Research Article

Invariant Means and Reversible Semigroup of Relatively Nonexpansive Mappings in Banach Spaces

Kyung Soo Kim

Graduate School of Education, Mathematics Education, Kyungnam University, Changwon 631-701, Republic of Korea

Correspondence should be addressed to Kyung Soo Kim; kksmj@kyungnam.ac.kr

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The purpose of this paper is to study modified Halpern type and Ishikawa type iteration for a semigroup of relatively nonexpansive mappings $\mathfrak{S} = \{T(s) : s \in S\}$ on a nonempty closed convex subset C of a Banach space with respect to a sequence of asymptotically left invariant means $\{\mu_n\}$ defined on an appropriate invariant subspace of $l^\infty(S)$, where S is a semigroup. We prove that, given some mild conditions, we can generate iterative sequences which converge strongly to a common element of the set of fixed points $F(\mathfrak{S})$, where $F(\mathfrak{S}) = \bigcap \{F(T(s)) : s \in S\}$.

1. Introduction

Let E be a real Banach space with the topological dual E^* and let C be a closed and convex subset of E . A mapping T of C into itself is called *nonexpansive* if $\|Tx - Ty\| \leq \|x - y\|$ for each $x, y \in C$.

Three classical iteration processes are often used to approximate a fixed point of a nonexpansive mapping. The first one is introduced by Halpern [1] and is defined as follows:

$$\begin{aligned} x_0 &= u \in C, \quad \text{chosen arbitrarily,} \\ x_{n+1} &= \alpha_n u + (1 - \alpha_n)Tx_n, \quad \forall n \geq 1, \end{aligned} \quad (1)$$

where $\{\alpha_n\}$ is a sequence in $[0, 1]$. He pointed out that the conditions $\lim_{n \rightarrow \infty} \alpha_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$ are necessary in the sense that if the iteration (1) converges to a fixed point of T , then these conditions must be satisfied. The second iteration process is known as Mann's iteration process [2] which is defined as follows:

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n)Tx_n, \quad \forall n \geq 1, \quad (2)$$

where the initial x_1 is taken in C arbitrary and the sequence $\{\alpha_n\}$ is in $[0, 1]$.

The third iteration process is referred to as Ishikawa's iteration process [3] which is defined as follows:

$$\begin{aligned} y_n &= \beta_n x_n + (1 - \beta_n)Tx_n, \\ x_{n+1} &= \alpha_n x_n + (1 - \alpha_n)Ty_n, \quad \forall n \geq 1, \end{aligned} \quad (3)$$

where the initial x_1 is taken in C arbitrary and $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $[0, 1]$.

In 2007, Lau et al. [4] proposed the following modification of Halpern's iteration (1) for amenable semigroups of nonexpansive mappings in a Banach space.

Theorem 1. *Let S be a left reversible semigroup and let $\mathfrak{S} = \{T(s) : s \in S\}$ be a representation of S as nonexpansive mappings from a compact convex subset C of a strictly convex and smooth Banach space E into C , let X be an amenable and $\mathfrak{S}$ -stable subspace of $l^\infty(S)$, and let $\{\mu_n\}$ be a strongly left regular sequence of means on X . Let $\{\alpha_n\}$ be a sequence in $[0, 1]$ such that $\lim_{n \rightarrow \infty} \alpha_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$. Let $x_1 = x \in C$ and let $\{x_n\}$ be the sequence defined by*

$$x_{n+1} = \alpha_n x + (1 - \alpha_n)T(\mu_n)x_n, \quad n \geq 2. \quad (4)$$

Then $\{x_n\}$ converges strongly to Px , where P denotes the unique sunny nonexpansive retraction of C onto $F(\mathfrak{S})$.

Let C be a closed and convex subset of E and let T be a mapping from C into itself. We denote by $F(T)$ the set of fixed

points of T . Point p in C is said to be an *asymptotic fixed point* of T [5] if C contains a sequence $\{x_n\}$ which converges weakly to p such that the strong $\lim_{n \rightarrow \infty} (Tx_n - x_n) = 0$. The set of asymptotic fixed points of T will be denoted by $\hat{F}(T)$. A mapping T from C into itself is called *relatively nonexpansive* [6–8], if $\hat{F}(T) = F(T)$ and $\phi(p, Tx) \leq \phi(p, x)$ for all $x \in C$ and $p \in F(T)$. The asymptotic behavior of relatively nonexpansive mappings was studied in [6, 7, 9].

Recently, Kim [10] proved a strong convergence theorem for relatively nonexpansive mappings in a Banach space by using the shrinking method.

Theorem 2. *Let S be a left reversible semigroup and let $\mathfrak{S} = \{T(s) : s \in S\}$ be a representation of S as relatively nonexpansive mappings from a nonempty, closed, and convex subset C of a uniformly convex and uniformly smooth Banach space E into C with $F(\mathfrak{S}) \neq \emptyset$. Let X be a subspace of $l^\infty(S)$ and let $\{\mu_n\}$ be a asymptotically left invariant sequence of means on X . Let $\{\alpha_n\}$ be a sequence in $[0, 1]$ such that $0 < \alpha_n < 1$ and $\lim_{n \rightarrow \infty} \alpha_n = 0$. Let $\{x_n\}$ be a sequence generated by the following algorithm:*

$$x_0 \in C, \quad \text{chosen arbitrarily,}$$

$$C_1 = C,$$

$$x_1 = \Pi_{C_1} x_0,$$

$$y_n = J^{-1}(\alpha_n Jx_1 + (1 - \alpha_n) JT_{\mu_n} x_n), \quad (5)$$

$$C_{n+1} = \{z \in C_n :$$

$$\phi(z, y_n) \leq \alpha_n \phi(z, x_1) + (1 - \alpha_n) \phi(z, x_n)\},$$

$$x_{n+1} = \Pi_{C_{n+1}} x_1, \quad \forall n \geq 1.$$

Then $\{x_n\}$ converges strongly to $\Pi_{F(\mathfrak{S})} x_1$, where $\Pi_{F(\mathfrak{S})}$ is the generalized projection from C onto $F(\mathfrak{S})$.

Let S be a semigroup. The purpose of this paper is to study modified Halpern type and Ishikawa type iterations for a semigroup of relatively nonexpansive mappings $\mathfrak{S} = \{T(s) : s \in S\}$ on a nonempty closed convex subset C of a Banach space with respect to a sequence of asymptotically left invariant means $\{\mu_n\}$ defined on an appropriate invariant subspace of $l^\infty(S)$. We prove that, given some mild conditions, we can generate iterative sequences which converge strongly to a common element of the set of fixed points $F(\mathfrak{S})$, where $F(\mathfrak{S}) = \bigcap \{F(T(s)) : s \in S\}$.

2. Preliminaries

A real Banach space E is said to be *strictly convex* if $\|(x + y)/2\| < 1$ for all $x, y \in E$ with $\|x\| = \|y\| = 1$ and $x \neq y$. It is said to be *uniformly convex* if $\lim_{n \rightarrow \infty} \|x_n - y_n\| = 0$ for any two sequences $\{x_n\}$ and $\{y_n\}$ in E such that $\|x_n\| = \|y_n\| = 1$ and $\lim_{n \rightarrow \infty} \|(x_n + y_n)/2\| = 1$. Let $U = \{x \in E : \|x\| = 1\}$ be the unit sphere of E . Then the Banach space E is said to be *smooth* if

$$\lim_{t \rightarrow 0} \frac{\|x + ty\| - \|x\|}{t} \quad (6)$$

exists for each $x, y \in U$. It is said to be *uniformly smooth* if the limit is attained uniformly for $x, y \in E$.

Let E be a real Banach space with norm $\|\cdot\|$ and let E^* be the dual space of E . Denote by $\langle \cdot, \cdot \rangle$ the duality product. We denote by J the normalized duality mapping from E to 2^{E^*} defined by

$$Jx = \{f^* \in E^* : \langle x, f^* \rangle = \|x\|^2 = \|f^*\|^2\}, \quad (7)$$

for $x \in E$. A Banach space E is said to have the Kadec-Klee property if a sequence $\{x_n\}$ of E satisfies that $x_n \rightharpoonup x$ and $\|x_n\| \rightarrow \|x\|$ and then $x_n \rightarrow x$, where $\rightharpoonup$ and $\rightarrow$ denote the weak convergence and the strong convergence, respectively.

We know the following:

- (1) the duality mapping J is monotone, that is, $\langle x - y, x^* - y^* \rangle \geq 0$ whenever $x^* \in Jx$ and $y^* \in Jy$;
- (2) if E is strictly convex, then J is one-to-one; that is, if $Jx \cap Jy$ is nonempty, then $x = y$;
- (3) if E is strictly convex, then J is strictly monotone; that is, $x = y$ whenever $\langle x - y, x^* - y^* \rangle = 0$, $x^* \in Jx$ and $y^* \in Jy$;
- (4) if E is uniformly convex, then E has the Kadec-Klee property;
- (5) if E is uniformly convex, then E is reflexive and strictly convex;
- (6) if E is smooth, then J is single-valued and norm-to-weak* continuous;
- (7) if E is uniformly smooth, then J is uniformly norm-to-norm continuous on bounded subsets of E ;
- (8) if E is reflexive, then J is onto;
- (9) if E is smooth and reflexive, then J is norm-to-weak continuous; that is, $Jx_n \rightharpoonup Jx$ whenever $x_n \rightarrow x$;
- (10) if E is smooth, strictly convex, and reflexive, then J is single-valued, one-to-one and onto; in this case, the inverse mapping J^{-1} coincides with the duality mapping on E ;
- (11) if E^* is strictly convex, then J is single-valued;
- (12) the norm of E^* is Fréchet differentiable if and only if E is strictly convex and reflexive Banach space which has the Kadec-Klee property.

For more details, see [11].

As well known, if C is a nonempty, closed, and convex subset of a Hilbert space H and $P_C : H \rightarrow C$ is the metric projection of H onto C , then P_C is nonexpansive (see, the reference therein). This fact actually characterizes Hilbert spaces. Consequently, it is not true to more general Banach spaces. In this connection, Alber [12] introduced a generalized projection operator Π_C in a Banach space E which is an analogue of the metric projection in Hilbert spaces. Consider the function defined by

$$\phi(x, y) = \|x\|^2 - 2\langle x, Jy \rangle + \|y\|^2, \quad (8)$$

for $x, y \in E$. Observe that, in a Hilbert space H , (8) reduces to

$$\phi(x, y) = \|x - y\|^2, \quad (9)$$

for $x, y \in H$. The generalized projection $\Pi_C : E \rightarrow C$ is a mapping that assigns an arbitrary point $x \in E$ to the minimum point of the functional $\phi(x, y)$; that is, $\Pi_C x = \bar{x}$, where $\bar{x}$ is the solution to the minimization problem:

$$\phi(\bar{x}, x) = \inf_{y \in C} \phi(y, x). \quad (10)$$

The existence and uniqueness of the operator Π_C follow from the properties of the functional $\phi(x, y)$ and strict monotonicity of the mapping J (see, e.g., [12, 13]). In a Hilbert space, $\Pi_C = P_C$. It is obvious from the definition of the function ϕ that

- (ϕ_1) $(\|x\| - \|y\|)^2 \leq \phi(x, y) \leq (\|x\| + \|y\|)^2$ for all $x, y \in E$,
- (ϕ_2) $\phi(x, y) = \phi(x, z) + \phi(z, y) + 2\langle x - z, Jz - Jy \rangle$ for all $x, y, z \in E$,
- (ϕ_3) $\phi(x, y) = \langle x, Jx - Jy \rangle + \langle y - x, Jy \rangle \leq \|x\| \|Jx - Jy\| + \|y - x\| \|Jy\|$ for all $x, y \in E$,
- (ϕ_4) if E is a reflexive, strictly convex, and smooth Banach space, then, for all $x, y \in E$,

$$\phi(x, y) = 0 \quad \text{iff } x = y. \quad (11)$$

For more details see [14].

Let S be a semigroup. We denote by $l^\infty(S)$ the Banach space of all bounded real-valued functionals on S with supremum norm. For each $s \in S$, we define the left and right translation operators $l(s)$ and $r(s)$ on $l^\infty(S)$ by

$$(l(s)f)(t) = f(st), \quad (r(s)f)(t) = f(ts), \quad (12)$$

for each $t \in S$ and $f \in l^\infty(S)$, respectively. Let X be a subspace of $l^\infty(S)$ containing 1. An element μ in the dual space X^* of X is said to be a *mean* on X if $\|\mu\| = \mu(1) = 1$. For $s \in S$, we can define a point evaluation δ_s by $\delta_s(f) = f(s)$ for each $f \in X$. It is well known that μ is mean on X if and only if

$$\inf_{s \in S} f(s) \leq \mu(f) \leq \sup_{s \in S} f(s), \quad (13)$$

for each $f \in X$.

Let X be a translation invariant subspace of $l^\infty(S)$ (i.e., $l(s)X \subset X$ and $r(s)X \subset X$ for each $s \in S$) containing 1. Then a mean μ on X is said to be *left invariant* (resp., *right invariant*) if

$$\mu(l(s)f) = \mu(f), \quad (\text{resp., } \mu(r(s)f) = \mu(f)) \quad (14)$$

for each $s \in S$ and $f \in X$. A mean μ on X is said to be *invariant* if μ is both left and right invariant [15–19]. X is said to be *left* (resp., *right*) *amenable* if X has a left (resp., right) invariant mean. X is amenable if X is left and right amenable. We call a semigroup S *amenable* if X is amenable. Further, amenable semigroups include all commutative semigroups

and solvable groups. However, the free group or semigroup of two generators is not left or right amenable (see [20–22]).

A net $\{\mu_\alpha\}$ of means on X is said to be *asymptotically left* (resp., *right*) *invariant* if

$$\begin{aligned} \lim_\alpha (\mu_\alpha(l(s)f) - \mu_\alpha(f)) &= 0, \\ (\text{resp., } \lim_\alpha (\mu_\alpha(r(s)f) - \mu_\alpha(f)) &= 0), \end{aligned} \quad (15)$$

for each $f \in X$ and $s \in S$, and it is said to be *left* (resp., *right*) *strongly asymptotically invariant* (or *strong regular*) if

$$\begin{aligned} \lim_\alpha \|l^*(s)\mu_\alpha - \mu_\alpha\| &= 0, \\ (\text{resp., } \lim_\alpha \|r^*(s)\mu_\alpha - \mu_\alpha\| &= 0), \end{aligned} \quad (16)$$

for each $s \in S$, where $l^*(s)$ and $r^*(s)$ are the adjoint operators of $l(s)$ and $r(s)$, respectively. Such nets were first studied by Day in [20] where they were called *weak* invariant* and *norm invariant*, respectively.

It is easy to see that if a semigroup S is left (resp., right) amenable, then the semigroup $S' = S \cup \{e\}$, where $es' = s'e = s'$ for all $s' \in S$, is also left (resp., right) amenable and converse.

From now on S denotes a semigroup with an identity e . S is called *left reversible* if any two right ideals of S have nonvoid intersection; that is, $aS \cap bS \neq \emptyset$ for $a, b \in S$. In this case, $(S, \leq)$ is a directed system when the binary relation “ $\leq$ ” on S is defined by $a \leq b$ if and only if $\{a\} \cup aS \supseteq \{b\} \cup bS$ for $a, b \in S$. It is easy to see that $t \leq ts$ for all $t, s \in S$. Further, if $t \leq s$ then $pt \leq ps$ for all $p \in S$. The class of left reversible semigroup includes all groups and commutative semigroups. If a semigroup S is left amenable, then S is left reversible. But the converse is not true [23–28].

Let S be a semigroup and let C be a closed and convex subset of E . Let $F(T)$ denote the fixed point set of T . Then $\mathfrak{F} = \{T(s) : s \in S\}$ is called a *representation of S as relatively nonexpansive mappings on C* if $T(s)$ is relatively nonexpansive with $T(e) = I$ and $T(st) = T(s)T(t)$ for each $s, t \in S$. We denote by $F(\mathfrak{F})$ the set of common fixed points of $\{T(s) : s \in S\}$; that is,

$$F(\mathfrak{F}) = \bigcap_{s \in S} F(T(s)) = \bigcap_{s \in S} \{x \in C : T(s)x = x\}. \quad (17)$$

We know that if μ is a mean on X and if for each $x^* \in E^*$ the function $s \mapsto \langle T(s)x, x^* \rangle$ is contained in X and C is weakly compact, then there exists a unique point x_0 of E such that $\mu\langle T(\cdot)x, x^* \rangle = \langle x_0, x^* \rangle$ for each $x^* \in E^*$. We denote such a point x_0 by $T_\mu x$. Note that $T_\mu x$ is contained in the closure of the convex hull of $\{T(s)x : s \in S\}$ for each $x \in C$. Note that $T_\mu z = z$ for each $z \in F(\mathfrak{F})$; see [29–31].

3. Lemmas

We need the following lemmas for the proof of our main results.

Lemma 3 (see [9]). *Let E be a strictly convex and smooth Banach space, let C be a closed convex subset of E , and let T*

be a relatively nonexpansive mapping from C into itself. Then $F(T)$ is closed and convex.

Lemma 4 (see [12, 32]). Let E be a reflexive, strictly convex, and smooth Banach space and let C be a nonempty, closed, and convex subset of E and $x \in E$. Then

$$\phi(y, \Pi_C x) + \phi(\Pi_C x, x) \leq \phi(y, x), \quad (18)$$

for all $y \in C$.

Lemma 5 (see [32]). Let E be a uniformly convex and smooth Banach space and let $\{x_n\}$, $\{y_n\}$ be two sequences of E . If $\lim_{n \rightarrow \infty} \phi(x_n, y_n) = 0$ and either $\{x_n\}$ or $\{y_n\}$ is bounded, then $\lim_{n \rightarrow \infty} \|x_n - y_n\| = 0$.

Lemma 6 (see [4, 33]). Let μ be a left invariant mean on X . Then $F(\mathfrak{S}) = F(T_\mu) \cap C_a$, where C_a denotes the set of almost periodic elements in C ; that is, all $x \in C$ such that $\{T(s)x : s \in S\}$ is relatively compact in the norm topology of E .

Lemma 7 (cf. [4, 10]). Let $\{\mu_n\}$ be an asymptotically left invariant sequence of means on X . If $z \in C_a$ and $\liminf_{n \rightarrow \infty} \|T_{\mu_n} z - z\| = 0$, then z is a common fixed point of $\mathfrak{S}$.

4. Strong Convergence Theorems

In this section, we will establish two strong convergence theorems of various iterative sequences for finding common fixed point of relatively nonexpansive mappings in a uniformly convex and uniformly smooth Banach spaces (cf. [34–36]).

We begin with a strong convergence theorem of modified Halpern's type.

Theorem 8. Let S be a left reversible semigroup and let $\mathfrak{S} = \{T(s) : s \in S\}$ be a representation of S as relatively nonexpansive mappings from a nonempty, closed, and convex subset C of a uniformly convex and uniformly smooth Banach space E into itself. Let X be a subspace of $l^\infty(S)$ and let $\{\mu_n\}$ be an asymptotically left invariant sequence of means on X . Let $\{\alpha_n\}$ be a sequence in $(0, 1)$ such that $\lim_{n \rightarrow \infty} \alpha_n = 0$. Let $\{x_n\}$ be a sequence generated by the following algorithm:

$$\begin{aligned} x_0 &\in C, \quad \text{chosen arbitrarily,} \\ x_{n+1} &= \Pi_C J^{-1} (\alpha_n J x_0 + (1 - \alpha_n) J T_{\mu_n} x_n), \quad \forall n \geq 0. \end{aligned} \quad (19)$$

If the interior of $F(\mathfrak{S})$ is nonempty, then $\{x_n\}$ converges strongly to some common fixed point $F(\mathfrak{S})$.

Proof. We show first that the sequence $\{x_n\}$ converges strongly in C .

From Lemma 3, we know $F(T)$ is closed and convex. So, we can define the generalized projection Π_C onto $F(\mathfrak{S})$. Most of all, we have

$$\begin{aligned} \|T_{\mu_n} x_n\| &= \sup \{ |\langle T_{\mu_n} x_n, x^* \rangle| : x^* \in E^*, \|x^*\| = 1 \} \\ &= \sup \{ |(\mu_n)_s \langle T(s) x_n, x^* \rangle| : x^* \in E^*, \|x^*\| = 1 \} \end{aligned}$$

$$\begin{aligned} &\leq \sup \{ (\mu_n)_s (\|T(s) x_n\| \|x^*\|) : x^* \in E^*, \|x^*\| = 1 \} \\ &= (\mu_n)_s \|T(s) x_n\|. \end{aligned} \quad (20)$$

Then, from the definition of relatively nonexpansive, we have

$$\begin{aligned} \phi(u, T_{\mu_n} x_n) &= \|u\|^2 - 2 \langle u, J T_{\mu_n} x_n \rangle + \|T_{\mu_n} x_n\|^2 \\ &= \|u\|^2 - 2 (\mu_n)_s \langle u, J T(s) x_n \rangle \\ &\quad + (\mu_n)_s \|T(s) x_n\|^2 \\ &= (\mu_n)_s \phi(u, T(s) x_n) \\ &\leq (\mu_n)_s \phi(u, x_n) = \phi(u, x_n), \end{aligned} \quad (21)$$

for all $u \in F(\mathfrak{S})$. From the convexity of $\|\cdot\|^2$ and (21), we get

$$\begin{aligned} \phi(u, x_{n+1}) &= \phi(u, \Pi_C J^{-1} (\alpha_n J x_0 + (1 - \alpha_n) J T_{\mu_n} x_n)) \\ &\leq \phi(u, J^{-1} (\alpha_n J x_0 + (1 - \alpha_n) J T_{\mu_n} x_n)) \\ &= \|u\|^2 - 2 \langle u, \alpha_n J x_0 + (1 - \alpha_n) J T_{\mu_n} x_n \rangle \\ &\quad + \|\alpha_n J x_0 + (1 - \alpha_n) J T_{\mu_n} x_n\|^2 \\ &\leq \|u\|^2 - 2 \alpha_n \langle u, J x_0 \rangle - 2 (1 - \alpha_n) \langle u, J T_{\mu_n} x_n \rangle \\ &\quad + \alpha_n \|x_0\|^2 + (1 - \alpha_n) \|T_{\mu_n} x_n\|^2 \\ &= \alpha_n \phi(u, x_0) + (1 - \alpha_n) \phi(u, T_{\mu_n} x_n) \\ &\leq \alpha_n \phi(u, x_0) + (1 - \alpha_n) \phi(u, x_n). \end{aligned} \quad (22)$$

So, we have

$$\begin{aligned} &(1 - \alpha_n) \{ \phi(u, x_{n+1}) - \phi(u, x_n) \} \\ &\leq \alpha_n \{ \phi(u, x_0) - \phi(u, x_{n+1}) \} \\ &\leq \alpha_n \phi(u, x_0). \end{aligned} \quad (23)$$

Since $\lim_{n \rightarrow \infty} \alpha_n = 0$, we obtain

$$\lim_{n \rightarrow \infty} \{ \phi(u, x_{n+1}) - \phi(u, x_n) \} \leq 0. \quad (24)$$

Therefore $\{\phi(u, x_n)\}$ is bounded and $\lim_{n \rightarrow \infty} \phi(u, x_n)$ exists. Then $\{x_n\}$ is also bounded. This implies that $\{T_{\mu_n} x_n\}$ is bounded. Since the interior of $F(\mathfrak{S})$ is nonempty, there exist $p \in F(\mathfrak{S})$ and $r > 0$ such that

$$p + r q \in F(\mathfrak{S}), \quad (25)$$

whenever $\|q\| \leq 1$. By (ϕ_2) , we have

$$\begin{aligned} \phi(u, x_n) &= \phi(u, x_{n+1}) + \phi(x_{n+1}, x_n) \\ &\quad + 2 \langle u - x_{n+1}, J x_{n+1} - J x_n \rangle, \end{aligned} \quad (26)$$

for any $u \in F(\mathfrak{F})$. This implies

$$\begin{aligned} & \langle x_{n+1} - u, Jx_n - Jx_{n+1} \rangle + \frac{1}{2} \phi(x_{n+1}, x_n) \\ &= \frac{1}{2} (\phi(u, x_n) - \phi(u, x_{n+1})). \end{aligned} \quad (27)$$

Also, we have

$$\begin{aligned} & \langle x_{n+1} - p, Jx_n - Jx_{n+1} \rangle \\ &= \langle x_{n+1} - (p + rq) + rq, Jx_n - Jx_{n+1} \rangle \\ &= \langle x_{n+1} - (p + rq), Jx_n - Jx_{n+1} \rangle \\ &\quad + r \langle q, Jx_n - Jx_{n+1} \rangle. \end{aligned} \quad (28)$$

On the other hand, by (24) and (25), we have that

$$\phi(p + rq, x_{n+1}) \leq \phi(p + rq, x_n). \quad (29)$$

From (27), we get

$$\begin{aligned} 0 &\leq \frac{1}{2} (\phi(p + rq, x_n) - \phi(p + rq, x_{n+1})) \\ &= \langle x_{n+1} - (p + rq), Jx_n - Jx_{n+1} \rangle \\ &\quad + \frac{1}{2} \phi(x_{n+1}, x_n) \\ &= \langle x_{n+1} - p, Jx_n - Jx_{n+1} \rangle \\ &\quad - r \langle q, Jx_n - Jx_{n+1} \rangle + \frac{1}{2} \phi(x_{n+1}, x_n). \end{aligned} \quad (30)$$

Then, by (27), we have

$$\begin{aligned} & r \langle q, Jx_n - Jx_{n+1} \rangle \\ &\leq \langle x_{n+1} - p, Jx_n - Jx_{n+1} \rangle + \frac{1}{2} \phi(x_{n+1}, x_n) \\ &= \frac{1}{2} (\phi(p, x_n) - \phi(p, x_{n+1})), \end{aligned} \quad (31)$$

for $p \in F(\mathfrak{F})$. Hence

$$\langle q, Jx_n - Jx_{n+1} \rangle \leq \frac{1}{2r} (\phi(p, x_n) - \phi(p, x_{n+1})). \quad (32)$$

Since q with $\|q\| \leq 1$ is arbitrary, by (24), we have

$$\|Jx_n - Jx_{n+1}\| \leq \frac{1}{2r} (\phi(p, x_n) - \phi(p, x_{n+1})). \quad (33)$$

So, we have

$$\begin{aligned} & \|Jx_{n+m} - Jx_n\| \\ &= \|Jx_{n+m} - Jx_{n+m-1} + Jx_{n+m-1} \\ &\quad - \cdots - Jx_{n+1} + Jx_{n+1} - Jx_n\| \end{aligned}$$

$$\begin{aligned} & \leq \sum_{i=n}^{n+m-1} \|Jx_i - Jx_{i+1}\| \\ &\leq \frac{1}{2r} \sum_{i=n}^{n+m-1} (\phi(p, x_i) - \phi(p, x_{i+1})) \\ &= \frac{1}{2r} (\phi(p, x_n) - \phi(p, x_{n+1})). \end{aligned} \quad (34)$$

We know that $\{\phi(p, x_n)\}$ converges. Hence, $\{Jx_n\}$ is a Cauchy sequence. Since E^* is complete, $\{Jx_n\}$ converges strongly to some point in E^* . Since E is uniformly convex, E^* has a Fréchet differentiable norm. Then J^{-1} is continuous on E^* . Hence $\{x_n\}$ converges strongly to some point v in C .

Now, we show that $v \in F(\mathfrak{F})$, where $v = \lim_{n \rightarrow \infty} \Pi_{F(\mathfrak{F})} x_n$. By (33) and the convergence of $\{\phi(p, x_n)\}$, we have

$$\lim_{n \rightarrow \infty} \|Jx_n - Jx_{n+1}\| = 0. \quad (35)$$

Since J^{-1} is uniformly norm-to-norm continuous on bounded sets, it follows that

$$\lim_{n \rightarrow \infty} \|x_n - x_{n+1}\| = 0. \quad (36)$$

Let $z_n = J^{-1}(\alpha_n Jx_0 + (1 - \alpha_n)JT_{\mu_n} x_n)$. Then, we have

$$\begin{aligned} & \|Jz_n - JT_{\mu_n} x_n\| \\ &= \|\alpha_n Jx_0 + (1 - \alpha_n)JT_{\mu_n} x_n - JT_{\mu_n} x_n\| \\ &= \alpha_n \|Jx_0 - JT_{\mu_n} x_n\|. \end{aligned} \quad (37)$$

Since $\lim_{n \rightarrow \infty} \alpha_n = 0$, we have

$$\lim_{n \rightarrow \infty} \|Jz_n - JT_{\mu_n} x_n\| = 0. \quad (38)$$

Since J^{-1} is uniformly norm-to-norm continuous on bounded sets, we get

$$\lim_{n \rightarrow \infty} \|z_n - T_{\mu_n} x_n\| = 0. \quad (39)$$

From $x_{n+1} = \Pi_C z_n$ and Lemma 4, we have

$$\begin{aligned} & \phi(T_{\mu_n} x_n, x_{n+1}) + \phi(x_{n+1}, z_n) \\ &= \phi(T_{\mu_n} x_n, \Pi_C z_n) + \phi(\Pi_C z_n, z_n) \\ &\leq \phi(T_{\mu_n} x_n, z_n). \end{aligned} \quad (40)$$

Since

$$\begin{aligned}
& \phi(T_{\mu_n}x_n, z_n) \\
&= \phi(T_{\mu_n}x_n, J^{-1}(\alpha_n Jx_0 + (1 - \alpha_n)JT_{\mu_n}x_n)) \\
&= \|T_{\mu_n}x_n\|^2 - 2\langle T_{\mu_n}x_n, \alpha_n Jx_0 + (1 - \alpha_n)JT_{\mu_n}x_n \rangle \\
&\quad + \|\alpha_n Jx_0 + (1 - \alpha_n)JT_{\mu_n}x_n\|^2 \\
&\leq \|T_{\mu_n}x_n\|^2 - 2\alpha_n \langle T_{\mu_n}x_n, Jx_0 \rangle \\
&\quad - 2(1 - \alpha_n) \langle T_{\mu_n}x_n, JT_{\mu_n}x_n \rangle \\
&\quad + \alpha_n \|x_0\|^2 + (1 - \alpha_n) \|T_{\mu_n}x_n\|^2 \\
&= \alpha_n \phi(T_{\mu_n}x_n, x_0) + (1 - \alpha_n) \phi(T_{\mu_n}x_n, T_{\mu_n}x_n) \\
&= \alpha_n \phi(T_{\mu_n}x_n, x_0)
\end{aligned} \tag{41}$$

and $\lim_{n \rightarrow \infty} \alpha_n = 0$, we have

$$\lim_{n \rightarrow \infty} \phi(T_{\mu_n}x_n, z_n) = 0. \tag{42}$$

From (40), we get

$$\lim_{n \rightarrow \infty} \phi(T_{\mu_n}x_n, x_{n+1}) = \lim_{n \rightarrow \infty} \phi(x_{n+1}, z_n) = 0. \tag{43}$$

By Lemma 5, we obtain

$$\lim_{n \rightarrow \infty} \|T_{\mu_n}x_n - x_{n+1}\| = \lim_{n \rightarrow \infty} \|x_{n+1} - z_n\| = 0. \tag{44}$$

Since $\|x_n - T_{\mu_n}x_n\| \leq \|x_n - x_{n+1}\| + \|x_{n+1} - z_n\| + \|z_n - T_{\mu_n}x_n\|$, from (36), (39), and (44), we have

$$\lim_{n \rightarrow \infty} \|x_n - T_{\mu_n}x_n\| = 0. \tag{45}$$

From Lemma 7, we have $x_n \in F(\mathfrak{S})$. Since $F(\mathfrak{S})$ is closed and $\lim_{n \rightarrow \infty} x_n = v$, we have $v \in F(\mathfrak{S})$, where $v = \lim_{n \rightarrow \infty} \Pi_{F(\mathfrak{S})}x_n$. $\square$

We now establish a convergence theorem of modified Ishikawa type.

Theorem 9. *Let S be a left reversible semigroup and let $\mathfrak{S} = \{T(s) : s \in S\}$ be a representation of S as relatively nonexpansive mappings from a nonempty, closed, and convex subset C of a uniformly convex and uniformly smooth Banach space E into itself. Let X be a subspace of $l^\infty(S)$ and let $\{\mu_n\}$ be an asymptotically left invariant sequence of means on X . Let $\{\alpha_n\}$ and $\{\beta_n\}$ be sequences of real numbers such that $\alpha_n, \beta_n \in (0, 1)$ and $\lim_{n \rightarrow \infty} \alpha_n = 0$, $\lim_{n \rightarrow \infty} \beta_n = 1$. Let $\{x_n\}$ be a sequence generated by the following algorithm:*

$$\begin{aligned}
& x_0 \in C, \quad \text{chosen arbitrarily,} \\
& y_n = J^{-1}(\beta_n Jx_n + (1 - \beta_n)JT_{\mu_n}x_n), \\
& x_{n+1} = \Pi_C J^{-1}(\alpha_n Jx_n + (1 - \alpha_n)JT_{\mu_n}y_n), \quad \forall n \geq 0.
\end{aligned} \tag{46}$$

If the interior of $F(\mathfrak{S})$ is nonempty, then $\{x_n\}$ converges strongly to some common fixed point $F(\mathfrak{S})$.

Proof. Firstly, we show that $\{x_n\}$ converges strongly in C .

From Lemma 3, we know $F(T)$ is closed and convex. So, we can define the generalized projection Π_C onto $F(\mathfrak{S})$. Let $u \in F(\mathfrak{S})$. From the definition of relatively nonexpansive and the convexity of $\|\cdot\|^2$, from (21), we have

$$\begin{aligned}
\phi(u, y_n) &= \phi(u, J^{-1}(\beta_n Jx_n + (1 - \beta_n)JT_{\mu_n}x_n)) \\
&\leq \beta_n \phi(u, x_n) + (1 - \beta_n) \phi(u, T_{\mu_n}x_n) \\
&\leq \phi(u, x_n),
\end{aligned} \tag{47}$$

for all $u \in F(\mathfrak{S})$. From (47), we obtain

$$\begin{aligned}
& \phi(u, x_{n+1}) \\
&= \phi(u, \Pi_C J^{-1}(\alpha_n Jx_n + (1 - \alpha_n)JT_{\mu_n}y_n)) \\
&\leq \phi(u, J^{-1}(\alpha_n Jx_n + (1 - \alpha_n)JT_{\mu_n}y_n)) \\
&\leq \alpha_n \phi(u, x_n) + (1 - \alpha_n) \phi(u, T_{\mu_n}y_n) \\
&\leq \alpha_n \phi(u, x_n) + (1 - \alpha_n) \phi(u, y_n) \\
&\leq \phi(u, x_n).
\end{aligned} \tag{48}$$

Hence, $\{\phi(u, x_n)\}$ is bounded and $\lim_{n \rightarrow \infty} \phi(u, x_n)$ exists. This implies that $\{x_n\}$, $\{T_{\mu_n}x_n\}$, and $\{y_n\}$ are bounded. Since the interior of $F(\mathfrak{S})$ is nonempty, similar to the proof of Theorem 8, we obtain that $\{x_n\}$ converges strongly to v in C .

Next, we show that $v \in F(\mathfrak{S})$, where $v = \lim_{n \rightarrow \infty} \Pi_{F(\mathfrak{S})}x_n$. Let

$$z_n = J^{-1}(\alpha_n Jx_n + (1 - \alpha_n)JT_{\mu_n}y_n). \tag{49}$$

From Lemma 4, we have

$$\begin{aligned}
& \phi(x_n, x_{n+1}) + \phi(x_{n+1}, z_n) \\
&= \phi(x_n, \Pi_C z_n) + \phi(\Pi_C z_n, z_n) \\
&\leq \phi(x_n, z_n).
\end{aligned} \tag{50}$$

Also,

$$\begin{aligned}
\phi(x_n, z_n) &= \phi(x_n, J^{-1}(\alpha_n Jx_n + (1 - \alpha_n)JT_{\mu_n}y_n)) \\
&\leq \alpha_n \phi(x_n, x_n) + (1 - \alpha_n) \phi(x_n, T_{\mu_n}y_n) \\
&\leq \alpha_n \phi(x_n, x_n) + (1 - \alpha_n) \phi(x_n, y_n) \\
&\leq \phi(x_n, y_n),
\end{aligned} \tag{51}$$

$$\begin{aligned}
\|Jx_n - Jy_n\| &= \|Jx_n - (\beta_n Jx_n + (1 - \beta_n)JT_{\mu_n}x_n)\| \\
&= (1 - \beta_n) \|Jx_n - JT_{\mu_n}x_n\|.
\end{aligned} \tag{52}$$

From $\lim_{n \rightarrow \infty} \beta_n = 1$ and (52), we have

$$\lim_{n \rightarrow \infty} \|Jx_n - Jy_n\| = 0. \tag{53}$$

Since J^{-1} is uniformly norm-to-norm continuous, we obtain

$$\lim_{n \rightarrow \infty} \|x_n - y_n\| = 0. \quad (54)$$

Hence,

$$\begin{aligned} \phi(x_n, y_n) &= \|x_n\|^2 - 2\langle x_n, Jy_n \rangle + \|y_n\|^2 \\ &= \|x_n\|^2 - 2\langle x_n, Jy_n - Jx_n \rangle \\ &\quad - 2\langle x_n, Jx_n \rangle + \|y_n\|^2 \\ &\leq \|y_n\|^2 - \|x_n\|^2 \\ &\quad + 2\|x_n\| \|Jy_n - Jx_n\| \\ &\leq \|y_n - x_n\| (\|y_n\| + \|x_n\|) \\ &\quad + 2\|x_n\| \|Jy_n - Jx_n\|. \end{aligned} \quad (55)$$

By (53) and (54), we have

$$\lim_{n \rightarrow \infty} \phi(x_n, y_n) = 0. \quad (56)$$

From (50) and (51), we obtain

$$\lim_{n \rightarrow \infty} \phi(x_n, x_{n+1}) = \lim_{n \rightarrow \infty} \phi(x_n, z_n) = 0. \quad (57)$$

From Lemma 5, we get

$$\lim_{n \rightarrow \infty} \|x_n - x_{n+1}\| = \lim_{n \rightarrow \infty} \|x_n - z_n\| = 0. \quad (58)$$

Since

$$\begin{aligned} \|Jz_n - JT_{\mu_n}y_n\| &= \|\alpha_n Jx_n + (1 - \alpha_n)JT_{\mu_n}y_n - JT_{\mu_n}y_n\| \\ &= \alpha_n \|Jx_n - JT_{\mu_n}x_n\| \end{aligned} \quad (59)$$

and $\lim_{n \rightarrow \infty} \alpha_n = 0$, we have

$$\lim_{n \rightarrow \infty} \|Jz_n - JT_{\mu_n}y_n\| = 0. \quad (60)$$

Since J^{-1} is uniformly norm-to-norm continuous, we obtain

$$\lim_{n \rightarrow \infty} \|z_n - T_{\mu_n}y_n\| = 0. \quad (61)$$

Since $\lim_{n \rightarrow \infty} \|x_n - z_n\| = 0$ and J is uniformly norm-to-norm continuous,

$$\lim_{n \rightarrow \infty} \|Jx_n - Jz_n\| = 0. \quad (62)$$

By (46) and (49), we have

$$\begin{aligned} JT_{\mu_n}x_n &= \frac{1}{1 - \beta_n} (Jy_n - \beta_n Jx_n), \\ JT_{\mu_n}y_n &= \frac{1}{1 - \alpha_n} (Jz_n - \alpha_n Jx_n). \end{aligned} \quad (63)$$

From (63), we obtain

$$\begin{aligned} &\|JT_{\mu_n}x_n - JT_{\mu_n}y_n\| \\ &= \left\| \frac{1}{1 - \beta_n} (Jy_n - \beta_n Jx_n) \right. \\ &\quad \left. - \frac{1}{1 - \alpha_n} (Jz_n - \alpha_n Jx_n) \right\| \\ &= \left\| Jy_n + \frac{\beta_n}{1 - \beta_n} (Jy_n - Jx_n) \right. \\ &\quad \left. - \left(Jz_n + \frac{\alpha_n}{1 - \alpha_n} (Jz_n - Jx_n) \right) \right\| \\ &\leq \|Jy_n - Jx_n\| + \|Jx_n - Jz_n\| \\ &\quad + \frac{\beta_n}{1 - \beta_n} \|Jy_n - Jx_n\| + \frac{\alpha_n}{1 - \alpha_n} \|Jz_n - Jx_n\| \\ &= \frac{1}{1 - \alpha_n} \|Jz_n - Jx_n\| + \frac{1}{1 - \beta_n} \|Jy_n - Jx_n\|. \end{aligned} \quad (64)$$

Combining (53), (62), and (64), we get

$$\lim_{n \rightarrow \infty} \|JT_{\mu_n}x_n - JT_{\mu_n}y_n\| = 0. \quad (65)$$

Since J^{-1} is uniformly norm-to-norm continuous, we have

$$\lim_{n \rightarrow \infty} \|T_{\mu_n}x_n - T_{\mu_n}y_n\| = 0. \quad (66)$$

Since

$$\begin{aligned} \|x_n - T_{\mu_n}x_n\| &\leq \|x_n - z_n\| + \|z_n - T_{\mu_n}y_n\| \\ &\quad + \|T_{\mu_n}y_n - T_{\mu_n}x_n\|, \end{aligned} \quad (67)$$

therefore, by (58), (61), (66), and (67), we obtain

$$\lim_{n \rightarrow \infty} \|x_n - T_{\mu_n}x_n\| = 0. \quad (68)$$

From Lemma 7, we have $x_n \in F(\mathfrak{S})$. Since $F(\mathfrak{S})$ is closed and $\lim_{n \rightarrow \infty} x_n = v$, we have $v \in F(\mathfrak{S})$, where $v = \lim_{n \rightarrow \infty} \Pi_{F(\mathfrak{S})} x_n$. $\square$

If we set $\beta_n = 1$, then the iteration (46) reduces modified Mann type. Hence we obtain the following corollary.

Corollary 10. *Let S be a left reversible semigroup and let $\mathfrak{S} = \{T(s) : s \in S\}$ be a representation of S as relatively nonexpansive mappings from a nonempty, closed, and convex subset C of a uniformly convex and uniformly smooth Banach space E into itself. Let X be a subspace of $l^\infty(S)$ and let $\{\mu_n\}$ be an asymptotically left invariant sequence of means on X . Let $\{\alpha_n\}$ be a sequence of real number such that $\alpha_n \in (0, 1)$ and $\lim_{n \rightarrow \infty} \alpha_n = 0$. Let $\{x_n\}$ be a sequence generated by the following algorithm:*

$$\begin{aligned} x_0 &\in C, \quad \text{chosen arbitrarily,} \\ x_{n+1} &= \Pi_C J^{-1} (\alpha_n Jx_n + (1 - \alpha_n)JT_{\mu_n}x_n), \quad \forall n \geq 0. \end{aligned} \quad (69)$$

If the interior of $F(\mathfrak{S})$ is nonempty, then $\{x_n\}$ converges strongly to some common fixed point $F(\mathfrak{S})$.

In a Hilbert space, J is the identity operator. Theorems 8 and 9 reduce to the following.

Corollary 11. *Let S be a left reversible semigroup and let $\mathfrak{S} = \{T(s) : s \in S\}$ be a representation of S as relatively nonexpansive mappings from a nonempty, closed, and convex subset C of a Hilbert space H into itself. Let X be a subspace of $l^\infty(S)$ and let $\{\mu_n\}$ be an asymptotically left invariant sequence of means on X . Let $\{\alpha_n\}$ be a sequence in $(0, 1)$ such that $\lim_{n \rightarrow \infty} \alpha_n = 0$. Let $\{x_n\}$ be a sequence generated by the following algorithm:*

$$\begin{aligned} x_0 &\in C, \quad \text{chosen arbitrarily,} \\ x_{n+1} &= P_C(\alpha_n x_0 + (1 - \alpha_n) T_{\mu_n} x_n), \quad \forall n \geq 0. \end{aligned} \quad (70)$$

If the interior of $F(\mathfrak{S})$ is nonempty, then $\{x_n\}$ converges strongly to some common fixed point $F(\mathfrak{S})$, where P_C is a metric projection.

Corollary 12. *Let S be a left reversible semigroup and let $\mathfrak{S} = \{T(s) : s \in S\}$ be a representation of S as relatively nonexpansive mappings from a nonempty, closed, and convex subset C of a Hilbert space H into itself. Let X be a subspace of $l^\infty(S)$ and let $\{\mu_n\}$ be an asymptotically left invariant sequence of means on X . Let $\{\alpha_n\}$ and $\{\beta_n\}$ be sequences of real numbers such that $\alpha_n, \beta_n \in (0, 1)$ and $\lim_{n \rightarrow \infty} \alpha_n = 0$, $\lim_{n \rightarrow \infty} \beta_n = 1$. Let $\{x_n\}$ be a sequence generated by the following algorithm:*

$$\begin{aligned} x_0 &\in C, \quad \text{chosen arbitrarily,} \\ y_n &= \beta_n x_n + (1 - \beta_n) T_{\mu_n} x_n, \\ x_{n+1} &= P_C(\alpha_n x_n + (1 - \alpha_n) T_{\mu_n} y_n), \quad \forall n \geq 0. \end{aligned} \quad (71)$$

If the interior of $F(\mathfrak{S})$ is nonempty, then $\{x_n\}$ converges strongly to some common fixed point $F(\mathfrak{S})$, where P_C is a metric projection.

Conflict of Interests

The author declares that there is no conflict of interests regarding the publication of this paper.

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