Research Article
On a Class of $q$-Bernoulli, $q$-Euler, and $q$-Genocchi Polynomials

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The main purpose of this paper is to introduce and investigate a class of $q$-Bernoulli, $q$-Euler, and $q$-Genocchi polynomials. The $q$-analogues of well-known formulas are derived. In addition, the $q$-analogue of the Srivastava-Pintér theorem is obtained. Some new identities, involving $q$-polynomials, are proved.

1. Introduction

Throughout this paper, we always make use of the classical definition of quantum concepts as follows.

The $q$-shifted factorial is defined by

\[ (a;q)_0 = 1, \quad (a;q)_n = \prod_{j=0}^{n-1} (1 - q^j a), \quad n \in \mathbb{N}, \]

\[ (a;q)_\infty = \prod_{j=0}^{\infty} (1 - q^j a), \quad |q| < 1, \quad a \in \mathbb{C}. \]  

It is known that

\[ (a;q)_n = \sum_{k=0}^{n} \binom{n}{k}_q q^{(1/2)k(k-1)} (-1)^k a^k. \]  

The $q$-numbers and $q$-factorial are defined by

\[ [a]_q = \frac{1 - q^a}{1 - q}, \quad (q \neq 1, \quad a \in \mathbb{C}); \]

\[ [0]_q = 1, \quad [n]_q! = [n]_q [n-1]_q! . \]  

The $q$-polynomial coefficient is defined by

\[ \binom{n}{k}_q = \frac{(q;q)_n}{(q;q)_{n-k}(q;q)_k}, \quad (k \leq n, \quad n \in \mathbb{N}). \]  

In the standard approach to the $q$-calculus two exponential functions are used, these $q$-exponential functions and improved type $q$-exponential function (see [1]) are defined as follows:

\[ e_q(z) = \sum_{n=0}^{\infty} \frac{z^n}{[n]_q!} = \prod_{k=0}^{\infty} \frac{1}{(1 - (1 - q)q^k z)}, \quad 0 < |q| < 1, \quad |z| < \frac{1}{1-q}. \]  

\[ E_q(z) = e_{1/q}(z) = \sum_{n=0}^{\infty} \frac{q^{(1/2)n(n-1)} z^n}{[n]_{1/q}!} = \prod_{k=0}^{\infty} \frac{1}{(1 + (1 - q)q^k z)}, \quad 0 < |q| < 1, \quad z \in \mathbb{C}, \]

\[ E_{1/q}(z) = e_q \left( \frac{z}{2} \right) E_q \left( \frac{z}{2} \right) = \sum_{n=0}^{\infty} \frac{(-1,q)_n}{2^n [n]_{1/q}!} z^n = \prod_{k=0}^{\infty} \frac{1}{(1 - (1 - q)q^k z/2)}, \quad 0 < q < 1, \quad |z| < \frac{2}{1-q}. \]  

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The form of improved type of $q$-exponential function $\mathcal{E}_q(z)$ motivated us to define a new $q$-addition and $q$-subtraction as
\[
(x\oplus_q y)^n := \sum_{k=0}^n \binom{n}{k} \frac{(-1)^k}{2^n} (1-q)^{n-k} x^k y^{n-k},
\]
for $n = 0, 1, 2, \ldots$, 
\[
(x\ominus_q y)^n := \sum_{k=0}^n \binom{n}{k} \frac{(-1)^k}{2^n} (1-q)^{n-k} x^k (y^{-1})^{n-k},
\]
for $n = 0, 1, 2, \ldots$.

It follows that
\[
\mathcal{E}_q(tx) \mathcal{E}_q(ty) = \left( \sum_{n=0}^{\infty} (x \oplus_q y)^n \frac{t^n}{[n]_q!} \right)^2,
\]
for $|t| < 2\pi$.

The $q$-Bernoulli numbers $\{B_m\}_{m \geq 0}$ are rational numbers in a sequence defined by the binomial recursion formula:
\[
\sum_{k=0}^m \binom{m}{k} B_k - B_m = \begin{cases} 1, & m = 1, \\ 0, & m > 1, \end{cases}
\]
or equivalently, the generating function
\[
\sum_{k=0}^{\infty} B_k \frac{t^k}{k!} = \frac{t}{e^t - 1}.
\]

$q$-Analagues of the Bernoulli numbers were first studied by Carlitz [2] in the middle of the last century when he introduced a new sequence $\{\beta_m\}_{m \geq 0}$:
\[
\sum_{k=0}^m \binom{m}{k} \beta_k q^{k+1} - \beta_m = \begin{cases} 1, & m = 1, \\ 0, & m > 1. \end{cases}
\]

Here and in the remainder of the paper, for the parameter $q$ we make the assumption that $|q| < 1$. Clearly we recover (8) if we let $q \to 1$ in (10). The $q$-binomial formula is known as
\[
(1 - a)_q^n = (a; q)_n = \prod_{j=0}^{n-1} (1 - q^j a).
\]

The above $q$-standard notation can be found in [3].

Carlitz has introduced the $q$-Bernoulli numbers and polynomials in [2]. Srivastava and Pintér proved some relations and theorems between the Bernoulli polynomials and Euler polynomials in [4]. They also gave some generalizations of these polynomials. In [4–16], the authors investigated some properties of the $q$-Euler polynomials and $q$-Genocchi polynomials. They gave some recurrence relations. In [17], Cenkci et al. gave the $q$-extension of Genocchi numbers in a different manner. In [18], Kim gave a new concept for the $q$-Genocchi numbers and polynomials. In [19], Simsek et al. investigated the $q$-Genocchi zeta function and $l$-function by using generating functions and Mellin transformation. There are numerous recent studies on this subject by, among many other authors, Cigler [20], Cenkci et al. [17, 21], Choi et al. [22], Cheon [23], Luo and Srivastava [8–10], Srivastava et al. [4, 24], Nalci and Pashaev [25] Gaboury and Kurt [26], Kim et al. [27], and Kurt [28].

We first give the definitions of the $q$-numbers and $q$-polynomials. It should be mentioned that the definition of $q$-Bernoulli numbers in Definition 1 can be found in [25].

**Definition 1.** Let $q \in \mathbb{C}, 0 < |q| < 1$. The $q$-Bernoulli numbers $b_{n,q}$ and polynomials $\Phi_{n,q}(x, y)$ are defined by means of the generating functions:
\[
\mathcal{B}(t) := \frac{t e_q (t/2) - e_q (-t/2)}{e_q (t/2) - e_q (-t/2)} = \frac{t}{\mathcal{E}_q(t) - 1},
\]
\[
= \sum_{n=0}^{\infty} b_{n,q} \frac{t^n}{[n]_q!}, \quad |t| < 2\pi,
\]
\[
\frac{t}{\mathcal{E}_q(t) - 1} \mathcal{E}_q(tx) \mathcal{E}_q(ty) = \sum_{n=0}^{\infty} \Phi_{n,q}(x, y) \frac{t^n}{[n]_q!}, \quad |t| < 2\pi.
\]

**Definition 2.** Let $q \in \mathbb{C}, 0 < |q| < 1$. The $q$-Euler numbers $e_{n,q}$ and polynomials $\mathcal{E}_{n,q}(x, y)$ are defined by means of the generating functions:
\[
\mathcal{E}(t) := \frac{2 e_q (t/2) - e_q (-t/2)}{e_q (t/2) + e_q (-t/2)} = \frac{2}{\mathcal{E}_q(t) + 1},
\]
\[
= \sum_{n=0}^{\infty} e_{n,q} \frac{t^n}{[n]_q!}, \quad |t| < \pi,
\]
\[
\frac{2}{\mathcal{E}_q(t) + 1} \mathcal{E}_q(tx) \mathcal{E}_q(ty) = \sum_{n=0}^{\infty} \Phi_{n,q}(x, y) \frac{t^n}{[n]_q!}, \quad |t| < \pi.
\]

**Definition 3.** Let $q \in \mathbb{C}, 0 < |q| < 1$. The $q$-Genocchi numbers $g_{n,q}$ and polynomials $\mathcal{G}_{n,q}(x, y)$ are defined by means of the generating functions:
\[
\mathcal{G}(t) := \frac{2 e_q (t/2) - e_q (-t/2)}{e_q (t/2) + e_q (-t/2)} = \frac{2}{\mathcal{E}_q(t) + 1},
\]
\[
= \sum_{n=0}^{\infty} g_{n,q} \frac{t^n}{[n]_q!}, \quad |t| < \pi,
\]
\[
\frac{2}{\mathcal{E}_q(t) + 1} \mathcal{E}_q(tx) \mathcal{E}_q(ty) = \sum_{n=0}^{\infty} \Phi_{n,q}(x, y) \frac{t^n}{[n]_q!}, \quad |t| < \pi.
\]
Note that Cigler [20] defined $q$-Genocchi numbers as
\[ t e_q(t) + e_q(-t) \]
Then comparing $g_{n,q}$ with $g_{n,q}$, we see that
\[ (-1)^{n-1} 2^{n-1} g_{2n+2,q} = (-q; q)_{2n+1} g_{2n+2,q}. \] (16)

**Definition 4.** Let $q \in \mathbb{C}, 0 < |q| < 1$. The $q$-tangent numbers $T_{n,q}$ are defined by means of the generating functions:
\[ \tan_q(t) = \frac{i}{t} \ln(1 - q e^{-it}) = \sum_{n=1}^{\infty} T_{n,q} \frac{(it)^n}{n!}. \]

It is obvious that, by letting $q$ tend to 1 from the left side, we lead to the classical definition of these polynomials:
\[ b_{n,q} := B_{n,q}(0), \quad \lim_{q \to 1} B_{n,q}(x) = B_n(x), \]
\[ \lim_{q \to 1} B_{n,q}(x, y) = B_n(x + y), \quad \lim_{q \to 1} B_{n,q} = B_n, \]
\[ e_{n,q} := E_{n,q}(0), \quad \lim_{q \to 1} E_{n,q}(x) = E_n(x), \]
\[ \lim_{q \to 1} E_{n,q}(x, y) = E_n(x + y), \quad \lim_{q \to 1} E_{n,q} = E_n, \]
\[ g_{n,q} := G_{n,q}(0), \quad \lim_{q \to 1} G_{n,q}(x) = G_n(x), \]
\[ \lim_{q \to 1} G_{n,q}(x, y) = G_n(x + y), \quad \lim_{q \to 1} G_{n,q} = G_n. \] (18)

Here $B_n(x), E_n(x), \text{and } G_n(x)$ denote the classical Bernoulli, Euler, and Genocchi polynomials, respectively, which are defined by
\[ \frac{t}{e^t - 1} e^{tx} = \sum_{n=0}^{\infty} B_n(x) \frac{t^n}{n!}, \quad \frac{2t}{e^t + 1} e^{tx} = \sum_{n=0}^{\infty} E_n(x) \frac{t^n}{n!}, \]
\[ \frac{2t}{e^t + 1} e^{tx} = \sum_{n=0}^{\infty} G_n(x) \frac{t^n}{n!}. \] (19)

The aim of the present paper is to obtain some results for the above newly defined $q$-polynomials. It should be mentioned that $q$-Bernoulli and $q$-Euler polynomials in our definitions are polynomials of $x$ and $y$ when $y = 0$, they are polynomials of $x$. First advantage of this approach is that for $q \to 1^-$, $B_{n,q}(x, y) (E_{n,q}(x, y), G_{n,q}(x, y))$ becomes the classical Bernoulli $B_n(x, y)$ (Euler $E_n(x, y)$, Genocchi $G_n(x, y)$) polynomial and we may obtain the $q$-analogue of well-known results, for example, Srivastava and Pintér [11], Cheon [23], and so forth. Second advantage is that, similar to the classical case, odd numbered terms of the Bernoulli numbers $b_{n,q}$ and the Genocchi numbers $g_{n,q}$ are zero, and even numbered terms of the Euler numbers $e_{n,q}$ are zero.

### 2. Preliminary Results

In this section we will provide some basic formulae for the $q$-Bernoulli, $q$-Euler, and $q$-Genocchi numbers and polynomials in order to obtain the main results of this paper in the next section.

**Lemma 5.** The $q$-Bernoulli numbers $b_{n,q}$ satisfy the following $q$-binomial recurrence:
\[ \sum_{k=0}^{n} \binom{n}{k} (-1, q)_{n-k} b_{k,q} - b_{n,q} = \begin{cases} 1, & n = 1, \\ 0, & n > 1. \end{cases} \] (20)

**Proof.** By a simple multiplication of (8) we see that
\[ \overset{\overline{B}}{B}(t) \overset{\overline{E}}{E}_q(t) = t + \overset{\overline{B}}{B}(t). \] (21)

So
\[ \sum_{n=0}^{\infty} \sum_{k=0}^{n} \binom{n}{k} (-1, q)_{n-k} b_{k,q} t^n = t + \sum_{n=0}^{\infty} b_{n,q} t^n. \] (22)

The statement follows by comparing $t^m$ coefficients. \( \square \)

We use this formula to calculate the first few $b_{k,q}$:
\[ b_{0,q} = 1, \]
\[ b_{1,q} = -\frac{1}{2}, \]
\[ b_{2,q} = \frac{1}{2} \frac{q(q+1)}{4 q^2 + q + 1} = \frac{q[2]_q}{4[3]_q}, \] (23)
\[ b_{3,q} = 0. \]

The similar property can be proved for $q$-Euler numbers
\[ \sum_{k=0}^{m} \binom{m}{k} (-1, q)_{m-k} e_{k,q} + e_{m,q} = \begin{cases} 2, & m = 0, \\ 0, & m > 0. \end{cases} \] (24)

and $q$-Genocchi numbers
\[ \sum_{k=0}^{m} \binom{m}{k} (-1, q)_{m-k} g_{k,q} + g_{m,q} = \begin{cases} 2, & m = 1, \\ 0, & m > 1. \end{cases} \] (25)

Using the above recurrence formulae we calculate the first few $e_{n,q}$ and $g_{n,q}$ terms as well:
\[ e_{0,q} = 1, \quad g_{0,q} = 0, \]
\[ e_{1,q} = -\frac{1}{2}, \quad g_{1,q} = 1, \]
\[ e_{2,q} = 0, \quad g_{2,q} = -\frac{[2]_q}{2} = -\frac{q+1}{2}, \] (26)
\[ e_{3,q} = \frac{[3]_q [2]_q - [4]_q}{8} = \frac{q(1+q)}{8}, \quad g_{3,q} = 0. \]
Remark 6. The first advantage of the new $q$-numbers $b_{k,q}, e_{k,q},$ and $g_{k,q}$ is that similar to classical case odd numbered terms of the Bernoulli numbers $b_{k,q}$ and the Genocchi numbers $g_{k,q}$ are zero, and even numbered terms of the Euler numbers $e_{k,q}$ are zero.

Next lemma gives the relationship between $q$-Genocchi numbers and $q$-Tangent numbers.

**Lemma 7.** For any $n \in \mathbb{N}$, we have
\[
\mathbf{T}_{2n+1, q} = g_{2n+2, q} \frac{(-1)^{k-1} 2^{2n+1}}{[2n + 2]_q}.
\] (27)

**Proof.** First we recall the definition of $q$-trigonometric functions:
\[
\begin{align*}
\cos_q t &= \frac{e_q(it) + e_q(-it)}{2}, & \sin_q t &= \frac{e_q(it) - e_q(-it)}{2i}, \\
i \tan_q t &= \frac{e_q(it) - e_q(-it)}{i e_q(it) + e_q(-it)}, & \cot_q t &= \frac{i e_q(it) + e_q(-it)}{e_q(it) - e_q(-it)}.
\end{align*}
\] (28)

Now by choosing $z = 2it$ in $\mathbf{B}(z)$, we get
\[
\mathbf{B}(2it) = \frac{2it}{e_q(2it)} - 1 = \frac{te_q(-it)}{\sin_q t} = \sum_{n=0}^{\infty} b_{n,q} \frac{(2it)^n}{[n]_q!}.
\] (29)

It follows that
\[
\mathbf{B}(2it) = \frac{te_q(-it)}{\sin_q t} = \frac{t}{\sin_q t} (\cos_q t - i \sin_q t) = t \cot_q t - it
\]
\[
= b_{0,q} + 2itb_{1,q} + \sum_{n=2}^{\infty} b_{n,q} \frac{(2it)^n}{[n]_q!}
\]
\[
= 1 - it + \sum_{n=2}^{\infty} b_{n,q} \frac{(2it)^n}{[n]_q!}.
\] (30)

By choosing $z = 2it$ in $\mathbf{B}(z)$, we get
\[
\mathbf{B}(2it) = \frac{4it}{e_q(2it)} + 1 = \frac{2ite_q(-it)}{\cos_q t} = \sum_{n=0}^{\infty} g_{n,q} \frac{(2it)^n}{[n]_q!},
\]
\[
\mathbf{B}(2it) = \frac{4it}{e_q(2it)} - 1 = \frac{2ite_q(-it)}{\cos_q t} = \sum_{n=0}^{\infty} b_{n,q} \frac{(2it)^n}{[n]_q!},
\]
\[
= 2i + 2t \tan_q t = g_{0,q} + 2itg_{1,q} + \sum_{n=2}^{\infty} g_{n,q} \frac{(2it)^n}{[n]_q!}
\]
\[
= 2i + \sum_{n=2}^{\infty} g_{n,q} \frac{(2it)^n}{[n]_q!}.
\] (32)

It follows that
\[
2t \tan_q t = \sum_{n=1}^{\infty} g_{2n,q} \frac{(2it)^{2n}}{[2n]_q!},
\]
\[
tan_q t = \sum_{n=1}^{\infty} g_{2n,q} \frac{(-1)^n(2it)^{2n-1}}{[2n]_q!}
\]
\[
= \sum_{n=1}^{\infty} g_{2n,q} \frac{(2it)^{2n-1}}{[2n]_q!} = \sum_{n=2}^{\infty} g_{2n+2,q} \frac{(2it)^{2n+1}}{[2n + 2]_q!}.
\] (33)

Thus
\[
tan_q t = -it \tan_q (it) = -i \sum_{n=1}^{\infty} g_{2n,q} \frac{(-1)^n(2it)^{2n-1}}{[2n]_q!} = \sum_{n=1}^{\infty} g_{2n+2,q} \frac{(2it)^{2n+1}}{[2n + 2]_q!}.
\] (34)

The following result is a $q$-analogue of the addition theorem, for the classical Bernoulli, Euler, and Genocchi polynomials.

**Lemma 8 (addition theorems).** For all $x, y \in \mathbb{C}$ we have
\[
\mathbf{B}_{n,q}(x, y) = \sum_{k=0}^{n} \binom{n}{k} b_{k,q}(x \oplus_q y)^{n-k},
\]
\[
\mathbf{B}_{n,q}(x, y) = \sum_{k=0}^{n} \binom{n}{k} \frac{(1,q)^{n-k}}{2^{n-k} [2n]_q} b_{k,q}(x)^{n-k}.
\]
\[ \mathcal{G}_{n,q}(x,y) = \sum_{k=0}^{n} \binom{n}{k} b_{k}(x) \mathcal{G}_{k,q}(y)^{n-k}, \]
\[ \mathcal{G}_{n,q}(x,y) = \sum_{k=0}^{n} \binom{n}{k} \frac{(-1,q)^{n-k}}{2^{n-k}} \mathcal{G}_{k,q}(y)^{n-k}, \]
\[ \mathcal{G}_{n,q}(x,y) = \sum_{k=0}^{n} \binom{n}{k} g_{k,q}(x) y^{n-k}, \]
\[ \mathcal{G}_{n,q}(x,y) = \sum_{k=0}^{n} \binom{n}{k} (-1,q)^{n-k} g_{k,q}(y)^{n-k}. \]

(35)

**Proof.** We prove only the first formula. It is a consequence of the following identity:
\[
\sum_{n=0}^{\infty} \binom{n}{k} \frac{t^n}{[n]_q!} = \frac{t}{[x]_q(t) - 1} \mathcal{E}_{q}(tx) \mathcal{E}_{q}(ty)
\]
\[
= \sum_{n=0}^{\infty} \binom{n}{k} \frac{t^n}{[n]_q!} \sum_{x=0}^{\infty} \binom{n}{x} b_{x,q} x^{n-x} t^n
\]
\[
= \sum_{n=0}^{\infty} \binom{n}{k} \frac{t^n}{[n]_q!} \sum_{x=0}^{\infty} \binom{n}{x} b_{x,q} x^{n-x} t^n.
\]

(36)

In particular, setting \( y = 0 \) in (35), we get the following formulae for \( q \)-Bernoulli, \( q \)-Euler and \( q \)-Genocchi polynomials, respectively:
\[ \mathcal{G}_{n,q}(x) = \sum_{k=0}^{n} \binom{n}{k} \frac{(-1,q)^{n-k}}{2^{n-k}} b_{k,q} x^{n-k}, \]
\[ \mathcal{G}_{n,q}(x) = \sum_{k=0}^{n} \binom{n}{k} \frac{(-1,q)^{n-k}}{2^{n-k}} b_{k,q} x^{n-k}, \]
\[ \mathcal{G}_{n,q}(x) = \sum_{k=0}^{n} \binom{n}{k} \frac{(-1,q)^{n-k}}{2^{n-k}} g_{k,q} x^{n-k}. \]

(37)

(38)

Setting \( y = 1 \) in (35), we get
\[ \mathcal{G}_{n,q}(x,1) = \sum_{k=0}^{n} \binom{n}{k} \frac{(-1,q)^{n-k}}{2^{n-k}} g_{k,q} x^{n-k}, \]
\[ \mathcal{G}_{n,q}(x,1) = \sum_{k=0}^{n} \binom{n}{k} \frac{(-1,q)^{n-k}}{2^{n-k}} g_{k,q} x^{n-k}, \]
\[ \mathcal{G}_{n,q}(x,1) = \sum_{k=0}^{n} \binom{n}{k} \frac{(-1,q)^{n-k}}{2^{n-k}} \mathcal{G}_{k,q}(x). \]

(39)

Clearly (39) is \( q \)-analogues of
\[ B_{n} (x+1) = \sum_{k=0}^{n} \binom{n}{k} B_{k}(x), \]
\[ E_{n} (x+1) = \sum_{k=0}^{n} \binom{n}{k} E_{k}(x), \]
\[ G_{n} (x+1) = \sum_{k=0}^{n} \binom{n}{k} G_{k}(x), \]

(40)

respectively.

**Lemma 9.** The odd coefficients of the \( q \)-Bernoulli numbers, except the first one, are zero. That means \( b_{n,q} = 0 \) where \( n = 2r + 1 \ (r \in \mathbb{N}) \).

**Proof.** It follows from the fact that the function
\[ f(t) = \sum_{n=0}^{\infty} \frac{b_{n,q}}{[n]_q!} t^n - b_{1,q} t \]
\[ = \frac{t}{[x]_q(t) - 1} + t = \frac{t}{[x]_q(t) - 1} \left( \frac{[x]_q(t) + 1}{2} \right) . \]

(41)

By using \( q \)-derivative we obtain the next lemma.

**Lemma 10.** One has
\[ D_{q,x} \mathcal{G}_{n,q}(x) = \frac{[n]_q}{2} \mathcal{G}_{n-1,q}(x) + \frac{[n]_q}{2} \mathcal{G}_{n-1,q}(qx), \]
\[ D_{q,x} \mathcal{G}_{n,q}(x) = \frac{[n]_q}{2} \mathcal{G}_{n-1,q}(x) + \frac{[n]_q}{2} \mathcal{G}_{n-1,q}(qx), \]
\[ D_{q,x} \mathcal{G}_{n,q}(x) = \frac{[n]_q}{2} \mathcal{G}_{n-1,q}(x) + \frac{[n]_q}{2} \mathcal{G}_{n-1,q}(qx). \]

(42)

**Lemma 11** (difference equations). One has
\[ \mathcal{G}_{n,q}(x,1) - \mathcal{G}_{n,q}(x) = \frac{(-1;q)_{n-1}}{2^{n-1}} [n]_q x^{n-1}, \ (n \geq 1), \]
\[ \mathcal{G}_{n,q}(x,1) + \mathcal{G}_{n,q}(x) = 2 \frac{(-1;q)_{n-1}}{2^{n}} x^{n}, \ (n \geq 0), \]
\[ \mathcal{G}_{n,q}(x,1) + \mathcal{G}_{n,q}(x) = 2 \frac{(-1;q)_{n-1}}{2^{n}} [n]_q x^{n-1}, \ (n \geq 1). \]

(43)

(44)

(45)

**Proof.** We prove the identity for the \( q \)-Bernoulli polynomials. From the identity
\[ t \frac{[x]_q(t)}{[x]_q(t) - 1} \mathcal{E}_{q}(tx) = t \mathcal{E}_{q}(tx) + \frac{t}{[x]_q(t) - 1} \mathcal{E}_{q}(tx), \]
\[ \frac{[x]_q(t)}{[x]_q(t) - 1} \frac{t}{[x]_q(t) - 1} \mathcal{E}_{q}(tx) = \frac{[x]_q(t)}{[x]_q(t) - 1} \mathcal{E}_{q}(tx) + \frac{t}{[x]_q(t) - 1} \mathcal{E}_{q}(tx), \]

(46)

it follows that
\[ \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{(-1,q)^{n-k}}{2^{n-k}} \mathcal{G}_{k,q}(x) \frac{t^n}{[n]_q!} \]
\[ = \sum_{n=0}^{\infty} \frac{(-1,q)^{n}}{2^n x^n} \frac{t^{n+1}}{[n]_q!} \]
\[ = \sum_{n=0}^{\infty} \mathcal{G}_{n,q}(x) \frac{t^n}{[n]_q!} . \]

(47)
From (43) and (37), (44) and (38), we obtain the following formulae.

**Lemma 12.** One has

\[ x^n = \frac{2^n}{(-1; q)_n q^{n+1}} \sum_{k=0}^{n} \frac{(-1)^k q^{n-k}}{2^{n-k}} \mathfrak{B}_{k,q}(x), \]

\[ x^n = \frac{2^{n-1}}{(-1; q)_n q^{n+1}} \sum_{k=0}^{n} \frac{(-1)^k q^{n-k}}{2^{n-k}} \mathfrak{B}_{k,q}(x) + \mathfrak{G}_{n,q}(x), \]

\[ x^n = \frac{2^{n-1}}{(-1; q)_n q^{n+1}} \left( \sum_{k=0}^{n} \frac{(-1)^k q^{n-k}}{2^{n-k}} \mathfrak{B}_{k,q}(x) + \mathfrak{G}_{n,q}(x) \right). \]

The above formulae are analogues of the following familiar expansions:

\[ x^n = \frac{1}{n+1} \sum_{k=0}^{n} \binom{n+1}{k} B_k(x), \]

\[ x^n = \frac{1}{2} \sum_{k=0}^{n} \binom{n}{k} E_k(x) + E_n(x), \]

\[ x^n = \frac{1}{2} \left( \sum_{k=0}^{n} \binom{n+1}{k} E_k(x) + E_{n+1}(x) \right), \]

respectively.

**Lemma 13.** The following identities hold true:

\[ \sum_{k=0}^{n} \binom{n}{k} \frac{(-1)^k q^{n-k}}{2^{n-k}} \mathfrak{B}_{k,q}(x,y) = \mathfrak{G}_{n,q}(x,y) \]

\[ = \left[ n \right]_q (x \oplus_q y)^{n-1}, \]

\[ \sum_{k=0}^{n} \binom{n}{k} \frac{(-1)^k q^{n-k}}{2^{n-k}} \left( \mathfrak{B}_{k,q}(x,y) + \mathfrak{G}_{n,q}(x,y) \right) = 2(x \oplus_q y)^n, \]

\[ \sum_{k=0}^{n} \binom{n}{k} \frac{(-1)^k q^{n-k}}{2^{n-k}} \mathfrak{B}_{k,q}(x,y) + \mathfrak{G}_{n,q}(x,y) = 2[n]_q (x \oplus_q y)^{n-1}. \]

**Proof.** We prove the identity for the -Bernoulli polynomials. From the identity

\[ t \mathfrak{E}_q(t) - \mathfrak{E}_q(t) \mathfrak{E}_q(t) = t \mathfrak{E}_q(t) \mathfrak{E}_q(t) + \frac{t}{\mathfrak{E}_q(t) - 1} \mathfrak{E}_q(t) \mathfrak{E}_q(t), \]

it follows that

\[ \sum_{n=0}^{\infty} \frac{n}{k} \binom{n}{k} \frac{(-1)^k q^{n-k} \mathfrak{B}_{k,q}(x,y)}{2^{n-k}} t^n = \frac{\sum_{n=0}^{\infty} \binom{n}{k} \frac{(-1)^k q^{n-k} \mathfrak{B}_{k,q}(x,y)}{2^n} x^k y^{n-k} t^{n-1}}{[n]_q!} \]

\[ + \sum_{n=0}^{\infty} \binom{n}{k} \frac{(-1)^k q^{n-k} \mathfrak{B}_{k,q}(x,y)}{2^n} x^k y^{n-k} t^{n-1} \]

\[ = \sum_{n=0}^{\infty} \frac{n}{k} \binom{n}{k} \frac{(-1)^k q^{n-k} \mathfrak{B}_{k,q}(x,y)}{2^{n-k}} x^k y^{n-k} t^{n-1} \]

\[ + \sum_{n=0}^{\infty} \binom{n}{k} \frac{(-1)^k q^{n-k} \mathfrak{B}_{k,q}(x,y)}{2^n} x^k y^{n-k} t^{n-1}. \]

\[ \square \]

3. Some New Formulae

The classical Cayley transformation \( z \rightarrow \text{Cay}(z,a) := (1 + az)/(1 - az) \) motivated us to derive the formula for \( \mathfrak{B}_q(x,y) \). In addition, by substituting \( \text{Cay}(z,(q-1)/2) \) in the generating formula we have

\[ \mathfrak{B}_q(x,y) \mathfrak{B}_q(t) = \left( \mathfrak{B}_q(t) q - \mathfrak{B}_q(t) \left( 1 + (1 - q) \frac{t}{2} \right) \right) \]

\[ \times \frac{1}{1 - q} \times \frac{2}{\mathfrak{E}_q(t) + 1}. \]

The right hand side can be presented by \( q \)-Euler numbers. Now equating coefficients of \( t^n \) we get the following identity. In the case that \( n = 0 \), we find the first improved \( q \)-Euler number which is exactly \( 1 \).

**Proposition 14.** For all \( n \geq 1 \),

\[ \sum_{k=0}^{n} \binom{n}{k} \frac{(-1)^k q^{n-k} \mathfrak{B}_{k,q}(x,y)}{2^{n-k}} \mathfrak{B}_{n-k,q}(x,y) = -q \sum_{k=0}^{n} \binom{n}{k} \frac{(-1)^k q^{n-k} \mathfrak{B}_{k,q}(x,y)}{2^{n-k}} \mathfrak{B}_{n-k,q}(x,y) \]

\[ = -q \sum_{k=0}^{n} \binom{n}{k} \frac{(-1)^k q^{n-k-1} \mathfrak{B}_{k,q}(x,y)}{2^{n-k-1}} \mathfrak{B}_{n-k,q}(x,y) \]

\[ \times 1 \frac{2}{\mathfrak{E}_q(t) + 1}. \]

Let us take a \( q \)-derivative from the generating function, after simplifying the equation, by knowing the quotient rule for quantum derivative, and also using

\[ \mathfrak{E}_q(t) = \frac{1 - (1 - q) t/2}{1 + (1 - q) t/2} \]

\[ D_q \left( \mathfrak{E}_q(t) \right) = \frac{\mathfrak{E}_q(t) + \mathfrak{E}_q(t)}{2}, \]

one has

\[ \mathfrak{B}_q(t) \mathfrak{B}_q(t) = \frac{2 + (1 - q) t}{2 \mathfrak{E}_q(t) (q - 1)} \left( \mathfrak{B}_q(t) - \mathfrak{B}_q(t) \right). \]

It is clear that \( \mathfrak{E}_q^{-1}(t) = \mathfrak{E}_q(t) \). Now, by equating coefficients of \( t^n \) we obtain the following identity.
Proposition 15. For all \( n \geq 1 \),

\[
\sum_{k=0}^{2n} \binom{2n}{k} \mathbf{B}_{k,q} \mathbf{B}_{2n-k,q} q^k
= -q \sum_{k=0}^{2n} \binom{2n}{k} \frac{(-1)^k q^{2n-k}}{2^{2n-k}} \mathbf{B}_{k,q} [k-1]_q (-1)^k
\]
\[+ \frac{q}{2} \sum_{k=0}^{2n-1} \binom{2n-1}{k} \frac{(-1)^k q^{2n-1-k}}{2^{2n-1-k}} \mathbf{B}_{k,q} \times [k-1]_q (-1)^k,
\]
\[
\sum_{k=0}^{2n+1} \binom{2n+1}{k} \mathbf{B}_{k,q} \mathbf{B}_{2n-k+1,q} q^k
= q \sum_{k=0}^{2n+1} \binom{2n+1}{k} \frac{(-1)^k q^{2n+1-k}}{2^{2n+1-k}} \mathbf{B}_{k,q} [k-1]_q (-1)^k
\]
\[- \frac{q}{2} \sum_{k=0}^{2n} \binom{2n}{k} \frac{(-1)^k q^{2n-k}}{2^{2n-k}} \mathbf{B}_{k,q} [k-1]_q (-1)^k.
\]

We may also derive a differential equation for \( \mathbf{B}_q(t) \). If we differentiate both sides of the generating function with respect to \( t \), after a little calculation we find that

\[
\frac{\partial}{\partial t} \mathbf{B}_q(t) = \mathbf{B}_q(t) \left( \frac{1}{t} - \frac{1}{\mathbf{B}_q(t)} \right) \left( \sum_{k=0}^{\infty} \frac{4t^k q^k}{k!} \right).
\]

If we differentiate \( \mathbf{B}_q(t) \) with respect to \( q \), we obtain, instead,

\[
\frac{\partial}{\partial q} \mathbf{B}_q(t) = -\mathbf{B}_q(t) \mathbf{B}_q(t) \sum_{k=0}^{\infty} \frac{4t^k q^k}{k!} \frac{(kq-1)-(k+1)q^k}{4-(1-q)^2 q^k}.
\]

Again, using the generating function and combining this with the derivative we get the partial differential equation.

Proposition 16. Consider the following:

\[
\frac{\partial}{\partial t} \mathbf{B}_q(t) - \frac{\partial}{\partial q} \mathbf{B}_q(t)
= \frac{\mathbf{B}_q(t)}{t} + \frac{\mathbf{B}_q(t) \mathbf{B}_q(t)}{t}
\times \sum_{k=0}^{\infty} \frac{4t^k (kq-1)-(k+1)q^k}{4-(1-q)^2 q^k}.
\]

4. Explicit Relationship between the \( q \)-Bernoulli and \( q \)-Euler Polynomials

In this section, we give some explicit relationship between the \( q \)-Bernoulli and \( q \)-Euler polynomials. We also obtain new formulae and some special cases for them. These formulae are extensions of the formulæ of Srivastava and Pintér, Cheon, and others.

We present natural \( q \)-extensions of the main results in the papers \([9, 11]\); see Theorems 17 and 19.

Theorem 17. For \( n \in \mathbb{N}_0 \), the following relationships hold true:

\[
\mathbf{B}_{n,q}(x,y) = \mathbf{B}_{n-k,q} \left[ \mathbf{B}_{k,q}(x) + \sum_{j=0}^{k} \binom{n}{j} \frac{(-1)^j q^j}{2^k-1} \frac{m^k-j}{m^k-j} \right]
\times \mathbf{E}_{n-k,q}(mx).
\]

Proof. Using the following identity

\[
\frac{t}{\mathbf{E}_q(t) - 1} \mathbf{E}_q(tx) \mathbf{E}_q(ty) = \frac{t}{\mathbf{E}_q(t) - 1} \mathbf{E}_q(tx) \cdot \frac{2}{\mathbf{E}_q(t/m) + 1} \mathbf{E}_q \left( \frac{t}{m} my \right)
\]

we have

\[
\sum_{n=0}^{\infty} \mathbf{B}_{n,q} (x,y) \frac{t^n}{[n]_q!}
= \frac{1}{[n]_q!} \sum_{n=0}^{\infty} \binom{(-1,n)_q}{n} \frac{m^n}{m^n} \frac{[n]_q!}{[n]_q!} \sum_{n=0}^{\infty} \mathbf{B}_{n,q} (x) \frac{t^n}{[n]_q!}
+ \frac{1}{[n]_q!} \sum_{n=0}^{\infty} \binom{(-1,n)_q}{n} \frac{m^n}{m^n} \frac{[n]_q!}{[n]_q!} \mathbf{B}_{n,q} (x) \frac{t^n}{[n]_q!}
=: I_1 + I_2.
\]

It is clear that

\[
I_2 = \frac{1}{[n]_q!} \sum_{n=0}^{\infty} \binom{(-1,n)_q}{n} \frac{m^n}{m^n} \frac{[n]_q!}{[n]_q!} \mathbf{B}_{n,q} (x) \mathbf{E}_{n,k,q}(my) \frac{t^n}{[n]_q!}
\]

\[
= \frac{1}{[n]_q!} \sum_{n=0}^{\infty} \binom{(-1,n)_q}{n} \frac{m^n}{m^n} \frac{[n]_q!}{[n]_q!} \mathbf{B}_{n,q} (x) \mathbf{E}_{n,k,q}(my) \frac{t^n}{[n]_q!}.
\]
On the other hand,

\[ I_1 = \frac{1}{2} \sum_{n=0}^{\infty} \mathcal{G}_{n,q}(my) \frac{t^n}{m^n[n]_q}, \]

\[ \times \sum_{n=0}^{\infty} \sum_{j=0}^{n} \binom{n}{j} \mathcal{G}_{j,q}(x) \frac{(-1, q)_{n-j}}{m^{n-j}2^{n-j}} \frac{t^n}{[n]_q!}, \]

\[ = \frac{1}{2} \sum_{n=0}^{\infty} \sum_{k=0}^{n} \binom{n}{k} \mathcal{G}_{n-k,q}(my) \times \left[ \sum_{j=0}^{k} \binom{k}{j} \frac{(-1, q)_{k-j}}{m^{k-j}2^{k-j}} \mathcal{G}_{j,q}(x) \right] \times \mathcal{G}_{n-k,q}(my) \frac{t^n}{[n]_q!}. \]

Therefore

\[ \sum_{n=0}^{\infty} \mathcal{G}_{n,q}(x, y) \frac{t^n}{[n]_q!} = \frac{1}{2} \sum_{n=0}^{\infty} \sum_{k=0}^{n} \binom{n}{k} m^{k-n} \times \left[ \mathcal{G}_{k,q}(x) + \sum_{j=0}^{k} \binom{k}{j} \frac{(-1, q)_{k-j}}{2^{k-j}} \mathcal{G}_{j,q}(x) \right] \times \mathcal{G}_{n-k,q}(my) \frac{t^n}{[n]_q!}. \]

(65)

It remains to equate the coefficients of \( t^n \). \( \square \)

Next we discuss some special cases of Theorem 17.

**Corollary 18.** For \( n \in \mathbb{N}_0 \), the following relationship holds true:

\[ \mathcal{G}_{n,q}(x, y) = \sum_{k=0}^{n} \binom{n}{k} \left( \mathcal{G}_{k,q}(x) + \frac{(-1, q)_{k-1}}{2^k} \mathcal{G}_{k-1,q}(x) \right) \mathcal{G}_{n-k,q}(y). \]

(67)

The formula (67) is a \( q \)-extension of the Cheon’s main result [23].

**Theorem 19.** For \( n \in \mathbb{N}_0 \), the following relationships

\[ \mathcal{G}_{n,q}(x, y) = \frac{1}{[n+1]_q} \times \sum_{k=0}^{n+1} \frac{1}{m^{n+1-k}} \left[ \frac{n+1}{k} \right]_q \times \left( \sum_{j=0}^{k} \binom{k}{j} \frac{(-1, q)_{k-j}}{m^{k-j}2^{k-j}} \mathcal{G}_{j,q}(y) - \mathcal{G}_{k,q}(y) \right) \times \mathcal{G}_{n+1-k,q}(mx) \]

hold true between the \( q \)-Bernoulli polynomials and \( q \)-Euler polynomials.

**Proof.** The proof is based on the following identity:

\[ \mathcal{G}_{q}(t) + 1 \mathcal{G}_{q}(tx) \mathcal{G}_{q}(ty) = \mathcal{G}_{q}(t/m) - \frac{1}{t} \mathcal{G}_{q}(t/m) - 1 \mathcal{G}_{q}(t/m). \]

(69)

Indeed

\[ \sum_{n=0}^{\infty} \mathcal{G}_{n,q}(x, y) \frac{t^n}{[n]_q!} = \sum_{n=0}^{\infty} \mathcal{G}_{n,q}(y) \frac{t^n}{[n]_q!} \sum_{n=0}^{\infty} \mathcal{G}_{n,q}(mx) \frac{t^n}{[m]_q!} \times \sum_{n=0}^{\infty} \mathcal{G}_{n,q}(mx) \mathcal{G}_{n-m,k,q}(mx) \frac{t^n}{[m]_q!}. \]

(70)

It follows that

\[ I_1 = \frac{1}{t} \sum_{n=0}^{\infty} \mathcal{G}_{n,q}(mx) \frac{t^n}{[m]_q!} \]

\[ = \frac{1}{t} \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{1}{m^{n-k}} \mathcal{G}_{k,q}(y) \mathcal{G}_{n-k,q}(mx) \frac{t^n}{[m]_q!}. \]

(71)

Next we give an interesting relationship between the \( q \)-Genocchi polynomials and the \( q \)-Bernoulli polynomials.
Theorem 20. For \( n \in \mathbb{N}_0 \), the following relationship

\[
\mathcal{G}_{nq}(x, y) = 1 + \sum_{k=0}^{n+1} \frac{1}{m^{n-k}} \left[ \binom{n+1}{k} \sum_{j=0}^{k} \frac{(-1,q)_{k-j} \mathcal{G}_{j,q}(x) - \mathcal{G}_{k,q}(x)}{m^{k-j} 2^{k-j}} \right] \\
\times \mathcal{B}_{n+1-k,q}(my),
\]

(72)

holds true between the \( q \)-Genocchi and the \( q \)-Bernoulli polynomials.

Proof. Using the following identity

\[
\frac{2t}{\mathcal{E}_q(t)} + \mathcal{E}_q(tx) \mathcal{E}_q(ty) = \frac{2t}{\mathcal{E}_q(t)} + \mathcal{E}_q(tx) \left( \mathcal{E}_q\left( \frac{t}{m} \right) - 1 \right) \frac{m}{t} \tag{73}
\]

we have

\[
\sum_{n=0}^{\infty} \frac{\mathcal{G}_{nq}(x, y)}{[n]_q!} \frac{t^n}{[n]_q!} = \mathcal{B}_{n+1}(x) \frac{t^n}{[n]_q!} \\
\times \sum_{n=0}^{\infty} \frac{(-1,q)_n}{m^n 2^n [n]_q!} \sum_{k=1}^{n} \frac{\mathcal{G}_{k,q}(x) \mathcal{B}_{n-k}(my)}{m^n [n]_q!} \frac{t^n}{[n]_q!} \\
\times \mathcal{B}_{n+1-k,q}(my) \frac{t^n}{[n]_q!}.
\]

The second identity can be proved in a like manner. \( \square \)

Conflict of Interests

The authors declare that there is no conflict of interests with any commercial identities regarding the publication of this paper.

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