Research Article

Some Connections between Class $\mathcal{U}$- and $\alpha$-Convex Functions

Edmond Aliaga$^1$ and Nikola Tuneski$^2$

$^1$ Department of Mathematics and Computer Sciences, Faculty of Mathematical and Natural Sciences, University of Prishtina, M. Terezan n.n., 10000 Prishtina, Kosovo
$^2$ Faculty of Mechanical Engineering, Ss. Cyril and Methodius University in Skopje, Karpoš II b.b., 1000 Skopje, Macedonia

Correspondence should be addressed to Nikola Tuneski; nikola.tuneski@mf.edu.mk

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The class $\mathcal{U}(\lambda, \mu)$ of normalized analytic functions that satisfy $|(z/f(z))^{1+\mu} \cdot f'(z) - 1| < \lambda$ for all $z$ in the open unit disk is studied and sufficient conditions for an $\alpha$-convex function to be in $\mathcal{U}(\lambda, \mu)$ are given.

1. Introduction

Let $\mathcal{A}$ denote the class of functions $f(z)$ which are analytic in the unit disk $D = \{ z : |z| < 1 \}$ with the normalization $f(0) = 0$ and $f'(0) = 1$.

For a function $f(z) \in \mathcal{A}$, we say that $f$ is starlike of order $\alpha$, $0 \leq \alpha < 1$, if and only if

$$\text{Re} \left\{ \frac{zf'(z)}{f(z)} \right\} > \alpha, \quad z \in D.$$  \hspace{1cm} (1)

We denote by $\mathcal{S}^*(\alpha)$ the class of all such functions. Also, we denote by $\mathcal{K}(\alpha)$ the class of convex functions of order $\alpha$, $0 \leq \alpha < 1$, that is, the class of functions $f(z) \in \mathcal{A}$ for which

$$\text{Re} \left\{ 1 + \frac{zf''(z)}{f'(z)} \right\} > \alpha, \quad z \in D.$$  \hspace{1cm} (2)

For $\alpha = 0$, we have the classes of $\mathcal{S}^*$ and $\mathcal{K}$ of starlike and convex functions, respectively. All of the above classes are subclasses of the class of univalent functions in $D$ and even more, $\mathcal{K} \subset \mathcal{S}^*$. For details see [1].

Further, for $f(z) \in \mathcal{A}$ and $\mu \in \mathbb{C}$, let us define the operator

$$U(f, \mu; z) = \left( \frac{z}{f(z)} \right)^{1+\mu} \cdot f'(z)$$  \hspace{1cm} (3)

and the class

$$\mathcal{U}(\lambda, \mu) = \left\{ f \in \mathcal{A} : \frac{z}{f(z)} \neq 0, \left| U(f, \mu; z) - 1 \right| < \lambda, z \in D \right\}.$$  \hspace{1cm} (4)

This class and its special cases $\mathcal{U}(\lambda) \equiv \mathcal{U}(\lambda, 1)$ and $\mathcal{U}(1) = \mathcal{U}(1, 1)$ are widely studied in the past decades ([2–12]). It is known [2, 12] that functions in $\mathcal{U}(\lambda)$ are univalent if $0 < \lambda \leq 1$, but not necessarily univalent if $\lambda > 1$. Further, Fournier and Ponnusamy [3] proved that assuming $\text{Re} \mu < 1$ the following equivalency holds:

$$\mathcal{U}(\lambda, \mu) \subset \mathcal{S}^* \iff 0 < \lambda \leq \frac{1 - \mu}{\sqrt{(1 - \mu)^2 + \mu^2}};$$  \hspace{1cm} (5)

that is, in general case, $\mathcal{U}(\lambda, \mu)$ is not a subset of $\mathcal{S}^*$. In particular,

$$\mathcal{U}(1, \mu) \subset \mathcal{S}^* \iff \mu = 0;$$  \hspace{1cm} (6)

that is, $\mathcal{U} \not\subset \mathcal{S}^*$, which can be also verified by the function

$$f(z) = \frac{z}{1 + (1/2)z + (1/2)z^2} \in \mathcal{U} \setminus \mathcal{S}^*.$$  \hspace{1cm} (7)
Finally, let us consider the classes
\[ M(\alpha, \gamma) = \{ f \in A : \Re J(f, \alpha; z) > \gamma, z \in D \}, \]
\[ \tilde{M}(\alpha, \beta) = \{ f \in A : \Re J(f, \alpha; z) - 1 < \beta, z \in D \}, \]
where
\[ J(f, \alpha; z) = (1 - \alpha) \frac{zf'(z)}{f(z)} + \alpha \left( 1 + \frac{zf''(z)}{f'(z)} \right). \]  
\[ \alpha, \gamma \in \mathbb{R}, \beta > 0. \]  
These classes make a "bridge" between the classes of starlike and convex functions (of some order). The class \( M(\alpha, \gamma) \) is in fact the class of \( \alpha \)-convex functions of order \( \gamma \) and \( \tilde{M}(\alpha, \beta) \), in the case when \( 0 < \beta \leq 1 \) is a subclass of \( M(\alpha, \gamma) \). Further, \( \alpha \)-convex functions of some order are also starlike ([13], page 10). Therefore, it gives rise to the question (which is studied in this paper) of finding the sufficient conditions for
\[ f \in \mathcal{U}(\lambda, -1/\alpha) \cap M(\alpha, \gamma), \]
\[ f \in \tilde{M}(\alpha, \beta) \implies f \in \mathcal{U}(\lambda, -1/\alpha). \]  

Let \( f(z) \) and \( q(z) \) be analytic in the unit disk. We say that \( f(z) \) is subordinate to \( q(z) \), and we write \( f(z) < q(z) \); if \( q(z) \) is univalent in \( D \), then \( f(0) = q(0) \) and \( f(D) \subseteq q(D) \). Further, we use the method of differential subordination introduced by Miller and Mocanu [14]. In fact, if \( \phi : C^2 \rightarrow C \) (\( C \) is the complex plane) is analytic in domain \( D \subseteq C \), if \( h(z) \) is univalent in \( D \), and if \( q(z) \) is analytic in \( D \) with \( q(z) \in D \), then \( q(z) \) satisfies a first-order differential subordination if
\[ \phi(p(z),zp'(z)) < h(z). \]  
The univalent function \( q(z) \) is called a dominant of the differential subordination (II) if \( p(z) < q(z) \) for all \( p(z) \) satisfying (II). If \( \tilde{q}(z) \) is a dominant of (II) and \( \tilde{q}(z) < q(z) \) for all dominants of (II), then we say that \( \tilde{q}(z) \) is the best dominant of the differential subordination (II).

We will make use of the following lemma.

**Lemma 1** (see [15]). Let \( q \) be univalent in the unit disk \( D \), and let \( \theta(w) \) and \( \phi(w) \) be analytic in a domain \( D \) containing \( q(D) \), with \( \phi(w) \neq 0 \) when \( w \in q(D) \). Set \( Q(z) = zq'(z)\phi(q(z)) \), \( h(z) = \theta(q(z)) + Q(z) \) and suppose that

(i) \( \theta \) is starlike in the unit disk \( D \);
(ii) \( \Re(zq'(z)/Q(z)) = \Re(\theta(q(z))/\phi(q(z))) \) \( + \Re((zQ'(z)/Q(z))) > 0 \), \( z \in D \).

If \( p \) is analytic in \( D \), with \( p(0) = q(0), p(D) \subseteq D \), and
\[ \theta(p(z)) + zp'(z)\phi(p(z)) \]
\[ < \theta(q(z)) + zq'(z)\phi(q(z)) = h(z), \]
then \( p(z) < q(z) \), and \( q \) is the best dominant of (12).

### 2. Main Results and Consequences

Now we will prove the following theorem that will further lead to connections between class \( \mathcal{U}(\lambda, \mu) \) and classes \( M(\alpha, \gamma) \) and \( \tilde{M}(\alpha, \beta) \).

**Theorem 2.** Let \( f \in A, 0 < \lambda \leq 1 \) and \( \alpha \neq 0 \). If \( f'(z) \neq 0 \) for all \( z \in D \), and if
\[ J(f, \alpha; z) < 1 + \frac{\alpha\lambda z}{1 + \lambda z} = h(z), \]
then
\[ U(f, -1/\alpha; z) < 1 + \lambda z; \]
that is, \( f \in \mathcal{U}(\lambda, -1/\alpha) \), and \( 1 + \lambda z \) is the best dominant of (13).

**Proof.** Let \( p(z) = U(f, -1/\alpha; z) = (z/\phi(z))^{1-1/\alpha}f'(z), q(z) = 1 + \lambda z, \theta(\omega) = 1, \) and \( \phi(\omega) = \alpha/\omega \), where \( \omega \in (D) \). Then \( q(z) \) is univalent in \( D \), \( \theta(\omega) \) and \( \phi(\omega) \) are analytic in domain \( D = C \setminus \{0\} \) which contains \( q(D) = \{1 + z : |z| < \lambda\} \). \( q(z) \) and \( D \) are analytic in \( D \), and \( \phi(\omega) \neq 0 \) when \( \omega \in q(D) \). On the other hand, let
\[ Q(z) = zq'(z)\phi(q(z)) = z\lambda \frac{\alpha}{q(z)} = \frac{\alpha\lambda z}{1 + \lambda z} = h(z) - 1. \]

Then
\[ \frac{zQ'(z)}{Q(z)} = \frac{zq'(z)}{Q(z)} = \frac{1}{1 + \lambda z}, \]
\[ \Re \left( \frac{zQ'(z)}{Q(z)} \right) = \Re \left( \frac{zq'(z)}{Q(z)} \right) = \Re \left( \frac{1}{q(z)} \right) > \frac{1}{1 + \lambda} > 0, \]
\[ z \in D. \]

The last inequality holds since \( \Re q(z) = 1 + \lambda \cdot \Re z < 1 + \lambda \) for all \( z \in D \) and \( q(D) \) does not contain the origin. So, conditions (i) and (ii) from Lemma 1 are satisfied.

Further, \( p \) is analytic in \( D \), \( p(0) = q(0) = 1 \). Also, \( p(z) \neq 0 \) for all \( z \in D \); that is, \( p(D) \subseteq D \), since \( f'(z) \neq 0 \) for all \( z \in D \) (condition of the theorem); \( zf(z) = 1 \neq 0 \) for \( z = 0 \) (because \( f \in A \) and \( f(z) \) has no poles on \( D \)). Hence from Lemma 1 and the fact that
\[ J(f, \alpha; z) = 1 + \alpha \cdot \frac{zp'(z)}{p(z)} = \theta(p(z)) + zp'(z)\phi(p(z)) \]
\[ < \frac{\alpha\lambda z}{1 + \lambda z} = \theta(q(z)) + zq'(z)\phi(q(z)), \]
we receive that \( p(z) < q(z) \), that is, relation (14), and we also receive that \( q(z) \) is the best dominant of (13). \( \square \)
Applying the definition of subordination to Theorem 2, we receive the following.

**Corollary 3.** Let \( f \in \mathcal{A}, 0 < \lambda \leq 1, \) and \( \alpha \neq 0. \) Also, let \( f(z)/z \neq 0 \) and \( f'(z) \neq 0 \) for all \( z \in \mathbb{D}. \) If

\[
|f(f, \alpha; z) - 1| < \frac{\lambda |\alpha|}{1 + \lambda}, \quad z \in \mathbb{D}, \tag{18}
\]

then \( f \in \mathcal{U}(\lambda, -1/\alpha) \); that is,

\[
|U(f, -\frac{1}{\alpha}; z) - 1| < \lambda, \quad z \in \mathbb{D}. \tag{19}
\]

This result is sharp; that is, the constant \( \lambda \) in inequality (19) cannot be replaced by a smaller one such that the implication still holds.

**Proof.** First, let us note that the function \( h \) defined by expression (13) is univalent in the unit disk such that

\[
\min \{|h(z) - 1| : |z| = 1\} = \frac{\lambda |\alpha|}{1 + \lambda}. \tag{20}
\]

So, the disk \( \{w : |w - 1| < \lambda |\alpha|/(1 + \lambda)\} \) is contained in \( h(\mathbb{D}) \) which, having in mind the definition of subordination, means that inequality (18) implies subordination (13). Further, from Theorem 2 follows subordination (14), which is equivalent to the inequality (19). Even more, Theorem 2 says that \( q(z) = 1 + \lambda z \) is the best dominant of (13).

In order to prove the sharpness of the result let us assume the opposite; that is, there exists \( \lambda_* > 0 \), \( 0 < \lambda_* < \lambda \), such that inequality (18) implies

\[
|U(f, -\frac{1}{\alpha}; z) - 1| < \lambda_*, \quad z \in \mathbb{D}; \tag{21}
\]

that is,

\[
U(f, -\frac{1}{\alpha}; z) < 1 + \lambda_* z \equiv q_*(z). \tag{22}
\]

On the other hand, inequality (18) implies subordination (13) with best dominant \( q(z) \), meaning that \( q(z) < q_*(z) \). This is a contradiction to the assumption \( \lambda_* < \lambda \) which proves the sharpness of the result.

Previous corollary can be written in the following, equivalent, form that gives conditions for inclusion of the class \( \mathcal{M}'(\alpha, \beta) \) into the class \( \mathcal{U}(\lambda, -1/\alpha) \).

**Corollary 4.** Let \( f \in \mathcal{A}, 0 < \lambda \leq 1, \) and \( \beta = \lambda |\alpha|/(1 + \lambda) \). Also, let \( f(z)/z \neq 0 \) and \( f'(z) \neq 0 \) for all \( z \in \mathbb{D}. \) Then

\[
f \in \mathcal{M}'(\alpha, \beta) \implies f \in \mathcal{U}(\lambda, -1/\alpha). \tag{23}
\]

The constant \( \lambda \), for the class \( \mathcal{U}(\lambda, -1/\alpha) \), cannot be replaced by a smaller one such that the inclusion still holds.

Next result gives the connection between classes \( \mathcal{M}(\alpha, \gamma) \) and \( \mathcal{U}(\lambda, -1/\alpha) \).

**Corollary 5.** Let \( f \in \mathcal{A}, 0 < \lambda \leq 1, \) and \( \alpha \neq 0, \) and

\[
\gamma := \begin{cases} 1 + \alpha - \frac{\alpha}{1 - \lambda}, & \text{if } \alpha > 0, \lambda \neq 1, \\ 1 + \alpha - \frac{\alpha}{1 + \lambda}, & \text{if } \alpha < 0. \end{cases} \tag{24}
\]

Also, let \( f(z)/z \neq 0 \) and \( f'(z) \neq 0 \) for all \( z \in \mathbb{D}. \) If subordination (13) holds, then

\[
f \in \mathcal{U}(\lambda, -\frac{1}{\alpha}) \cap \mathcal{M}(\alpha, \gamma). \tag{25}
\]

**Proof.** It is easy to check that all conditions of Theorem 2 are fulfilled; hence, \( f \in \mathcal{U}(\lambda, -1/\alpha) \).

It remains to verify that \( f \in \mathcal{M}(\alpha, \gamma) \); that is, subordination (13) implies

\[
\Re J(f, \alpha; z) > \gamma, \quad z \in \mathbb{D}. \tag{26}
\]

Having in mind the definition of subordination and the fact that \( h(z) = 1 + (\alpha z)/(1 + \lambda z) \) is univalent, it is enough to show that \( \Re h(z) \geq \gamma \) for all \( z \in \mathbb{D}. \) The last is true because

\[
\inf \{|\Re h(z) : z \in \mathbb{D}|\} = \min \{|\Re h(z) : |z| = 1| \}
\]

\[
= \min \{|h(1), h(-1)| \} = \gamma. \quad \Box
\]

**Remark 6.** The case \( \alpha > 0 \) and \( \lambda = 1 \) is not covered by the previous corollary since then \( \inf \{|\Re h(z) : z \in \mathbb{D}|\} = -\infty. \)

### 3. Examples

Now we will apply the results from the previous section on specific functions \( f \in \mathcal{A} \) and receive interesting conclusions.

**Example 1.** Let \( 0 < \lambda \leq 1, \alpha \neq 0, \) and \( \alpha \in \mathbb{R} \). Consider the following.

(i) Let

\[
\gamma := \begin{cases} 1 + \alpha - \frac{\alpha}{1 - \lambda}, & \text{if } \alpha > 0, \lambda \neq 1, \\ 1 + \alpha - \frac{\alpha}{1 + \lambda}, & \text{if } \alpha < 0. \end{cases} \tag{28}
\]

If \( a \neq 0, a \neq -1, \lambda \leq |1 + a|, aa > 0, \) and \( |\alpha| > |a|, \) then

\[
f(z) = z \cdot \left(1 + \frac{\lambda}{1 + a}ight)^{a} \in \mathcal{U}(\lambda, -\frac{1}{a}) \cap \mathcal{M}(\alpha, \gamma). \tag{29}
\]

(ii) If \( |\alpha| < 1 \) and one of the following two sets of conditions holds:

\[
\alpha > 0, \quad |\alpha| (1 + \lambda) (1 - |\alpha| + \alpha) < \lambda \alpha (1 - |\alpha|) \tag{30}
\]

or

\[
\alpha < 0, \quad |\alpha| (1 + \lambda) (1 + |\alpha| + \alpha) < \lambda |\alpha| (1 + |\alpha|), \tag{31}
\]

then
\[ f(z) = z \cdot e^{\alpha z} \in \mathcal{U}\left(\lambda, -\frac{1}{\alpha}\right). \] (32)

In both cases, power is taken by its principal value.

**Proof.** (i) For the function \( f(z) = z(1 + (\lambda/(1 + \alpha))z)^{\alpha} \), we have \( f(0) = 0 \) and

\[ f'(z) = \left(1 + \frac{\lambda}{1 + \alpha} z\right)^{\alpha} \cdot \left(1 + \frac{\alpha \lambda z}{1 + \alpha + \lambda z}\right) \Rightarrow f'(0) = 1. \] (33)

Condition \( \lambda \leq |1 + \alpha| \) guarantees that \( 1 + \alpha + \lambda z \neq 0 \) for all \( z \in \mathbb{D} \); hence, \( f \) is an analytic function and \( f \in \mathcal{A} \). For the function \( f \), it is easy to verify that

\[ J(f, \alpha; z) = 1 + \frac{\alpha \lambda z}{1 + \lambda z}. \] (35)

Further, from the definition of subordination, we have that \( \alpha z < \alpha z \) when \( |\alpha| > |\alpha| \) and \( \alpha > 0 \); that is, both are positive or both negative. Therefore,

\[ J(f, \alpha; z) = 1 + \frac{\alpha \lambda z}{1 + \lambda z} < 1 + \frac{\alpha \lambda z}{1 + \lambda z}. \] (36)

So, all conditions of Corollary 5 bring us to the conclusion that \( f \in \mathcal{F}(1, -1/\alpha) \cap \mathcal{M}(\alpha, \gamma) \).

(ii) It is easy to verify that \( f \in \mathcal{A} \) and that

\[ J(f, \alpha; z) = 1 + \alpha z \left(1 + \frac{\alpha}{1 + \alpha z}\right). \] (37)

Therefore,

\[ \Delta = \sup \left\{ |J(f, \alpha; z) - 1| : z \in \mathbb{D} \right\} = \max \left\{ |J(f, \alpha; z) - 1| : |z| = 1 \right\} = |\alpha| \cdot \max \left\{ 1 + \frac{\alpha}{1 + \alpha z} : z \in \mathbb{D} \right\} = |\alpha| \cdot \left\{ \begin{array}{ll}
1 + \frac{\alpha}{1 - |\alpha|} & \text{if } \alpha > 0, \\
1 + \frac{\alpha}{1 + |\alpha|} & \text{if } \alpha < 0.
\end{array} \right. \]

Further, if one of the conditions (30) or (31) holds, then \( \Delta \leq (\lambda|\alpha|)/(1 + \lambda) \); that is, we receive inequality (18). Finally, we have shown that all conditions of Corollary 3 are fulfilled, which leads to \( f(z) \in \mathcal{F}(\lambda, -1/\alpha) \).

The following example exhibits some concrete conclusions that can be obtained from the results of the previous section by specifying the values of \( \lambda \) and/or \( \alpha \).

**Example 2.** Let \( f \in \mathcal{A}, 0 < \lambda \leq 1 \) and let \( f(z)/z \neq 0 \) and \( f'(z) \neq 0 \) for all \( z \in \mathbb{D} \). Consider the following:

(i) \( f \in \mathcal{F}(\alpha, |\alpha|/2) \Rightarrow f \in \mathcal{F}(1, -1/\alpha) (\lambda = 1 \text{ in Corollary 4}); \)

(ii) \( f \in \mathcal{F}(1, -1/\alpha) \cap \mathcal{M}(\alpha, 1 + \alpha/2) (\lambda = 1 \text{ and } \alpha < 0 \text{ in Corollary 5}). \)

**Conflict of Interests**

The authors declare that there is no conflict of interests regarding the publication of this paper.

**References**


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