, \eta \in \mathbb{N}$.

In [5], Dutta introduced the sequence spaces $c_0(\|\cdot\|, \|\Delta^\mu\|, p)$, $c_0(\|\cdot\|, \|\Delta^\eta\|, p)$, $\ell_{\infty}(\|\cdot\|, \|\Delta^\mu\|, p)$, $m(\|\cdot\|, \|\Delta^\mu\|, p)$, $\Delta^\mu$, and $m_0(\|\cdot\|, \|\Delta^\eta\|, p)$, where $\eta, \mu \in \mathbb{N}$ and $\Delta^\eta x_k = \Delta^{\eta-1} x_k - \Delta^{\eta-1} x_{k-\eta}$ and $\Delta^0 x_k = x_k$ for all $k, \eta \in \mathbb{N}$, which is equivalent to the following binomial representation:

$$\Delta^\mu x_k = \sum_{v=0}^{\mu} (-1)^v \binom{\mu}{v} x_{k-v} \quad (4)$$

The difference sequence spaces have been studied by authors [6–14] and references therein. Başar and Altay [15] introduced the generalized difference matrix $B = (b_{mk})$ for all $k, m \in \mathbb{N}$, which is a generalization of $\Delta^1$-difference operator by

$$b_{mk} = \begin{cases} r, & k = m \\ s, & k = m - 1 \\ 0, & (k > m) \text{ or } (0 \leq k < m - 1). \end{cases} \quad (5)$$

Başar and Kayıkçı [16] defined the matrix $B^\mu(b_{mk}^\mu)$ which reduced the difference matrix $\Delta^\mu$ in case $r = 1$, $s = -1$. The generalized $B^\mu$-difference operator is equivalent to the following binomial representation:

$$B^\mu x = B^\mu(x_k) = \sum_{v=0}^{\mu} r^v \binom{\mu}{v} x_{k-v} \quad (6)$$
Let $\land = (\land_k)$ be a sequence of nonzero scalars. Then, for a sequence space $E_\land$, the multiplier sequence space $E_\land$, associated with the multiplier sequence $\land$, is defined as

$$E_\land = \{ x = (x_k) \in \ell^\infty : (\land_k x_k) \in E \}. \quad (7)$$

An Orlicz function $M$ is a function, $M : [0, \infty) \rightarrow [0, \infty)$, which is continuous, nondecreasing, and convex with $M(0) = 0$, $M(x) > 0$ for $x > 0$, and $M(x) \rightarrow \infty$ as $x \rightarrow \infty$.

We say that an Orlicz function $M$ satisfies the $\Delta_2$-condition if there exists $K > 2$ and $x_0 \geq 0$ such that $M(2x) \leq KM(x)$ for all $x \geq x_0$. The $\Delta_2$-condition is equivalent to $M(Lx) \leq KLM(x)$ for all $x > x_0$ and for $L, K > 1$.

Lindenstrauss and Tzafriri [17] used the idea of Orlicz function to define the following sequence space:

$$\ell_M = \left\{ x \in \ell^\infty : \sum_{k=1}^{\infty} M \left( \frac{|x_k|}{\rho} \right) < \infty \right\}. \quad (8)$$

which is called an Orlicz sequence space. The space $\ell_M$ is a Banach space with the norm

$$\|x\| = \inf \left\{ \rho > 0 : \sum_{k=1}^{\infty} M \left( \frac{|x_k|}{\rho} \right) \leq 1 \right\}. \quad (9)$$

It is shown in [17] that every Orlicz sequence space $\ell_M$ contains a subspace isomorphic to $\ell_p (\rho \geq 1)$.

A sequence $\mathscr{A} = (M_k)$ of Orlicz functions is called a Musielak-Orlicz function; see [18, 19]. A sequence $\mathscr{N} = (N_k)$ defined by

$$N_k (v) = \sup \{ |v| u - M_k (u) : u \geq 0 \}, \quad k = 1, 2, \ldots, \quad (10)$$

called the complimentary function of a Musielak-Orlicz function $\mathscr{A}$. For a given Musielak-Orlicz function $\mathscr{A}$, the Musielak-Orlicz sequence space $t_{\mathscr{A}}$ and its subspace $h_{\mathscr{A}}$ are defined as follows:

$$t_{\mathscr{A}} = \left\{ x \in \ell^\infty : I_{\mathscr{A}} (cx) < \infty \text{ for some } c > 0 \right\},$$

$$h_{\mathscr{A}} = \left\{ x \in \ell^\infty : I_{\mathscr{A}} (cx) < \infty \text{ for all } c > 0 \right\},$$

where $I_{\mathscr{A}}$ is a convex modular defined by

$$I_{\mathscr{A}} (x) = \sum_{k=1}^{\infty} M_k (x_k), \quad x = (x_k) \in t_{\mathscr{A}}. \quad (12)$$

We consider $t_{\mathscr{A}}$ equipped with the Luxemburg norm

$$\|x\| = \inf \left\{ k > 0 : I_{\mathscr{A}} \left( \frac{x}{k} \right) \leq 1 \right\} \quad (13)$$

or equipped with the Orlicz norm

$$\|x\|_0 = \inf \left\{ \frac{1}{k} (1 + I_{\mathscr{A}} (kx)) : k > 0 \right\}. \quad (14)$$

By a lacunary sequence $\theta = (i_r)$, $r = 0, 1, 2, \ldots$, where $i_0 = 0$, we mean an increasing sequence of nonnegative integers $h_r = (i_r - r_{i_r}) \rightarrow \infty (r \rightarrow \infty)$. The intervals determined by $\theta$ are denoted by $L_r = (i_{r-1}, i_r]$ and the ratio $i_r/i_{r-1}$ will be denoted by $q_r$. The space of lacunary strongly convergent sequences $N_\theta$ was defined by Freedman et al. [20] as follows:

$$N_\theta = \left\{ x = (x_k) : \lim_{r \rightarrow \infty} \frac{1}{h_r} \sum_{k \in L_r} |x_k - L| = 0, \text{ for some } L \right\}. \quad (15)$$

The concept of 2-normed spaces was initially developed by Gähler [21] in the mid-1960’s, while that of $n$-normed spaces one can see in Misiak [22]. Since then, many others have studied this concept and obtained various results; see Gunawan [23, 24] and Gunawan and Mashadi [25]. For more details about sequence spaces see [26–33] and references therein. Let $n \in \mathbb{N}$ and $X$ be linear space over the field $\mathbb{K}$, where $\mathbb{K}$ is the field of real or complex numbers of dimension $d$, where $d \geq n \geq 2$.

A real valued function $\|\cdot, \ldots, \cdot\|$ on $X^n$ satisfying the following four conditions:

(1) $\|(x_1, x_2, \ldots, x_n)\| = 0$ if and only if $x_1, x_2, \ldots, x_n$ are linearly dependent in $X$;

(2) $\|(x_1, x_2, \ldots, x_n)\|$ is invariant under permutation;

(3) $\|(\alpha x_1, x_2, \ldots, x_n)\| = |\alpha| \|(x_1, x_2, \ldots, x_n)\|$ for any $\alpha \in \mathbb{K}$;

(4) $\|(x_1 + x'_1, x_2, \ldots, x_n)\| \leq \|(x_1, x_2, \ldots, x_n)\| + \|(x'_1, x_2, \ldots, x_n)\|$ is called an $n$-norm on $X$ and the pair $(X, \|\cdot, \ldots, \cdot\|)$ is called an $n$-normed space over the field $\mathbb{K}$. For example, we may take $X = \mathbb{R}^n$ being equipped with the Euclidean $n$-norm $\|(x_1, x_2, \ldots, x_n)\|_E = \sqrt{\sum_{i=1}^{n} x_i^2}$ of the $n$-dimensional parallelepiped spanned by the vectors $x_1, x_2, \ldots, x_n$ which may be given explicitly by the formula

$$\|(x_1, x_2, \ldots, x_n)\|_E = \left| \det (x_i) \right|, \quad (16)$$

where $x_i = (x_1, x_2, x_3, \ldots, x_n) \in \mathbb{R}^n$ for each $i = 1, 2, 3, \ldots, n$ and $\|\cdot\|_E$ denotes the Euclidean norm. Let $(X, \|\cdot, \ldots, \cdot\|)$ be an $n$-normed space of dimension $d \geq n \geq 2$ and $\{a_1, a_2, \ldots, a_n\}$ linearly independent set in $X$. Then the following function $\|(\cdot, \ldots, \cdot)\|_{\infty}$ on $X^{n-1}$ defined by

$$\|(x_1, x_2, \ldots, x_n)\|_{\infty} = \max \{ \|(x_1, x_2, \ldots, x_{n-1}, a_i)\| : i = 1, 2, \ldots, n \} \quad (17)$$

defines an $(n-1)$ norm on $X$ with respect to $\{a_1, a_2, \ldots, a_n\}$.

A sequence $(x_k)$ in an $n$-normed space $(X, \|\cdot, \ldots, \cdot\|)$ is said to converge to some $L \in X$ if

$$\lim_{k \rightarrow \infty} \|(x_k - L, z_1, \ldots, z_{n-1})\| = 0, \quad (18)$$

for every $z_1, \ldots, z_{n-1} \in X$.

A sequence $(x_k)$ in a normed space $(X, \|\cdot, \ldots, \cdot\|)$ is said to be Cauchy if

$$\lim_{r \rightarrow \infty} \|(x_k - x_p, z_1, \ldots, z_{n-1})\| = 0, \quad (19)$$

for every $z_1, \ldots, z_{n-1} \in X$. 

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If every Cauchy sequence in $X$ converges to some $L \in X$ then $X$ is said to be complete with respect to the $n$-norm. Any complete $n$-normed space is said to be $n$-Banach space.

Let $(X, ||\cdot||)$ be an $n$-normed space and let $s(w-x)$ denote the space of $X$-valued sequences. Let $p = (p_k)$ be any bounded sequence of positive real numbers and $\mathcal{M} = (M_k)$ a Musielak-Orlicz function. We define the following sequence spaces in this paper:

\[
\begin{align*}
    w_0^\theta (\mathcal{M}, B_{\alpha}^\mu, p, ||\cdot||, \ldots, ||) & = \left\{ x = (x_k) \in s(w-x) : \lim_{r \to \infty} \frac{1}{h_r} \times \sum_{k \in I_r} M_k \left( \left\| \left( \frac{B_{\alpha}^\mu x_k - L}{\rho}, z_1, \ldots, z_{n-1} \right) \right\| \right)^{p_k} = 0, \ \rho > 0 \right\}, \\
    w_\infty^\theta (\mathcal{M}, B_{\alpha}^\mu, p, ||\cdot||, \ldots, ||) & = \left\{ x = (x_k) \in s(w-x) : \lim_{r \to \infty} \frac{1}{h_r} \times \sum_{k \in I_r} M_k \left( \left\| \left( \frac{B_{\alpha}^\mu x_k - L}{\rho}, z_1, \ldots, z_{n-1} \right) \right\| \right)^{p_k} < \infty, \ \rho > 0 \right\};
\end{align*}
\]

when $p_k = 1$, for all $k$, we get

\[
\begin{align*}
    w_0^\theta (\mathcal{M}, B_{\alpha}^\mu, ||\cdot||, \ldots, ||) & = \left\{ x = (x_k) \in s(w-x) : \lim_{r \to \infty} \frac{1}{h_r} \times \sum_{k \in I_r} M_k \left( \left\| \left( \frac{B_{\alpha}^\mu x_k - L}{\rho}, z_1, \ldots, z_{n-1} \right) \right\| \right) = 0, \ \rho > 0 \right\}, \\
    w_\infty^\theta (\mathcal{M}, B_{\alpha}^\mu, ||\cdot||, \ldots, ||) & = \left\{ x = (x_k) \in s(w-x) : \lim_{r \to \infty} \frac{1}{h_r} \times \sum_{k \in I_r} M_k \left( \left\| \left( \frac{B_{\alpha}^\mu x_k - L}{\rho}, z_1, \ldots, z_{n-1} \right) \right\| \right) < \infty, \ \rho > 0 \right\};
\end{align*}
\]

(21)
The following inequality will be used throughout the paper. If
\[ 0 \leq p_k \leq \sup_{k} p_k = H, \quad k = \max(1, 2^{H-1}) \],
then
\[ |a_k + b_k|^{p_k} \leq K \left( |a_k|^{p_k} + |b_k|^{p_k} \right) \tag{23} \]
for all \( k \) and \( a_k, b_k \in \mathbb{C} \). Also \( |a|^{p_k} \leq \max(1, |a|^{|H|}) \) for all \( a \in \mathbb{C} \).

## 2. Main Results

**Theorem 1.** Let \( \mathcal{M} = (\mathcal{M}_k) \) be a Musielak-Orlicz function and \( p = (p_k) \) a bounded sequence of positive real numbers; the spaces \( \omega_0^p(\mathcal{M}, B^\mu_k, p, \|\cdot\|, \ldots, \|\cdot\|) \), \( \omega^p(\mathcal{M}, B^\mu_k, p, \|\cdot\|, \ldots, \|\cdot\|) \), and \( \omega_\infty^p(\mathcal{M}, B^\mu_k, p, \|\cdot\|, \ldots, \|\cdot\|) \) are linear over the field of complex numbers \( \mathbb{C} \).

**Proof.** Let \( x = (x_k), y = (y_k) \in \omega_0^p(\mathcal{M}, B^\mu_k, p, \|\cdot\|, \ldots, \|\cdot\|) \), and \( \alpha, \beta \in \mathbb{C} \). Then there exist positive real numbers \( \rho_1 \) and \( \rho_2 \) such that
\[
\lim_{r \to \infty} \frac{1}{h_r} \sum_{k \in \mathbb{N}} M_k \left( \left\| \frac{B^\mu_k x_k}{\rho_1}, z_1, \ldots, z_{n-1} \right\| \right)^{p_k} = 0, \tag{24}
\]
Define \( \rho_3 = \max(2|\alpha|\rho_1, 2|\beta|\rho_2) \). Since \( \|\cdot\|, \ldots, \|\cdot\| \) is an \( n \)-norm on \( X \) and \( M_k \)'s are nondecreasing and convex functions so by using inequality (23) we have
\[
\lim_{r \to \infty} \frac{1}{h_r} \sum_{k \in \mathbb{N}} M_k \left( \left\| \frac{B^\mu_k(\alpha x_k + \beta y_k)}{\rho_3}, z_1, \ldots, z_{n-1} \right\| \right)^{p_k} \\
\leq \lim_{r \to \infty} \frac{1}{h_r} \sum_{k \in \mathbb{N}} M_k \left( \left\| \frac{B^\mu_k\alpha x_k}{\rho_3}, z_1, \ldots, z_{n-1} \right\| \right)^{p_k} \\
+ \left( \left\| \frac{B^\mu_k\beta y_k}{\rho_3}, z_1, \ldots, z_{n-1} \right\| \right)^{p_k} \\
\leq K \lim_{r \to \infty} \frac{1}{h_r} \sum_{k \in \mathbb{N}} \frac{1}{2p_k} M_k \left( \left\| \frac{B^\mu_k x_k}{\rho_1}, z_1, \ldots, z_{n-1} \right\| \right)^{p_k} \\
+ K \lim_{r \to \infty} \frac{1}{h_r} \sum_{k \in \mathbb{N}} \frac{1}{2p_k} M_k \left( \left\| \frac{B^\mu_k y_k}{\rho_2}, z_1, \ldots, z_{n-1} \right\| \right)^{p_k} \\
= 0. \tag{25}
\]
Thus, we have \( \alpha x + \beta y \in \omega_0^p(\mathcal{M}, B^\mu_k, p, \|\cdot\|, \ldots, \|\cdot\|) \). Hence \( \omega_0^p(\mathcal{M}, B^\mu_k, p, \|\cdot\|, \ldots, \|\cdot\|) \) is a linear space. Similarly, we can prove that \( \omega^p(\mathcal{M}, B^\mu_k, p, \|\cdot\|, \ldots, \|\cdot\|) \) and \( \omega_\infty^p(\mathcal{M}, B^\mu_k, p, \|\cdot\|, \ldots, \|\cdot\|) \) are linear spaces. This completes the proof of the theorem. \( \square \)

**Theorem 2.** Let \( \mathcal{M} = (\mathcal{M}_k) \) be a Musielak-Orlicz function and \( p = (p_k) \) a bounded sequence of positive real numbers; the space \( \omega_0^p(\mathcal{M}, B^\mu_k, p, \|\cdot\|, \ldots, \|\cdot\|) \) is a topological linear space paranormed by
\[
g(x) = \inf \left\{ \rho_0^{p_k/M} : \left( \frac{1}{h_r} \sum_{k \in \mathbb{N}} M_k \left( \left\| \frac{B^\mu_k x_k}{\rho_k}, z_1, \ldots, z_{n-1} \right\| \right)^{p_k} \right)^{1/M} \leq 1 \right\}, \tag{26}
\]
where \( M = \max(1, \sup_k p_k < \infty) \).

**Proof.** Clearly \( g(x) \geq 0 \) for \( x = (x_k) \in \omega_0^p(\mathcal{M}, B^\mu_k, p, \|\cdot\|, \ldots, \|\cdot\|) \). Since \( M_k(0) = 0 \), we get \( g(0) = 0 \). Again, if \( g(x) = 0 \), then
\[
g(x) = \inf \left\{ \rho_0^{p_k/M} : \left( \frac{1}{h_r} \sum_{k \in \mathbb{N}} M_k \left( \left\| \frac{B^\mu_k x_k}{\rho_k}, z_1, \ldots, z_{n-1} \right\| \right)^{p_k} \right)^{1/M} \leq 1 \right\} = 0. \tag{27}
\]
This implies that, for a given \( \varepsilon > 0 \), there exist some \( \rho_\varepsilon \) (0 < \( \rho_\varepsilon < \varepsilon \)) such that
\[
\left( \frac{1}{h_r} \sum_{k \in \mathbb{N}} M_k \left( \left\| \frac{B^\mu_k x_k}{\rho_\varepsilon}, z_1, \ldots, z_{n-1} \right\| \right)^{p_k} \right)^{1/M} \leq 1. \tag{28}
\]
Thus
\[
\left( \frac{1}{h_r} \sum_{k \in \mathbb{N}} M_k \left( \left\| \frac{B^\mu_k x_k}{\varepsilon}, z_1, \ldots, z_{n-1} \right\| \right)^{p_k} \right)^{1/M} \leq 1 \tag{29}
\]
for each \( r \), and suppose that \( x_k \neq 0 \) for each \( k \in \mathbb{N} \). This implies that \( B^\mu_k x_k \neq 0 \) for each \( k \in \mathbb{N} \). Let \( \varepsilon \to 0 \), then
\[
\left( \left\| \frac{B^\mu_k x_k}{\rho_\varepsilon}, z_1, \ldots, z_{n-1} \right\| \right)^{p_k} \to \infty; \tag{30}
\]
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which is a contradiction. Therefore, $B^\mu_k x_k = 0$ for each $k$ and thus $x_k = 0$ for each $k \in \mathbb{N}$. Let $\rho_1 > 0$ and $\rho_2 > 0$ be such that

\[
\left( \frac{1}{h_r k_e l_r} \sum_{k \in I_r} M_k \left( \left\| \left( \frac{B^\mu_k x_k}{\rho_1}, z_1, \ldots, z_{n-1} \right) \right\| \right)^{p_k} \right)^{1/M} \leq 1,
\]

\[
\left( \frac{1}{h_r k_e l_r} \sum_{k \in I_r} M_k \left( \left\| \left( \frac{B^\mu_k y_k}{\rho_2}, z_1, \ldots, z_{n-1} \right) \right\| \right)^{p_k} \right)^{1/M} \leq 1
\]

for each $r$.

Let $\rho = \rho_1 + \rho_2$; then by using Minkowski’s inequality, we have

\[
\left( \frac{1}{h_r k_e l_r} \sum_{k \in I_r} M_k \left( \left\| \left( \frac{B^\mu_k (x_k + y_k)}{\rho}, z_1, \ldots, z_{n-1} \right) \right\| \right)^{p_k} \right)^{1/M} \leq \left( \frac{1}{h_r k_e l_r} \sum_{k \in I_r} M_k \right)^{1/M}
\]

\[
\times \left( \left( \frac{\rho_1}{\rho_1 + \rho_2} \right) \left( \left\| \left( \frac{B^\mu_k x_k}{\rho_1}, z_1, \ldots, z_{n-1} \right) \right\| \right)^{p_k} \right)^{1/M} + \left( \frac{\rho_2}{\rho_1 + \rho_2} \right) \left( \left\| \left( \frac{B^\mu_k y_k}{\rho_2}, z_1, \ldots, z_{n-1} \right) \right\| \right)^{p_k} \leq 1
\]

(33)

Therefore, $g(x + y) \leq g(x) + g(y)$.

Finally, we prove that the scalar multiplication is continuous. Let $\nu$ be any complex number. By definition,

\[
g(\nu x) = \inf \left\{ \rho^{p/M} : \frac{1}{h_r k_e l_r} \sum_{k \in I_r} M_k \left( \left\| \left( \frac{\nu B^\mu_k x_k}{\rho}, z_1, \ldots, z_{n-1} \right) \right\| \right)^{p_k} \right\}^{1/M} \leq 1
\]

(34)
Then
\[ g(\|t\|) = \inf \left\{ \left( \frac{1}{t_r} \sum_{k \in I_r} M_k \right)^{p/M} : \frac{1}{t_r} \sum_{k \in I_r} M_k \sup_{t \in B} \left( \left\| \frac{B_x^k t - L}{t} - z_1, \ldots, z_{n-1} \right\| \right)^{p_k} \leq 1 \right\}. \]

(35)

where \( t = \rho / \|t\| \). Since \( \|t\| \leq \max(1, \|t\| \sup p_k) \), we have
\[ g(\|x\|) = \max \left( 1, \|x\| \sup p_k \right) \inf \left\{ \left( \frac{1}{t_r} \sum_{k \in I_r} M_k \right)^{p/M} \times \left( \left\| \frac{B_x^k t - L}{t} - z_1, \ldots, z_{n-1} \right\| \right)^{p_k} \right\}. \]

(36)

So, the fact that scalar multiplication is continuous follows from the above inequality. This completes the proof of theorem. □

**Theorem 3.** Let \( \mathcal{M} = (M_k) \) be a Musielak-Orlicz function. If \( \sup_m (M_k(x))^{p_k} < \infty \) for all fixed \( x > 0 \), then \( w^\beta(\mathcal{M}, B^\mu_k, p, \| \cdot \|, \cdot, \cdot, \cdot) \subset w^\beta(\mathcal{M}, B^\mu_k, p, \| \cdot \|, \cdot, \cdot, \cdot) \) if and only if
\[ \lim_{r \to \infty} \frac{1}{h_r(m)} \sum_{k \in I_r} M_k(t)^{p_k} = \infty, \quad \text{for some } t > 0. \]

(40)

Proof. Let \( x = (x_k) \in w^\beta(\mathcal{M}, B^\mu_k, p, \| \cdot \|, \cdot, \cdot, \cdot). \) Then there exists some positive number \( \rho_1 \) such that
\[ \lim_{r \to \infty} \frac{1}{h_r(m)} \sum_{k \in I_r} M_k \left( \left\| \frac{B_x^k t - L}{t} - z_1, \ldots, z_{n-1} \right\| \right)^{p_k} = 0. \]

(37)

Define \( \rho = 2\rho_1 \). Since \( M_k \) is nondecreasing and convex by using inequality (23), we have
\[ \lim_{r \to \infty} \frac{1}{h_r(m)} \sum_{k \in I_r} M_k \left( \left\| \frac{B_x^k t - L}{t} - z_1, \ldots, z_{n-1} \right\| \right)^{p_k} \leq K \left\{ \lim_{r \to \infty} \frac{1}{h_r(m)} \sum_{k \in I_r} M_k \left( \left\| \frac{B_x^k t - L}{t} - z_1, \ldots, z_{n-1} \right\| \right)^{p_k} \right\}. \]

(38)

Hence \( x = (x_k) \in w^\beta(\mathcal{M}, B^\mu_k, p, \| \cdot \|, \cdot, \cdot, \cdot, \cdot, \cdot). \) This completes the proof of the theorem. □

**Theorem 4.** Let \( \mathcal{M} = (M_k) \) be a Musielak-Orlicz function and \( 0 < h = \inf p_k \). Then
\[ w^\beta(\mathcal{M}, B^\mu_k, p, \| \cdot \|, \cdot, \cdot, \cdot, \cdot, \cdot) \subset w^\beta(\mathcal{M}, B^\mu_k, p, \| \cdot \|, \cdot, \cdot, \cdot, \cdot, \cdot) \]

if and only if
\[ \lim_{r \to \infty} \frac{1}{h_r(m)} \sum_{k \in I_r} M_k(t)^{p_k} = \infty, \quad \text{for some } t > 0. \]

Proof. Let \( w^\beta(\mathcal{M}, B^\mu_k, p, \| \cdot \|, \cdot, \cdot, \cdot, \cdot, \cdot) \subset w^\beta(\mathcal{M}, B^\mu_k, p, \| \cdot \|, \cdot, \cdot, \cdot, \cdot, \cdot) \). Suppose (40) does not hold. Therefore there is a subinterval \( I_{(m)} \) of the set of intervals \( I \), and a number \( n_0 \), where \( n_0 = \| (B_x^k t - L, z_1, \ldots, z_{n-1}) \| \) for all \( k \), such that
\[ \frac{1}{h_r(m)} \sum_{k \in I_{(m)}} M_k(t)^{p_k} \leq N < \infty, \quad m = 1, 2, 3, \ldots. \]

(41)

Let us define \( x = (x_k) \) as follows:
\[ B_x^k t_k = \begin{cases} \rho n_0, & k \in I_{(m)}, \\ 0, & k \notin I_{(m)}. \end{cases} \]

(42)

Thus by (41), \( x = (x_k) \in w^\beta(\mathcal{M}, B^\mu_k, p, \| \cdot \|, \cdot, \cdot, \cdot, \cdot, \cdot) \). But \( x = (x_k) \notin w^\beta(\mathcal{M}, B^\mu_k, p, \| \cdot \|, \cdot, \cdot, \cdot, \cdot, \cdot) \). Hence (40) must hold.

Conversely, suppose that (40) holds and let \( x = (x_k) \in w^\beta(\mathcal{M}, B^\mu_k, p, \| \cdot \|, \cdot, \cdot, \cdot, \cdot, \cdot) \). Then,
\[ \frac{1}{h_r(m)} \sum_{k \in I_r} M_k \left( \left\| \frac{B_x^k t - L}{t} - z_1, \ldots, z_{n-1} \right\| \right)^{p_k} \leq N < \infty. \]

(43)

Suppose that \( x = (x_k) \notin w^\beta(\mathcal{M}, B^\mu_k, p, \| \cdot \|, \cdot, \cdot, \cdot, \cdot, \cdot) \). Then for some number \( \varepsilon > 0 \), there is a number \( N_0 \) such that, for a subinterval \( I_{(m)} \) of the set of intervals \( I_r \),
\[ \left\| \frac{B_x^k t - L}{t} - z_1, \ldots, z_{n-1} \right\| > \varepsilon \quad \text{for } N \geq N_0. \]

(44)
We have \( M_k(\|\langle B^{\mu}\wedge x_k/\rho, z_1, \ldots, z_{n-1} \rangle \|) \geq M(\varepsilon)^{p_k} \), which contradicts (40) by using (43). Hence we get
\[
w^0_{\infty}(\mathcal{M}, B^{\mu}, p, \|, \|, \|) \subset w^0_{\infty}(\mathcal{M}, B^{\mu}, p, \|, \|, \|).
\]
This completes the proof.

**Theorem 5.** Let \( 0 < h = \inf p_k \leq \sup p_k = H < \infty \). For any Musielak-Orlicz function \( \mathcal{M} = (M_k) \) which satisfies \( \Delta_2 \)-condition, one has

\[
\begin{align*}
(i) & \quad w^0_0(\langle B^{\mu}, p, \|, \|, \| \rangle) \subset w^0_0(\mathcal{M}, B^{\mu}, p, \|, \|, \|) \\
(ii) & \quad w^0(\langle B^{\mu}, p, \|, \|, \| \rangle) \subset w^0(\mathcal{M}, B^{\mu}, p, \|, \|, \|) \\
(iii) & \quad w^0_0(\langle B^{\mu}, p, \|, \|, \| \rangle) \subset w^0_0(\mathcal{M}, B^{\mu}, p, \|, \|, \|).
\end{align*}
\]

**Proof.** Let \( x = (x_k) \in u^0_0(\langle B^{\mu}, p, \|, \|, \| \rangle) \). Then, we have
\[
\frac{1}{h_r} \sum_{k \in I_r} \left( \left\| \frac{B^{\mu}_x}{\rho} x_k, z_1, \ldots, z_{n-1} \right\| \right)^{p_k} \to 0 \quad \text{as} \quad r \to \infty.
\]

(46)

Let \( \varepsilon > 0 \), and choose \( \delta \) with \( 0 < \delta \leq 1 \) such that \( M_k < \varepsilon \) for \( 0 < t \leq \delta \). We can write
\[
\begin{align*}
\frac{1}{h_r} \sum_{k \in I_r} \left( \left\| \frac{B^{\mu}_x}{\rho} x_k, z_1, \ldots, z_{n-1} \right\| \right)^{p_k}
= & \quad \frac{1}{h_r} \sum_{k \in I_r, \|B^{\mu}_x(x_k/\rho, z_1, \ldots, z_{n-1})\| \leq \delta} \left( \left\| \frac{B^{\mu}_x}{\rho} x_k, z_1, \ldots, z_{n-1} \right\| \right)^{p_k} \\
& + \frac{1}{h_r} \sum_{k \in I_r, \|B^{\mu}_x(x_k/\rho, z_1, \ldots, z_{n-1})\| > \delta} \left( \left\| \frac{B^{\mu}_x}{\rho} x_k, z_1, \ldots, z_{n-1} \right\| \right)^{p_k}.
\end{align*}
\]

(47)

For the first summation above, we can write
\[
\frac{1}{h_r} \sum_{k \in I_r, \|B^{\mu}_x(x_k/\rho, z_1, \ldots, z_{n-1})\| \leq \delta} \left( \left\| \frac{B^{\mu}_x}{\rho} x_k, z_1, \ldots, z_{n-1} \right\| \right)^{p_k} < \max(\varepsilon, \varepsilon^h).
\]

(48)

By using continuity of \( M_k \), for the second summation we can write
\[
\left\| \frac{B^{\mu}_x}{\rho} x_k, z_1, \ldots, z_{n-1} \right\| < 1 + \frac{\left\| \langle B^{\mu}_x(x_k/\rho, z_1, \ldots, z_{n-1}) \rangle \right\|}{\delta}.
\]

(49)

Since each \( M_k \) is nondecreasing and convex and satisfies \( \Delta_2 \)-condition, it follows that
\[
\begin{align*}
\frac{1}{h_r} \sum_{k \in I_r} M_k \left( \left\| \frac{B^{\mu}_x}{\rho} x_k, z_1, \ldots, z_{n-1} \right\| \right)^{p_k} & \leq \max(\varepsilon, \varepsilon^h) \\
& + \max \left\{ 1, \left[ \frac{2M_k \left( \left\| \langle B^{\mu}_x(x_k/\rho, z_1, \ldots, z_{n-1}) \rangle \right\| \right)}{\delta} \right]^h \right\} \left( \frac{1}{h_r} \sum_{k \in I_r} \left( \left\| \frac{B^{\mu}_x}{\rho} x_k, z_1, \ldots, z_{n-1} \right\| \right)^{p_k} \right) \\
& \times \left( \frac{1}{h_r} \sum_{k \in I_r} \left( \left\| \frac{B^{\mu}_x}{\rho} x_k, z_1, \ldots, z_{n-1} \right\| \right)^{p_k} \right).
\end{align*}
\]

(50)

Taking limit as \( \varepsilon \to 0 \) and \( r \to \infty \), it follows that \( x = (x_k) \in u^0_0(\mathcal{M}, B^{\mu}, p, \|, \|, \|) \). Hence \( u^0_0(\langle B^{\mu}, p, \|, \|, \| \rangle) \subset u^0_0(\mathcal{M}, B^{\mu}, p, \|, \|, \|) \). Similarly, we can prove (ii) and (iii). This completes the proof of the theorem.

**Theorem 6.** Let \( \mathcal{M} = (M_k) \) be a Musielak-Orlicz function. Then the following statements are equivalent:

\[
\begin{align*}
(i) & \quad w^0_0(\langle B^{\mu}, p, \|, \|, \| \rangle) \subset w^0_0(\mathcal{M}, B^{\mu}, p, \|, \|, \|) \\
(ii) & \quad w^0(\langle B^{\mu}, p, \|, \|, \| \rangle) \subset w^0(\mathcal{M}, B^{\mu}, p, \|, \|, \|) \\
(iii) & \quad \sup \{1/h_r \sum_{k \in I_r} M_k(\varepsilon)^{p_k} < \infty \text{ for all } t > 0, \text{ where } t = \|B^{\mu}_x(x_k/\rho, z_1, \ldots, z_{n-1})\| \}.
\end{align*}
\]

**Proof.** (i) \(\Rightarrow\) (ii) Suppose (i) holds. In order to prove (ii) we have to show that
\[
w^0(\langle B^{\mu}, p, \|, \|, \| \rangle) \subset w^0(\mathcal{M}, B^{\mu}, p, \|, \|, \|).
\]

(51)

Let \( x = (x_k) \in w^0_0(\mathcal{M}, B^{\mu}, p, \|, \|, \|) \). Then for a given \( \varepsilon > 0 \) there exists \( s > s_\varepsilon \) such that
\[
\frac{1}{h_r} \sum_{k \in I_r} \left( \left\| \frac{B^{\mu}_x}{\rho} x_k, z_1, \ldots, z_{n-1} \right\| \right)^{p_k} < \varepsilon.
\]

(52)

Hence there exists \( K > 0 \) such that
\[
\sup \left\{ \frac{1}{h_r} \sum_{k \in I_r} \left( \left\| \frac{B^{\mu}_x}{\rho} x_k, z_1, \ldots, z_{n-1} \right\| \right)^{p_k} \right\} < K.
\]

(53)

This shows that \( x = (x_k) \in w^0_0(\mathcal{M}, B^{\mu}, p, \|, \|, \|) \).

(ii) \(\Rightarrow\) (iii) Suppose (ii) holds and (iii) fails to hold. Then for some \( t > 0 \),
\[
\sup \left\{ \frac{1}{h_r} \sum_{k \in I_r} M_k(\varepsilon)^{p_k} = \infty \right\}.
\]

(54)

and, therefore, we can find a subinterval \( I_{r(m)} \) of the set of intervals \( I_r \) such that
\[
\frac{1}{h_r(m)} \sum_{k \in I_{r(m)}} M_k \left( \frac{1}{m} \right)^{p_k} \geq m, \quad m = 1, 2, 3, \ldots
\]

(55)
Let us define \( x = (x_k) \) as follows:
\[
B^{\rho}_{r}x_k = \begin{cases} 
\rho m, & k \in I_r(m) \\
0, & k \notin I_r(m). 
\end{cases}
\] (56)

Thus \( x = (x_k) \in w_0^\theta(B^\rho_{r}, p, ||.||, ||.) \). But by (55), \( x = (x_k) \notin w_0^\theta(\mathcal{M}, B^\rho_{r}, p, ||.||, ||.) \) which contradicts (ii). Hence (iii) must hold.

(iii) \( \Rightarrow \) (i) Let (iii) hold. Suppose that \( x = (x_k) \notin w_0^\theta(\mathcal{M}, B^\rho_{r}, p, ||.||, ||.) \). Then for \( x = (x_k) \in w_0^\theta(B^\rho_{r}, p, ||.||, ||.) \)
\[
\sup_r \frac{1}{h_r} \sum_{k \in I_r} M_k \left( \left\| \frac{B^\rho_{r} x_k}{\rho} , z_1, \ldots, z_{n-1} \right\| \right)^p_k = \infty.
\] (57)

Let \( t = \frac{\|B^\rho_{r} x_k\|/\rho, z_1, \ldots, z_{n-1}}{\|B^\rho_{r} x_k\|/\rho \cdot z_1, \ldots, z_{n-1}} \) for each \( k \), and then by (57)
\[
\sup_r \frac{1}{h_r} \sum_{k \in I_r} M_k (M_k(t))_p^k > \infty, \text{ which contradicts (iii). Hence (i) must hold. This completes the proof of the theorem.} \]

**Theorem 7.** Let \( \mathcal{M} = (M_k) \) be a Musielak-Orlicz function. Then the following statements are equivalent:

(i) \( w_0^\theta(\mathcal{M}, B^\rho_{r}, p, ||.||, ||.) \subset w_0^\theta(B^\rho_{r}, p, ||.||, ||.) \);

(ii) \( w_0^\theta(\mathcal{M}, B^\rho_{r}, p, ||.||, ||.) \subset w_0^\theta(B^\rho_{r}, p, ||.||, ||.) \);

(iii) \( \inf_r (1/h_r) \sum_{k \in I_r} M_k (M_k(t)_p^k > 0 \text{ for all } t > 0.} \)

*Proof. (i) \( \Rightarrow \) (ii) is obvious

(ii) \( \Rightarrow \) (iii) Let (ii) hold and let (iii) fail to hold. Then
\[
\inf_r \frac{1}{h_r} \sum_{k \in I_r} M_k (M_k(t)_p^k = 0 \text{ for some } t > 0,}
\] (58)

and we can find a subinterval \( I_{r(m)} \) of the set of intervals \( I_r \)
\[
\frac{1}{h_r} (m) \sum_{k \in I_{r(m)}} M_k (M_k(m)_p^m < \frac{1}{m}, \text{ } m = 1, 2, 3, \ldots}
\] (59)

Let us define \( x = (x_k) \) as follows:
\[
B^{\rho}_{r}x_k = \begin{cases} 
\rho m, & k \in I_{r(m)} \\
0, & k \notin I_{r(m)}.
\end{cases}
\] (60)

Thus by (iii), \( x = (x_k) \in w_0^\theta(\mathcal{M}, B^\rho_{r}, p, ||.||, ||. ||) \). But \( x = (x_k) \notin w_0^\theta(\mathcal{M}, B^\rho_{r}, p, ||.||, ||. ||) \).

(iii) \( \Rightarrow \) (i) Let (iii) hold. Suppose that \( x = (x_k) \in w_0^\theta(\mathcal{M}, B^\rho_{r}, p, ||.||, ||. ||) \). Therefore,
\[
\frac{1}{h_r} \sum_{k \in I_r} M_k \left( \left\| \frac{B^\rho_{r} x_k}{\rho} , z_1, \ldots, z_{n-1} \right\| \right)^p_k \rightarrow 0 \text{ as } r \rightarrow \infty.
\] (61)

Again suppose \( x = (x_k) \notin w_0^\theta(B^\rho_{r}, p, ||.||, ||.) \) for some number \( \varepsilon > 0 \) and a subinterval \( I_{r(m)} \) of the set of intervals \( I_r \), we have
\[
\left\| \frac{B^\rho_{r} x_k}{\rho} , z_1, \ldots, z_{n-1} \right\| \geq \varepsilon \forall k.
\] (62)

Then, from properties of the Orlicz function, we can write
\[
M_k \left( \left\| \frac{B^\rho_{r} x_k}{\rho} , z_1, \ldots, z_{n-1} \right\| \right)^p_k \geq M_k (e)^p_k.
\] (63)

Consequently, by (61), we have \( \lim_{r \rightarrow \infty} \sum_{k \in I_r} M_k (e)^p_k = 0 \), which contradicts (iii). Hence (i) must hold. This completes the proof of the theorem. \( \square \)

**Theorem 8.** (i) If \( 0 < \inf \ p_k \leq p_k \leq 1 \text{ for all } k \), then \( w_0^\theta(\mathcal{M}, B^\rho_{r}, p, ||.||, ||.) \subset w_0^\theta(\mathcal{M}, B^\rho_{r}, p, ||.||, ||.) \).

(ii) If \( 1 \leq p_k \leq \sup p_k = H < \infty \), then \( w_0^\theta(\mathcal{M}, B^\rho_{r}, p, ||.||, ||.) \subset w_0^\theta(\mathcal{M}, B^\rho_{r}, p, ||.||, ||.) \).

*Proof. (i) Let \( x \in w_0^\theta(\mathcal{M}, B^\rho_{r}, p, ||.||, ||.) \). Since \( 0 < \inf \ p_k \leq 1 \), we get
\[
\frac{1}{h_r} \sum_{k \in I_r} M_k \left( \left\| \frac{B^\rho_{r} x_k}{\rho} - L , z_1, \ldots, z_{n-1} \right\| \right)^p_k
\]
\[
\leq \frac{1}{h_r} \sum_{k \in I_r} M_k \left( \left\| \frac{B^\rho_{r} x_k}{\rho} , z_1, \ldots, z_{n-1} \right\| \right)^p_k,
\] (64)

and hence \( x \in w_0^\theta(\mathcal{M}, B^\rho_{r}, p, ||.||, ||.) \).

(ii) Let \( 1 \leq p_k \leq \sup p_k = H < \infty \) and \( x = (x_k) \in w_0^\theta(\mathcal{M}, B^\rho_{r}, p, ||.||, ||.) \). Then for each \( 0 < \varepsilon < 1 \) there exists a positive integer \( s_0 \) such that
\[
\frac{1}{h_r} \sum_{k \in I_r} M_k \left( \left\| \frac{B^\rho_{r} x_k}{\rho} - L , z_1, \ldots, z_{n-1} \right\| \right)^p_k
\]
\[
\leq \varepsilon < 1 \forall r > s_0.
\] (65)

This implies that
\[
\frac{1}{h_r} \sum_{k \in I_r} M_k \left( \left\| \frac{B^\rho_{r} x_k}{\rho} , z_1, \ldots, z_{n-1} \right\| \right)^p_k
\]
\[
\leq \frac{1}{h_r} \sum_{k \in I_r} M_k \left( \left\| \frac{B^\rho_{r} x_k}{\rho} , z_1, \ldots, z_{n-1} \right\| \right)^p_k.
\] (66)

Therefore \( x = (x_k) \in w_0^\theta(\mathcal{M}, B^\rho_{r}, ||.||, ||.) \). This completes the proof of the theorem. \( \square \)

**Conflict of Interests**

The authors declare that there is no conflict of interests regarding the publication of this paper.

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