Research Article

Some Connections between Class $\mathcal{U}$ - and α -Convex Functions

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The class $\mathcal{U}(\lambda, \mu)$ of normalized analytic functions that satisfy $|(z/f(z))^{1+\mu} \cdot f'(z) - 1| < \lambda$ for all z in the open unit disk is studied and sufficient conditions for an α -convex function to be in $\mathcal{U}(\lambda, \mu)$ are given.

1. Introduction

Let $\mathcal{A}$ denote the class of functions $f(z)$ which are analytic in the unit disk $\mathbb{D} = \{z : |z| < 1\}$ with the normalization $f(0) = 0$ and $f'(0) = 1$.

For a function $f(z) \in \mathcal{A}$, we say that f is starlike of order α , $0 \leq \alpha < 1$, if and only if

$$\operatorname{Re} \left\{ \frac{zf'(z)}{f(z)} \right\} > \alpha, \quad z \in \mathbb{D}. \quad (1)$$

We denote by $\mathcal{S}^*(\alpha)$ the class of all such functions. Also, we denote by $\mathcal{K}(\alpha)$ the class of convex functions of order α , $0 \leq \alpha < 1$, that is, the class of functions $f(z) \in \mathcal{A}$ for which

$$\operatorname{Re} \left\{ 1 + \frac{zf''(z)}{f'(z)} \right\} > \alpha, \quad z \in \mathbb{D}. \quad (2)$$

For $\alpha = 0$, we have the classes of $\mathcal{S}^*$ and $\mathcal{K}$ of starlike and convex functions, respectively. All of the above classes are subclasses of the class of univalent functions in $\mathbb{D}$ and even more, $\mathcal{K} \subset \mathcal{S}^*$. For details see [1].

Further, for $f(z) \in \mathcal{A}$ and $\mu \in \mathbb{C}$, let us define the operator

$$U(f, \mu; z) = \left(\frac{z}{f(z)} \right)^{1+\mu} \cdot f'(z) \quad (3)$$

and the class

$$\begin{aligned} \mathcal{U}(\lambda, \mu) \\ = \left\{ f \in \mathcal{A} : \frac{z}{f(z)} \neq 0, |U(f, \mu; z) - 1| < \lambda, z \in \mathbb{D} \right\}. \end{aligned} \quad (4)$$

This class and its special cases $\mathcal{U}(\lambda) \equiv \mathcal{U}(\lambda, 1)$ and $\mathcal{U} \equiv \mathcal{U}(1) = \mathcal{U}(1, 1)$ are widely studied in the past decades ([2–12]). It is known [2, 12] that functions in $\mathcal{U}(\lambda)$ are univalent if $0 < \lambda \leq 1$, but not necessarily univalent if $\lambda > 1$. Further, Fournier and Ponnusamy [3] proved that assuming $\operatorname{Re} \mu < 1$ the following equivalency holds:

$$\mathcal{U}(\lambda, \mu) \subset \mathcal{S}^* \iff 0 < \lambda \leq \frac{|1 - \mu|}{\sqrt{(1 - \mu)^2 + \mu^2}}; \quad (5)$$

that is, in general case, $\mathcal{U}(\lambda, \mu)$ is not a subset of $\mathcal{S}^*$. In particular,

$$\mathcal{U}(1, \mu) \subset \mathcal{S}^* \iff \mu = 0; \quad (6)$$

that is, $\mathcal{U} \not\subset \mathcal{S}^*$, which can be also verified by the function

$$f(z) = \frac{z}{1 + (1/2)z + (1/2)z^3} \in \mathcal{U} \setminus \mathcal{S}^*. \quad (7)$$

Finally, let us consider the classes

$$\begin{aligned}\mathcal{M}(\alpha, \gamma) &= \{f \in \mathcal{A} : \operatorname{Re} J(f, \alpha; z) > \gamma, z \in \mathbb{D}\}, \\ \mathcal{M}'(\alpha, \beta) &= \{f \in \mathcal{A} : |J(f, \alpha; z) - 1| < \beta, z \in \mathbb{D}\},\end{aligned}\quad (8)$$

where

$$J(f, \alpha; z) \equiv (1 - \alpha) \frac{zf'(z)}{f(z)} + \alpha \left(1 + \frac{zf''(z)}{f'(z)} \right), \quad (9)$$

$\alpha, \gamma \in \mathbb{R}$, and $\beta > 0$. These classes make a “bridge” between the classes of starlike and convex functions (of some order). The class $\mathcal{M}(\alpha, \gamma)$ is in fact the class of α -convex functions of order γ and $\mathcal{M}'(\alpha, \beta)$, in the case when $0 < \beta \leq 1$ is a subclass of $\mathcal{M}(\alpha, \gamma)$. Further, α -convex functions of some order are also starlike ([13], page 10). Therefore, it gives rise to the question (which is studied in this paper) of finding the sufficient conditions for

$$\begin{aligned}f &\in \mathcal{U}\left(\lambda, -\frac{1}{\alpha}\right) \cap \mathcal{M}(\alpha, \gamma), \\ f &\in \mathcal{M}'(\alpha, \beta) \implies f \in \mathcal{U}\left(\lambda, -\frac{1}{\alpha}\right).\end{aligned}\quad (10)$$

Let $f(z)$ and $g(z)$ be analytic in the unit disk. We say that $f(z)$ is subordinate to $g(z)$, and we write $f(z) < g(z)$; if $g(z)$ is univalent in $\mathbb{D}$, then $f(0) = g(0)$ and $f(\mathbb{D}) \subseteq g(\mathbb{D})$. Further, we use the method of differential subordination introduced by Miller and Mocanu [14]. In fact, if $\phi : \mathbb{C}^2 \rightarrow \mathbb{C}$ ($\mathbb{C}$ is the complex plane) is analytic in domain $D \subset \mathbb{C}$, if $h(z)$ is univalent in $\mathbb{D}$, and if $p(z)$ is analytic in $\mathbb{D}$ with $(p(z), zp'(z)) \in D$, when $z \in \mathbb{D}$, then we say that $p(z)$ satisfies a first-order differential subordination if

$$\phi(p(z), zp'(z)) < h(z). \quad (11)$$

The univalent function $q(z)$ is called a dominant of the differential subordination (11) if $p(z) < q(z)$ for all $p(z)$ satisfying (11). If $\tilde{q}(z)$ is a dominant of (11) and $\tilde{q}(z) < q(z)$ for all dominants of (11), then we say that $\tilde{q}(z)$ is the best dominant of the differential subordination (11).

We will make use of the following lemma.

Lemma 1 (see [15]). *Let q be univalent in the unit disk $\mathbb{D}$, and let $\theta(w)$ and $\phi(w)$ be analytic in a domain D containing $q(\mathbb{D})$, with $\phi(w) \neq 0$ when $w \in q(\mathbb{D})$. Set $Q(z) = zq'(z)\phi(q(z))$, $h(z) = \theta(q(z)) + Q(z)$ and suppose that*

- (i) Q is starlike in the unit disk $\mathbb{D}$;
- (ii) $\operatorname{Re}(zh'(z))/Q(z) = \operatorname{Re}[(\theta'(q(z))/\phi(q(z))) + (zQ'(z)/Q(z))] > 0, z \in \mathbb{D}$.

If p is analytic in $\mathbb{D}$, with $p(0) = q(0)$, $p(\mathbb{D}) \subseteq D$, and

$$\begin{aligned}\theta(p(z)) + zp'(z)\phi(p(z)) \\ < \theta(q(z)) + zq'(z)\phi(q(z)) = h(z),\end{aligned}\quad (12)$$

then $p(z) < q(z)$, and q is the best dominant of (12).

2. Main Results and Consequences

Now we will prove the following theorem that will further lead to connections between class $\mathcal{U}(\lambda, \mu)$ and classes $\mathcal{M}'(\alpha, \beta)$ and $\mathcal{M}(\alpha, \gamma)$.

Theorem 2. *Let $f \in \mathcal{A}$, $0 < \lambda \leq 1$ and $\alpha \neq 0$. If $f'(z) \neq 0$ for all $z \in \mathbb{D}$, and if*

$$J(f, \alpha; z) < 1 + \frac{\alpha\lambda z}{1 + \lambda z} \equiv h(z), \quad (13)$$

then

$$U\left(f, -\frac{1}{\alpha}; z\right) < 1 + \lambda z; \quad (14)$$

that is, $f \in \mathcal{U}(\lambda, -1/\alpha)$, and $1 + \lambda z$ is the best dominant of (13).

Proof. Let $p(z) = U(f, -1/\alpha; z) = (z/f(z))^{1-1/\alpha} \cdot f'(z)$, $q(z) = 1 + \lambda z$, $\theta(w) = 1$, and $\phi(w) = \alpha/w$, where $w \in (\mathbb{D})$. Then $q(z)$ is univalent in $\mathbb{D}$, $\theta(w)$ and $\phi(w)$ are analytic in domain $D = \mathbb{C} \setminus \{0\}$ which contains $q(\mathbb{D}) = \{1 + z : |z| < \lambda\}$ ($q(z) = 1 + \lambda z$ and $\mathbb{D}$ is the unit disk), and $\phi(w) \neq 0$ when $w \in q(\mathbb{D})$. On the other hand, let

$$Q(z) = zq'(z)\phi(q(z)) = z\lambda \frac{\alpha}{q(z)} = \frac{\alpha\lambda z}{1 + \lambda z} = h(z) - 1. \quad (15)$$

Then

$$\begin{aligned}\frac{zQ'(z)}{Q(z)} &= \frac{zh'(z)}{Q(z)} = \frac{1}{1 + \lambda z}, \\ \operatorname{Re}\left\{\frac{zQ'(z)}{Q(z)}\right\} &= \operatorname{Re}\left\{\frac{zh'(z)}{Q(z)}\right\} = \operatorname{Re}\frac{1}{q(z)} > \frac{1}{1 + \lambda} > 0, \\ & z \in \mathbb{D}.\end{aligned}\quad (16)$$

The last inequality holds since $\operatorname{Re} q(z) = 1 + \lambda \operatorname{Re} z < 1 + \lambda$ for all $z \in \mathbb{D}$ and $q(\mathbb{D})$ does not contain the origin. So, conditions (i) and (ii) from Lemma 1 are satisfied.

Further, p is analytic in $\mathbb{D}$ and $p(0) = q(0) = 1$. Also, $p(z) \neq 0$ for all $z \in \mathbb{D}$; that is, $p(\mathbb{D}) \subseteq D$, since $f'(z) \neq 0$ for all $z \in \mathbb{D}$ (condition of the theorem); $z/f(z) = 1 \neq 0$ for $z = 0$ (because $f \in \mathcal{A}$) and $f(z)$ has no poles on $\mathbb{D}$. Hence from Lemma 1 and the fact that

$$\begin{aligned}J(f, \alpha; z) &= 1 + \alpha \cdot \frac{zp'(z)}{p(z)} = \theta(p(z)) + zp'(z)\phi(p(z)) \\ &< \frac{\alpha\lambda z}{1 + \lambda z} = \theta(q(z)) + zq'(z)\phi(q(z)),\end{aligned}\quad (17)$$

we receive that $p(z) < q(z)$, that is, relation (14), and we also receive that $q(z)$ is the best dominant of (13). $\square$

Applying the definition of subordination to Theorem 2, we receive the following.

Corollary 3. Let $f \in \mathcal{A}$, $0 < \lambda \leq 1$, and $\alpha \neq 0$. Also, let $f(z)/z \neq 0$ and $f'(z) \neq 0$ for all $z \in \mathbb{D}$. If

$$|J(f, \alpha; z) - 1| < \frac{\lambda |\alpha|}{1 + \lambda}, \quad z \in \mathbb{D}, \quad (18)$$

then $f \in \mathcal{U}(\lambda, -1/\alpha)$; that is,

$$\left| U\left(f, -\frac{1}{\alpha}; z\right) - 1 \right| < \lambda, \quad z \in \mathbb{D}. \quad (19)$$

This result is sharp; that is, the constant λ in inequality (19) cannot be replaced by a smaller one such that the implication still holds.

Proof. First, let us note that the function h defined by expression (13) is univalent in the unit disk such that

$$\min \{|h(z) - 1| : |z| = 1\} = \frac{\lambda |\alpha|}{1 + \lambda}. \quad (20)$$

So, the disk $\{w : |w - 1| < \lambda |\alpha|/(1 + \lambda)\}$ is contained in $h(\mathbb{D})$ which, having in mind the definition of subordination, means that inequality (18) implies subordination (13). Further, from Theorem 2 follows subordination (14), which is equivalent to the inequality (19). Even more, Theorem 2 says that $q(z) = 1 + \lambda z$ is the best dominant of (13).

In order to prove the sharpness of the result let us assume the opposite; that is, there exists λ_* , $0 < \lambda_* < \lambda$, such that inequality (18) implies

$$\left| U\left(f, -\frac{1}{\alpha}; z\right) - 1 \right| < \lambda_*, \quad z \in \mathbb{D}; \quad (21)$$

that is,

$$U\left(f, -\frac{1}{\alpha}; z\right) < 1 + \lambda_* z \equiv q_*(z). \quad (22)$$

On the other hand, inequality (18) implies subordination (13) with best dominant $q(z)$, meaning that $q(z) < q_*(z)$. This is a contradiction to the assumption $\lambda_* < \lambda$ which proves the sharpness of the result. $\square$

Previous corollary can be written in the following, equivalent, form that gives conditions for inclusion of the class $\mathcal{M}'(\alpha, \beta)$ into the class $\mathcal{U}(\lambda, -1/\alpha)$.

Corollary 4. Let $f \in \mathcal{A}$, $0 < \lambda \leq 1$, $\alpha \neq 0$, and $\beta = \lambda |\alpha|/(1 + \lambda)$. Also, let $f(z)/z \neq 0$ and $f'(z) \neq 0$ for all $z \in \mathbb{D}$. Then

$$f \in \mathcal{M}'(\alpha, \beta) \implies f \in \mathcal{U}\left(\lambda, -\frac{1}{\alpha}\right). \quad (23)$$

The constant λ , for the class $\mathcal{U}(\lambda, -1/\alpha)$, cannot be replaced by a smaller one such that the inclusion still holds.

Next result gives the connection between classes $\mathcal{M}(\alpha, \gamma)$ and $\mathcal{U}(\lambda, -1/\alpha)$.

Corollary 5. Let $f \in \mathcal{A}$, $0 < \lambda \leq 1$, $\alpha \neq 0$, and

$$\gamma := \begin{cases} 1 + \alpha - \frac{\alpha}{1 - \lambda}, & \text{if } \alpha > 0, \lambda \neq 1, \\ 1 + \alpha - \frac{\alpha}{1 + \lambda}, & \text{if } \alpha < 0. \end{cases} \quad (24)$$

Also, let $f(z)/z \neq 0$ and $f'(z) \neq 0$ for all $z \in \mathbb{D}$. If subordination (13) holds, then

$$f \in \mathcal{U}\left(\lambda, -\frac{1}{\alpha}\right) \cap \mathcal{M}(\alpha, \gamma). \quad (25)$$

Proof. It is easy to check that all conditions of Theorem 2 are fulfilled; hence, $f \in \mathcal{U}(\lambda, -1/\alpha)$.

It remains to verify that $f \in \mathcal{M}(\alpha, \gamma)$; that is, subordination (13) implies

$$\operatorname{Re} J(f, \alpha; z) > \gamma, \quad z \in \mathbb{D}. \quad (26)$$

Having in mind the definition of subordination and the fact that $h(z) = 1 + (\alpha \lambda z)/(1 + \lambda z)$ is univalent, it is enough to show that $\operatorname{Re} h(z) \geq \gamma$ for all $z \in \mathbb{D}$. The last is true because

$$\begin{aligned} \inf \{\operatorname{Re} h(z) : z \in \mathbb{D}\} &= \min \{\operatorname{Re} h(z) : |z| = 1\} \\ &= \min \{h(1), h(-1)\} = \gamma. \end{aligned} \quad (27)$$

$\square$

Remark 6. The case $\alpha > 0$ and $\lambda = 1$ is not covered by the previous corollary since then $\inf \{\operatorname{Re} h(z) : z \in \mathbb{D}\} = -\infty$.

3. Examples

Now we will apply the results from the previous section on specific functions $f \in \mathcal{A}$ and receive interesting conclusions.

Example 1. Let $0 < \lambda \leq 1$, $\alpha \neq 0$, and $a \in \mathbb{R}$. Consider the following.

(i) Let

$$\gamma := \begin{cases} 1 + \alpha - \frac{\alpha}{1 - \lambda}, & \text{if } \alpha > 0, \lambda \neq 1 \\ 1 + \alpha - \frac{\alpha}{1 + \lambda}, & \text{if } \alpha < 0. \end{cases} \quad (28)$$

If $a \neq 0$, $a \neq -1$, $\lambda \leq |1 + a|$, $a\alpha > 0$, and $|a| > |\alpha|$, then

$$f(z) = z \cdot \left(1 + \frac{\lambda}{1 + a} z\right)^a \in \mathcal{U}\left(\lambda, -\frac{1}{\alpha}\right) \cap \mathcal{M}(\alpha, \gamma). \quad (29)$$

(ii) If $|a| < 1$ and one of the following two sets of conditions holds:

$$\alpha > 0, \quad |a|(1 + \lambda)(1 - |a| + \alpha) < \lambda \alpha (1 - |a|) \quad (30)$$

or

$$\alpha < 0, \quad |a|(1 + \lambda)(1 + |a| + \alpha) < \lambda |\alpha| (1 + |a|), \quad (31)$$

then

$$f(z) = z \cdot e^{az} \in \mathcal{U}\left(\lambda, -\frac{1}{\alpha}\right). \quad (32)$$

In both cases, power is taken by its principal value.

Proof. (i) For the function $f(z) = z(1 + (\lambda/(1+a))z)^a$, we have $f(0) = 0$ and

$$f'(z) = \left(1 + \frac{\lambda}{1+a}z\right)^a \cdot \left(1 + \frac{\alpha\lambda z}{1+\alpha+\lambda z}\right) \Rightarrow f'(0) = 1. \quad (33)$$

Condition $\lambda \leq |1+a|$ guarantees that $1+\alpha+\lambda z \neq 0$ for all $z \in \mathbb{D}$; hence, f is an analytic function and $f \in \mathcal{A}$. For the function f , it is easy to verify that

$$\begin{aligned} J(f, \alpha; z) &\equiv (1-\alpha) \frac{zf'(z)}{f(z)} + \alpha \left(1 + \frac{zf''(z)}{f'(z)}\right) \\ &= 1 + \frac{a\lambda z}{1+\lambda z}. \end{aligned} \quad (34)$$

Further, from the definition of subordination, we have that $az < \alpha z$ when $|a| > |\alpha|$ and $a\alpha > 0$; that is, both are positive or both negative. Therefore,

$$J(f, \alpha; z) = 1 + \frac{a\lambda z}{1+\lambda z} < 1 + \frac{\alpha\lambda z}{1+\lambda z}. \quad (35)$$

So, all conditions of Corollary 5 bring us to the conclusion that $f \in \mathcal{U}(\lambda, -1/\alpha) \cap \mathcal{M}(\alpha, \gamma)$.

(ii) It is easy to verify that $f \in \mathcal{A}$ and that

$$\begin{aligned} J(f, \alpha; z) &\equiv (1-\alpha) \frac{zf'(z)}{f(z)} + \alpha \left(1 + \frac{zf''(z)}{f'(z)}\right) \\ &= 1 + az \left(1 + \frac{\alpha}{1+az}\right). \end{aligned} \quad (36)$$

Therefore,

$$\begin{aligned} \Delta &\equiv \sup \{|J(f, \alpha; z) - 1| : z \in \mathbb{D}\} \\ &= \max \{|J(f, \alpha; z) - 1| : |z| = 1\} \\ &= |a| \cdot \max \left\{1 + \frac{\alpha}{1+az} : z \in \mathbb{D}\right\} \\ &= |a| \cdot \begin{cases} 1 + \frac{\alpha}{1-|a|}, & \text{if } \alpha > 0, \\ 1 + \frac{\alpha}{1+|a|}, & \text{if } \alpha < 0. \end{cases} \end{aligned} \quad (37)$$

Further, if one of the conditions (30) or (31) holds, then $\Delta \leq (\lambda|\alpha|)/(1+\lambda)$; that is, we receive inequality (18). Finally, we have shown that all conditions of Corollary 3 are fulfilled, which leads to $f(z) \in \mathcal{U}(\lambda, -1/\alpha)$. $\square$

The following example exhibits some concrete conclusions that can be obtained from the results of the previous section by specifying the values of λ and/or α .

Example 2. Let $f \in \mathcal{A}$, $0 < \lambda \leq 1$ and let $f(z)/z \neq 0$ and $f'(z) \neq 0$ for all $z \in \mathbb{D}$. Consider the following:

- (i) $f \in \mathcal{M}'(\alpha, |\alpha|/2) \Rightarrow f \in \mathcal{U}(1, -1/\alpha)$ ($\lambda = 1$ in Corollary 4);
- (ii) $f \in \mathcal{U}(1, -1/\alpha) \cap \mathcal{M}(\alpha, 1 + \alpha/2)$ ($\lambda = 1$ and $\alpha < 0$ in Corollary 5).

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

References

- [1] P. L. Duren, *Univalent Functions*, vol. 259, Springer, New York, NY, USA, 1983.
- [2] L. A. Aksent'ev, "Sufficient conditions for univalence of regular functions," *Izvestiya Vysših Učebnyh Zavedenii Matematika*, vol. 4, no. 3, pp. 3–7, 1958 (Russian).
- [3] R. Fournier and S. Ponnusamy, "A class of locally univalent functions defined by a differential inequality," *Complex Variables and Elliptic Equations*, vol. 52, no. 1, pp. 1–8, 2007.
- [4] M. Nunokawa, M. Obradović, and S. Owa, "One criterion for univalence," *Proceedings of the American Mathematical Society*, vol. 106, no. 4, pp. 1035–1037, 1989.
- [5] M. Obradović and S. Ponnusamy, "Product of univalent functions," *Mathematical and Computer Modelling*, vol. 57, no. 3–4, pp. 793–799, 2013.
- [6] M. Obradović and S. Ponnusamy, "Criteria for univalent functions in the unit disk," *Archiv der Mathematik*, vol. 100, no. 2, pp. 149–157, 2013.
- [7] M. Obradović and S. Ponnusamy, "On harmonic combination of univalent functions," *Bulletin of the Belgian Mathematical Society. Simon Stevin*, vol. 19, no. 3, pp. 461–472, 2012.
- [8] M. Obradović and S. Ponnusamy, "New criteria and distortion theorems for univalent functions," *Complex Variables, Theory and Application*, vol. 44, no. 3, pp. 173–191, 2001.
- [9] M. Obradović, S. Ponnusamy, V. Singh, and P. Vasundhara, "Univalence, starlikeness and convexity applied to certain classes of rational functions," *Analysis*, vol. 22, no. 3, pp. 225–242, 2002.
- [10] M. Obradović and S. Ponnusamy, "Univalence and starlikeness of certain transforms defined by convolution of analytic functions," *Journal of Mathematical Analysis and Applications*, vol. 336, no. 2, pp. 758–767, 2007.
- [11] M. Obradović and S. Ponnusamy, "Coefficient characterization for certain classes of univalent functions," *Bulletin of the Belgian Mathematical Society. Simon Stevin*, vol. 16, no. 2, pp. 251–263, 2009.
- [12] S. Ozaki and M. Nunokawa, "The Schwarzian derivative and univalent functions," *Proceedings of the American Mathematical Society*, vol. 33, pp. 392–394, 1972.
- [13] S. S. Miller and P. T. Mocanu, *Differential Subordinations*, vol. 225 of *Theory and Applications*, Marcel Dekker, New York, NY, USA, 2000.
- [14] S. S. Miller and P. T. Mocanu, "Differential subordinations and univalent functions," *The Michigan Mathematical Journal*, vol. 28, no. 2, pp. 157–172, 1981.
- [15] S. S. Miller and P. T. Mocanu, "On some classes of first-order differential subordinations," *The Michigan Mathematical Journal*, vol. 32, no. 2, pp. 185–195, 1985.