Research Article

A Double Inequality for the Trigamma Function and Its Applications

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We prove that $p = 1$ and $q = 2$ are the best possible parameters in the interval $(0, \infty)$ such that the double inequality $(e^{p/(x+1)} - e^{-p/x})/2p < \psi'(x+1) < (e^{q/(x+1)} - e^{-q/x})/2q$ holds for $x > 0$. As applications, some new approximation algorithms for the circumference ratio π and Catalan constant $G = \sum_{n=0}^{\infty} ((-1)^n / (2n+1)^2)$ are given. Here, ψ' is the trigamma function.

1. Introduction

For real and positive values of x , the classical Euler's gamma function Γ and its logarithmic derivative ψ , the so-called psi function, are defined as

$$\Gamma(x) = \int_0^{\infty} t^{x-1} e^{-t} dt, \quad \psi(x) = \frac{\Gamma'(x)}{\Gamma(x)}. \quad (1)$$

For extension of these functions to complex variables and for basic properties, see [1]. The derivatives $\psi', \psi'', \psi''', \dots$ are known as polygamma functions (see [2]). In particular, ψ' is called trigamma function.

Recently, the bounds for the trigamma function using exponential functions have attracted the attention of many researchers. For example, Elezović et al. [3] proved that the inequality

$$\psi'(x) < e^{-\psi(x)} \quad (2)$$

holds for all $x > 0$. In [4, Theorem 2.7], Batir proved that $a^* = 1/2$ and $b^* = \pi^2 e^{-2\gamma}/6$ are the best possible constants such that the double inequality

$$(x + a^*) e^{-2\psi(x+1)} < \psi'(x+1) < (x + b^*) e^{-2\psi(x+1)} \quad (3)$$

holds for all $x > 0$, where γ is Euler's constant. Batir [4] also showed that

$$\begin{aligned} & \frac{1}{2} \left(\frac{2}{x^2} - 1 + e^{2/(x+1)} - e^{-2\psi(x+1)} \right) \\ & < \psi'(x+1) \\ & < \frac{1}{2} \left(\frac{2}{x^2} + 1 - e^{-2/x} + e^{-2\psi(x+1)} \right) \end{aligned} \quad (4)$$

for all $x > 0$. In [5, (1.11)], Guo and Qi established that

$$\frac{\vartheta e^{\vartheta/x}}{x^2 (e^{\vartheta/x} - 1)} < \psi'(x) < \frac{\theta e^{\theta/x}}{x^2 (e^{\theta/x} - 1)} \quad (5)$$

if $x > 0$ and $0 < \vartheta \leq 1, \theta \geq 2$. They [6, Lemma 2] found a very simple upper bound for trigamma function in terms of exponential function as follows:

$$\psi'(x) < e^{1/x} - 1 \quad (6)$$

for all $x > 0$. The inequality (6) was generalized in [7, Theorem 3.1], [8, Theorem 1.1], and [9, Theorem 1.1] to a complete monotonicity which reads that the difference $e^{1/x} - \psi'(x)$ is completely monotonic on $(0, \infty)$. Many other new results involving the psi and trigamma functions can be found in the literature [10, 11].

Suppose that $m \in (0, \infty)$ and g and g_m are the real functions defined on $(0, \infty)$. $m = \lambda$ is said to be the best possible constant in $(0, \infty)$ such that the inequality $g(x) > (<) g_m(x)$ holds for all $x > 0$ if $g_\lambda(x) \geq (\leq) g_\mu(x)$ on $(0, \infty)$, or $g_\lambda(x)$ and $g_\mu(x)$ are not comparable on $(0, \infty)$, and $\lim_{x \rightarrow \infty} (g(x) - g_\lambda(x)) / (g(x) - g_\mu(x)) = 0$ for any $\mu \in (0, \infty)$ satisfies $g(x) > (<) g_\mu(x)$ on the interval $(0, \infty)$.

The main purpose of this paper is to find the best possible constants $p, q \in (0, \infty)$ such that the double inequality

$$\theta(x, p) < \psi'(x+1) < \theta(x, q) \quad (7)$$

or equivalently

$$\frac{1}{x^2} + \theta(x, p) < \psi'(x) < \frac{1}{x^2} + \theta(x, q) \quad (8)$$

holds for all $x > 0$, where

$$\theta(x, m) = \frac{e^{m/(x+1)} - e^{-m/x}}{2m}, \quad m > 0. \quad (9)$$

Our main result is the following Theorem 1.

Theorem 1. $p = 1$ and $q = 2$ are the best possible constants in the interval $(0, \infty)$ such that the double inequality (7) or (8) holds for all $x > 0$.

From Theorem 1, we clearly see the following.

Corollary 2. The double inequality

$$\sinh \frac{1}{x+1} < \psi'(x+1) < \frac{1}{2} \sinh \frac{2}{x} \quad (10)$$

holds for all $x > 0$.

2. Lemmas

Lemma 3. Let the function θ be defined on $(0, \infty)^2$ by (9). Then the function θ is strictly decreasing with respect to m on $(0, 1]$ and strictly increasing on $[3/2, +\infty)$.

Proof. It follows from (9) that

$$\begin{aligned} & \frac{m^2 x}{2(m+x)} e^{m/x} \frac{\partial \theta}{\partial m} \\ &= \left(1 - \frac{x(x-m+1)}{(x+1)(m+x)} \exp\left(\frac{m}{x} + \frac{m}{x+1}\right) \right) \\ &=: h_m(x), \end{aligned} \quad (11)$$

$$\begin{aligned} \frac{\partial h_m}{\partial x} &= \frac{m^2 \exp((m/x)((2x+1)/(x+1)))}{x^3 (x+1)^3 (m+x)^2} \\ &\quad \times ((3-2m)x^2 + (3-2m)x + (1-m)). \end{aligned}$$

If $m \in (0, 1]$, then $\partial h_m / \partial x > 0$; that is, h_m is strictly increasing with respect to $x > 0$. Therefore,

$$h_m(x) < \lim_{x \rightarrow \infty} h_m(x) = 0, \quad (12)$$

which implies that $\partial \theta / \partial m < 0$.

If $m \geq 3/2$, then $\partial h_m / \partial x < 0$; that is, h_m is strictly decreasing with respect to $x > 0$, which leads to the conclusion that

$$h_m(x) > \lim_{x \rightarrow \infty} h_m(x) = 0. \quad (13)$$

□

Lemma 4. Let the function θ be defined on $(0, \infty)^2$ by (9) and

$$F_m(x) = \psi'(x+1) - \theta(x, m). \quad (14)$$

Then the equation

$$F_m(0^+) = \psi'(1) - \frac{e^m}{2m} = 0 \quad (15)$$

has two roots

$$m_1 = 0.5023 \dots, \quad m_2 = 1.7510 \dots \quad (16)$$

such that $F_m(0^+) > 0$ for $m \in (m_1, m_2)$ and $F_m(0^+) < 0$ for $m \in (0, m_1) \cup (m_2, \infty)$.

Proof. Differentiation yields

$$\frac{d}{dm} F_m(0^+) = -\frac{1}{2m^2} e^m (m-1), \quad (17)$$

which reveals that the function $m \mapsto F_m(0^+)$ is strictly increasing on $(0, 1)$ and strictly decreasing on $(1, \infty)$. Therefore, Lemma 4 follows from the piecewise monotonicity of the function $m \mapsto F_m(0^+)$ and the numerical computations results:

$$\begin{aligned} F_{0.5023}(0^+) &= \frac{1}{6} \pi^2 - \frac{e^{0.5023}}{1.0046} = -1.685 \dots \times 10^{-5} < 0, \\ F_{0.5024}(0^+) &= \frac{1}{6} \pi^2 - \frac{e^{0.5024}}{1.0048} = 1.46 \dots \times 10^{-4} > 0, \\ F_{1.751}(0^+) &= \frac{1}{6} \pi^2 - \frac{e^{1.751}}{3.502} = 5.68 \dots \times 10^{-5} > 0, \\ F_{1.7511}(0^+) &= \frac{1}{6} \pi^2 - \frac{e^{1.7511}}{3.5022} = -1.374 \dots \times 10^{-5} < 0. \end{aligned} \quad (18)$$

□

Lemma 5. Let $m \geq 0$, $m_1 = 0.5023 \dots$, and $m_2 = 1.7510 \dots$ and let $F_m(x)$ be defined as in Lemma 4. Then the following statements are true:

- (i) if the inequality $F_m(x) \geq 0$ holds for all $x > 0$, then $m \in [1, m_2]$;
- (ii) if the inequality $F_m(x) \leq 0$ holds for all $x > 0$, then $m \in (0, m_1] \cup [2, \infty)$.

Proof. It follows from the series formulas that

$$\psi'(x) = \frac{1}{x} + \frac{1}{2x^2} + \frac{1}{6x^3} - \frac{1}{30x^5} + \cdots, \quad (19)$$

$$e^{m/(x+1)} = 1 + \frac{m}{x+1} + \frac{1}{2} \left(\frac{m}{x+1} \right)^2 + \frac{1}{6} \left(\frac{m}{x+1} \right)^3 + \frac{1}{24} \left(\frac{m}{x+1} \right)^5 + \cdots, \quad (20)$$

$$e^{-m/x} = 1 - \frac{m}{x} + \frac{1}{2} \left(-\frac{m}{x} \right)^2 + \frac{1}{6} \left(-\frac{m}{x} \right)^3 + \frac{1}{24} \left(-\frac{m}{x} \right)^5 + \cdots, \quad (21)$$

and we get

$$\begin{aligned} & \lim_{x \rightarrow \infty} \frac{F_m(x)}{x^{-3}} \\ &= \lim_{x \rightarrow \infty} \left(\left(\frac{1}{(x+1)} + \frac{1}{2(x+1)^2} + \frac{1}{6(x+1)^3} \right. \right. \\ & \quad \left. \left. - \left(\left(1 + \frac{m}{x+1} + \frac{1}{2} \left(\frac{m}{x+1} \right)^2 + \frac{1}{6} \left(\frac{m}{x+1} \right)^3 \right) \right. \right. \right. \\ & \quad \left. \left. - \left(1 - \frac{m}{x} + \frac{1}{2} \left(-\frac{m}{x} \right)^2 + \frac{1}{6} \left(-\frac{m}{x} \right)^3 \right) \right) \right) \\ & \quad \left. \times (2m)^{-1} \right) \\ & \quad \times (x^{-3})^{-1} \\ &= -\frac{1}{12} \lim_{x \rightarrow \infty} \left(((2m^2 - 6m + 4)x^3 + (3m^2 - 9m + 6)x^2 \right. \\ & \quad \left. + (3m^2 - 3m)x + m^2) \right. \\ & \quad \left. \times ((x+1)^3)^{-1} \right) \\ &= -\frac{1}{12} (2m^2 - 6m + 4) = -\frac{1}{6} (m-1)(m-2). \end{aligned} \quad (22)$$

(i) If inequality $F_m(x) \geq 0$ holds for all $x > 0$, then, from

$$\begin{aligned} F_m(0^+) &= \psi'(1) - \frac{e^m}{2m} \geq 0, \\ \lim_{x \rightarrow \infty} \frac{F_m(x)}{x^{-3}} &= -\frac{1}{6} (m-1)(m-2) \geq 0 \end{aligned} \quad (23)$$

and Lemma 4, we clearly see that $m \in [m_1, m_2] \cap [1, 2] = [1, m_2]$.

(ii) If inequality $F_m(x) \leq 0$ holds for all $x > 0$, then

$$\begin{aligned} F_m(0^+) &= \psi'(1) - \frac{e^m}{2m} \leq 0, \\ \lim_{x \rightarrow \infty} \frac{F_m(x)}{x^{-3}} &= -\frac{1}{6} (m-1)(m-2) \leq 0 \end{aligned} \quad (24)$$

and Lemma 4 lead to the conclusion that

$$\begin{aligned} m &\in ((0, m_1] \cup [m_2, \infty)) \cap ((0, 1] \cup [2, \infty)) \\ &= (0, m_1] \cup [2, \infty). \end{aligned} \quad (25)$$

□

Lemma 6. Let the function θ be defined on $(0, \infty)^2$ by (9). Then $\theta(x, 1)$ and $\theta(x, m)$ are not comparable for all $x > 0$ if $m \in (1, 2)$.

Proof. For $x > 0$ and $m > 0$, let

$$\begin{aligned} G_{1,m}(x) &= \theta(x, 1) - \theta(x, m) \\ &= \frac{e^{1/(x+1)} - e^{-1/x}}{2} \\ & \quad - \frac{e^{m/(x+1)} - e^{-m/x}}{2m}. \end{aligned} \quad (26)$$

Then simple computation leads to

$$G_{1,m}(0^+) = \frac{e}{2} - \frac{e^m}{2m}. \quad (27)$$

From (20) and (21), we have

$$\lim_{x \rightarrow \infty} \frac{G_{1,m}(x)}{x^{-3}} = -\frac{1}{6} (m-1)(m-2). \quad (28)$$

Differentiation yields

$$\frac{d}{dm} G_{1,m}(0^+) = -\frac{e^m}{2m^2} (m-1), \quad (29)$$

which shows that $m \mapsto G_{1,m}(0^+)$ is strictly decreasing on $(1, \infty)$. Therefore,

$$G_{1,m}(0^+) < G_{1,1}(0^+) = 0 \quad (30)$$

if $m \in (1, 2)$. On the other hand, we clearly see that $\lim_{x \rightarrow \infty} [x^3 G_{1,m}(x)] > 0$ for $m \in (1, 2)$. □

Lemma 7. Let $n \in \mathbb{N} = \{1, 2, 3, \dots\}$ and $m \in \mathbb{N} \cup \{0\}$ with $n > m$ and let $P_n(t)$ be the polynomial of degree n defined by

$$P_n(t) = \sum_{i=m+1}^n a_i t^i - \sum_{i=0}^m a_i t^i, \quad (31)$$

where $a_n, a_m > 0$ and $a_i \geq 0$ for $0 \leq i \leq n-1$ with $i \neq m$. Then, there exists $t_{m+1} \in (0, \infty)$ such that $P_n(t_{m+1}) = 0$ and $P_n(t) < 0$ for $t \in (0, t_{m+1})$ and $P_n(t) > 0$ for $t \in (t_{m+1}, \infty)$.

Proof. Differentiating $P_n(t)$ gives

$$\begin{aligned} P_n^{(k)}(t) &= \sum_{i=m+1}^n \frac{i!}{(i-k)!} a_i t^{i-k} \\ & \quad - \sum_{i=k}^m \frac{i!}{(i-k)!} a_i t^{i-k} \quad \text{for } 1 \leq k \leq m, \end{aligned} \quad (32)$$

$$P_n^{(m+1)}(t) = \sum_{i=m+1}^n \frac{i!}{(i-m-1)!} a_i t^{i-m-1} > 0.$$

Note that

$$P_n^{(k)}(\infty) = \infty, \quad P_n^{(k)}(0^+) = -a_k \quad (33)$$

for $1 \leq k \leq m$.

From $P_n^{(m+1)}(t) > 0$, we clearly see that $P_n^{(m)}(t)$ is strictly increasing on $(0, \infty)$. Then $P_n^{(m)}(\infty) > 0$ and $P_n^{(m)}(0^+) = -a_m < 0$ lead to the conclusion that there exists $t_1 \in (0, \infty)$ such that $P_n^{(m)}(t_1) = 0$ and $P_n^{(m)}(t) < 0$ for $t \in (0, t_1)$ and $P_n^{(m)}(t) > 0$ for $t \in (t_1, \infty)$. Therefore, $P_n^{(m-1)}(t)$ is strictly decreasing on $(0, t_1)$ and strictly increasing on (t_1, ∞) .

It follows from the piecewise monotonicity of $P_n^{(m-1)}(t)$ and $P_n^{(m-1)}(\infty) = \infty$ that $P_n^{(m-1)}(t) < P_n^{(m-1)}(0^+) = -a_{m-1} \leq 0$ for $t \in (0, t_1)$, and there exists $t_2 \in (t_1, \infty)$ such that $P_n^{(m-1)}(t_2) = 0$ and $P_n^{(m-1)}(t) < 0$ for $t \in (0, t_2)$ and $P_n^{(m-1)}(t) > 0$ for $t \in (t_2, \infty)$. Therefore, $P_n^{(m-2)}(t)$ is strictly decreasing on $(0, t_2)$ and strictly increasing on (t_2, ∞) .

After repeating the same steps as above $m+1$ times, we deduce that there exists $t_{m+1} \in (t_m, \infty) \subset (0, \infty)$ such that $P_n(t_{m+1}) = 0$ and $P_n(t) < 0$ for $t \in (0, t_{m+1})$ and $P_n(t) > 0$ for $t \in (t_{m+1}, \infty)$. $\square$

Lemma 8. Let the function θ be defined on $(0, \infty)^2$ by (9). Then there exists $m_0 \in (2/5, 9/20)$ such that $\theta(x, 2)$ and $\theta(x, m)$ are not comparable for all $x > 0$ if $m \in (m_0, 1)$, and $\theta(x, 2) < \theta(x, m)$ for all $x > 0$ if $m \in (0, m_0]$.

Proof. For $x > 0$ and $m > 0$, let

$$\begin{aligned} G_{2,m}(x) &= \theta(x, 2) - \theta(x, m) \\ &= \frac{e^{2/(x+1)} - e^{-2/x}}{4} \\ &\quad - \frac{e^{m/(x+1)} - e^{-m/x}}{2m}. \end{aligned} \quad (34)$$

Then simple computation leads to

$$G_{2,m}(0^+) = \frac{e^2}{4} - \frac{e^m}{2m}. \quad (35)$$

(i) We prove that there exists $m_0 \in (2/5, 9/20)$ such that $\theta(x, 2)$ and $\theta(x, m)$ are not comparable for all $x > 0$ if $m \in (m_0, 1)$. For this end, it suffices to prove that there exists $m_0 \in (2/5, 9/20)$ such that $G_{2,m}(0^+) > 0$ and $\lim_{x \rightarrow \infty} x^3 G_{2,m}(x) < 0$ if $m \in (m_0, 1)$.

Indeed, it follows from

$$\frac{d}{dm} G_{2,m}(0^+) = -\frac{e^m}{2m^2} (m-1) \quad (36)$$

that the function $m \mapsto G_{2,m}(0^+)$ is strictly increasing on $(0, 1)$. Numerical computations show that

$$\begin{aligned} G_{2,2/5}(0^+) &= \frac{1}{4}e^2 - \frac{5}{4}e^{2/5} < 0, \\ G_{2,9/20}(0^+) &= \frac{1}{4}e^2 - \frac{10}{9}e^{9/20} > 0. \end{aligned} \quad (37)$$

Therefore, there exists $m_0 \in (2/5, 9/20)$ such that

$$G_{2,m}(0^+) = \frac{e^2}{4} - \frac{e^m}{2m} = 0 \quad (38)$$

and $G_{2,m}(0^+) < 0$ for $m \in (0, m_0)$ and $G_{2,m}(0^+) > 0$ for $m \in (m_0, 1)$.

On the other hand, it follows from $m \in (m_0, 1)$ together with (20) and (21) that

$$\lim_{x \rightarrow \infty} \frac{G_{2,m}(x)}{x^{-3}} = -\frac{1}{6}(m-1)(m-2) < 0. \quad (39)$$

(ii) We prove that $\theta(x, 2) < \theta(x, m)$ for all $x > 0$ if $m \in (0, m_0]$. From Lemma 3, we know that $m \mapsto \theta(x, m)$ is strictly decreasing on $(0, 1]$, so it suffices to prove that $\theta(x, 2) < \theta(x, m_0)$ for all $x > 0$.

Let

$$\begin{aligned} g_1(x) &= e^{m_0/x} G_{2,m_0}(x) \\ &= \frac{1}{4} \exp\left(\frac{m_0}{x} + \frac{2}{x+1}\right) - \frac{1}{4} \exp\left(\frac{m_0}{x} - \frac{2}{x}\right) \\ &\quad + \frac{1}{2m_0} \left(1 - \exp\left(\frac{m_0}{x} + \frac{m_0}{x+1}\right)\right). \end{aligned} \quad (40)$$

Then

$$\lim_{x \rightarrow \infty} g_1(x) = 0. \quad (41)$$

Differentiation yields

$$\begin{aligned} &\frac{4}{2-m_0} x^2 e^{(2-m_0)/x} \times g_1'(x) \\ &= \frac{2(2x^2 + 2x + 1)}{(2-m_0)(x+1)^2} \exp\left(\frac{2}{x} + \frac{m_0}{x+1}\right) - 1 \\ &\quad - \frac{(m_0+2)x^2 + 2m_0x + m_0}{(2-m_0)(x+1)^2} \exp\left(\frac{2}{x} + \frac{2}{x+1}\right) \\ &:= g_2(x), \end{aligned} \quad (42)$$

$$\begin{aligned} g_2'(x) &= 2 \left((2m_0+2)x^4 + (2m_0+10)x^3 \right. \\ &\quad \left. + (m_0+14)x^2 + 8x + 2 \right) \\ &\quad \times \left((2-m_0)x^2(x+1)^4 \right)^{-1} \\ &\quad \times \exp\left(\frac{m_0}{x+1} + \frac{2}{x}\right) g_3(x), \end{aligned} \quad (43)$$

where

$$\begin{aligned} g_3(x) &= \left((2m_0+2)x^4 + (6m_0+2)x^3 \right. \\ &\quad \left. + (7m_0+2)x^2 + 4m_0x + m_0 \right) \\ &\quad \times \left((2m_0+2)x^4 + (2m_0+10)x^3 \right. \\ &\quad \left. + (m_0+14)x^2 + 8x + 2 \right)^{-1} \\ &\quad \times e^{(2-m_0)/(x+1)} - 1. \end{aligned} \quad (44)$$

Note that

$$\lim_{x \rightarrow \infty} g_2(x) = 0, \quad (45)$$

$$\lim_{x \rightarrow 0^+} g_3(x) = \frac{1}{2} m_0 e^{2-m_0} - 1 = 0, \quad (46)$$

$$\lim_{x \rightarrow \infty} g_3(x) = 0, \quad (47)$$

where the second equation in (46) follows from (38).

Differentiating $g_3(x)$ leads to

$$\begin{aligned} g_3'(x) &= \left((2 - m_0) e^{(2-m_0)/(x+1)} \right) \\ &\quad \times \left((x+1)^2 (8x + m_0 x^2 + 2m_0 x^3 \right. \\ &\quad \left. + 2m_0 x^4 + 14x^2 + 10x^3 + 2x^4 + 2)^2 \right)^{-1} \\ &\quad \times g_4(x), \end{aligned} \quad (48)$$

where

$$\begin{aligned} g_4(x) &= (4 - 4m_0^2) x^8 + (16 - 16m_0^2) x^7 \\ &\quad + (-28m_0^2 - 24m_0 + 32) x^6 \\ &\quad + (-28m_0^2 - 72m_0 + 40) x^5 \\ &\quad - (17m_0^2 + 104m_0 - 42) x^4 \\ &\quad - (6m_0^2 + 88m_0 - 36) x^3 \\ &\quad - (m_0^2 + 46m_0 - 18) x^2 \\ &\quad - (14m_0 - 4) x - 2m_0 \\ &:= a_8 x^8 + a_7 x^7 + a_6 x^6 + a_5 x^5 - a_4 x^4 \\ &\quad - a_3 x^3 - a_2 x^2 - a_1 x - a_0. \end{aligned} \quad (49)$$

We assert that there exists a unique $x_4^* \in (0, \infty)$ such that $g_4(x) < 0$ for $x \in (0, x_4^*)$ and $g_4(x) > 0$ for $x \in (x_4^*, \infty)$, which leads to the conclusion that $g_3(x)$ is strictly decreasing on $(0, x_4^*)$ and strictly increasing on $[x_4^*, \infty)$. To this end, it is enough to verify that the coefficients of $g_4(x)$ satisfy the conditions of Lemma 7. In fact, it follows from $m_0 \in (2/5, 9/20) := (m_{0-}, m_{0+})$ that we have

$$\begin{aligned} a_8 &= 4 - 4m_0^2 = 4(1 - m_0^2) > 0, \\ a_7 &= 16 - 16m_0^2 = 16(1 - m_0^2) > 0, \\ a_6 &= -28m_0^2 - 24m_0 + 32 \\ &> -28m_{0+}^2 - 24m_{0+} + 32 = \frac{1553}{100} > 0, \end{aligned}$$

$$\begin{aligned} a_5 &= -28m_0^2 - 72m_0 + 40 \\ &> -28m_{0+}^2 - 72m_{0+} + 40 = \frac{193}{100} > 0, \\ a_4 &= (17m_0^2 + 104m_0 - 42) \\ &> 17m_{0-}^2 + 104m_{0-} - 42 = \frac{58}{25} > 0, \\ a_3 &= (6m_0^2 + 88m_0 - 36) \\ &> 6m_{0-}^2 + 88m_{0-} - 36 = \frac{4}{25} > 0, \\ a_2 &= (m_0^2 + 46m_0 - 18) \\ &> m_{0-}^2 + 46m_{0-} - 18 = \frac{14}{25} > 0, \\ a_1 &= (14m_0 - 4) \\ &> 14m_{0-} - 4 = \frac{8}{5} > 0, \\ a_0 &= 2m_0 > 0. \end{aligned} \quad (50)$$

From the piecewise monotonicity of $g_3(x)$ together with (46) and (47), we clearly see that

$$\begin{aligned} g_3(x) &< \lim_{x \rightarrow 0^+} g_3(x) = 0 \quad \text{for } x \in (0, x_4^*), \\ g_3(x) &< \lim_{x \rightarrow \infty} g_3(x) = 0 \quad \text{for } x \in [x_4^*, \infty); \end{aligned} \quad (51)$$

that is, $g_3(x) < 0$ for $x \in (0, \infty)$. Then (43) and (45) lead to the conclusion that $g_2(x) > \lim_{x \rightarrow \infty} g_2(x) = 0$ for $x \in (0, \infty)$, which implies that $g_1(x)$ is strictly increasing on $(0, \infty)$ and $g_1(x) < \lim_{x \rightarrow \infty} g_1(x) = 0$ for $x \in (0, \infty)$.

Therefore, $\theta(x, 2) < \theta(x, m_0)$ follows easily from (40) and $g_1(x) < 0$. $\square$

Lemma 9 (see [12, pp. 258–260]). *Let $x > 0$ and $n \in \{0, 1, 2, \dots\}$. Then*

$$\psi^{(n)}(x+1) - \psi^{(n)}(x) = \frac{(-1)^n n!}{x^{n+1}}. \quad (52)$$

From the proof of [4, Theorem 2.6], we get the following.

Lemma 10. *The inequality*

$$\psi'(x+1) > \frac{2x+1}{2x^2+2x+2/3} \quad (53)$$

holds for $x > -1$.

The following lemma can be derived immediately from the proof of [4, Theorem 2.1].

Lemma 11 (see [4, Theorem 2.1]). *Let y be the function defined on $(0, \infty)$ by*

$$y(x) = e^{2\psi(x)}. \quad (54)$$

Then $y'''(x) > 0$ for $x \in (0, \infty)$.

The well-known Hermite-Hadamard inequality for convex function can be stated as follows.

Lemma 12 (see [13]). *Let $I \subseteq \mathbb{R}$ be an interval, $a, b \in I$ with $a < b$, and let $f : I \rightarrow \mathbb{R}$ be a convex function. Then*

$$f\left(\frac{a+b}{2}\right) \leq \frac{\int_a^b f(x) dx}{b-a} \leq \frac{f(a) + f(b)}{2}. \quad (55)$$

3. Proofs of Theorem 1

Proof of Theorem 1. We divide the proof into four parts.

(I) We prove the first inequality in (7); that is,

$$F_1(x) = \psi'(x+1) - \frac{e^{1/(x+1)} - e^{-1/x}}{2} > 0, \quad (56)$$

where $F_m(x)$ is defined by (14).

It follows from Lemma 10 that

$$\begin{aligned} F_1(x) &= \psi'(x+1) - \frac{e^{1/(x+1)} - e^{-1/x}}{2} \\ &> \frac{2x+1}{2x^2+2x+2/3} - \frac{e^{1/(x+1)} - e^{-1/x}}{2} \\ &:= e^{-1/x} H(x), \end{aligned} \quad (57)$$

where

$$\begin{aligned} H(x) &= \frac{2x+1}{2x^2+2x+2/3} e^{1/x} \\ &\quad - \frac{\exp((1/(x+1)) + (1/x)) - 1}{2}. \end{aligned} \quad (58)$$

We clearly see that it is enough to prove that $H(x) > 0$ for $x > 0$.

Differentiating $H(x)$ gives

$$\begin{aligned} H'(x) &= \frac{1}{2x^2} e^{1/x} \\ &\quad \times \left(\frac{2x^2+2x+1}{(x+1)^2} e^{1/(x+1)} \right. \\ &\quad \left. - \frac{3(x+1)(6x^3+6x^2+4x+1)}{(3x^2+3x+1)^2} \right) \\ &= \frac{1}{2x^2} e^{1/x} L \\ &\quad \times \left(\frac{2x^2+2x+1}{(x+1)^2} e^{1/(x+1)}, \right. \\ &\quad \left. \frac{3(x+1)(6x^3+6x^2+4x+1)}{(3x^2+3x+1)^2} \right) \times h(x), \end{aligned} \quad (59)$$

where $L(a, b) = (b-a)/(\ln a - \ln b)$ denotes the logarithmic mean of positive numbers a and b , and

$$\begin{aligned} h(x) &= \frac{1}{x+1} + \ln \frac{2x^2+2x+1}{(x+1)^2} \\ &\quad - \ln \frac{3(x+1)(6x^3+6x^2+4x+1)}{(3x^2+3x+1)^2}. \end{aligned} \quad (60)$$

Differentiating $h(x)$ leads to

$$\begin{aligned} h'(x) &= (x(2x+1)(7x^2+7x+3)) \\ &\quad \times ((x+1)^2(3x^2+3x+1)(2x^2+2x+1) \\ &\quad \times (6x^3+6x^2+4x+1))^{-1} > 0 \end{aligned} \quad (61)$$

for $x > 0$, which means that h is strictly increasing on $(0, \infty)$ and $h(x) < \lim_{x \rightarrow \infty} h(x) = 0$. It in turn implies that H is strictly decreasing on $(0, \infty)$. Therefore, $H(x) > \lim_{x \rightarrow \infty} H(x) = 0$ for $x > 0$.

(II) We prove that $m = 1$ is the best possible constant such that $\psi'(x+1) > \theta(x, m)$ for all $x > 0$.

From Lemma 5, we know that $m \in [1, m_2]$ if $\psi'(x+1) > \theta(x, m)$ for all $x > 0$, where $m_2 = 1.7510\dots$. It follows from Lemma 6 that $\theta(x, 1)$ and $\theta(x, m)$ are not comparable for all $x > 0$ if $m \in (1, 2)$; that is to say, $\theta(x, m^*)$ is not a better lower bound of $\psi'(x+1)$ than $\theta(x, 1)$ even if there exists $m^* \in (1, m_2]$ such that $\psi'(x+1) > \theta(x, m^*)$.

For any $m^* \in (1, m_2]$, (22) leads to

$$\lim_{x \rightarrow \infty} [x^3 F_{m^*}(x)] = \frac{1}{6} (m^* - 1) (2 - m^*). \quad (62)$$

It follows from (19), (20), and (21) that we get

$$\lim_{x \rightarrow \infty} [x^5 F_1(x)] = \frac{1}{24}. \quad (63)$$

(III) We prove the second inequality (7); that is,

$$F_2(x) = \psi'(x+1) - \frac{e^{2/(x+1)} - e^{-2/x}}{4} < 0 \quad (64)$$

for $x > 0$, where $F_m(x)$ is defined by (14).

Lemma 11 implies that the function $x \mapsto y'(x) = 2\psi'(x)e^{2\psi(x)}$ is strictly convex on $(0, \infty)$. Then, making use of Lemma 12, we get

$$y'\left(\frac{x+x+2}{2}\right) < \frac{\int_x^{x+2} y'(t) dt}{x+2-x} \quad (65)$$

for $x > 0$. That is,

$$\begin{aligned} 2\psi'(x+1)e^{2\psi(x+1)} &< \frac{e^{2\psi(x+2)} - e^{2\psi(x)}}{2}, \\ \psi'(x+1) &< \frac{1}{4} [e^{2(\psi(x+2)-\psi(x+1))} - e^{2(\psi(x)-\psi(x+1))}]. \end{aligned} \quad (66)$$

Therefore, inequality (64) follows easily from (52) and the last inequality above.

(IV) We prove that $m = 2$ is the best possible constant such that $\psi'(x+1) < \theta(x, m)$ for all $x > 0$.

From Lemma 5, we know that $m \in (0, m_1] \cup [2, \infty)$ if $\psi'(x+1) < \theta(x, m)$ for all $x > 0$, where $m_1 = 0.5023 \dots$.

It follows from Lemma 8 that there exists $m_0 \in (2/5, 9/20) \subset (0, m_1)$ such that $\theta(x, 2)$ and $\theta(x, m)$ are not comparable for all $x > 0$ if $m \in (m_0, m_1]$ and $\theta(x, 2) < \theta(x, m)$ for all $x > 0$ if $m \in (0, m_0]$. Lemma 3 leads to the conclusion that $\theta(x, 2) < \theta(x, m)$ for all $x > 0$ if $m \in (2, \infty)$.

If there exists $m^* \in (m_0, m_1]$ such that $F_{m^*}(x) = \psi'(x+1) - \theta(x, m^*) < 0$ for all $x > 0$, then, from (19), (20), (21), and (22), we get

$$\begin{aligned} \lim_{x \rightarrow \infty} [x^3 F_{m^*}(x)] &= -\frac{1}{6} (1 - m^*) (2 - m^*), \\ \lim_{x \rightarrow \infty} [x^7 F_2(x)] &= -\frac{1}{45}. \end{aligned} \quad (67)$$

□

From the above proof and Lemma 3 we get the following.

Corollary 13. Let the function θ be defined on $(0, \infty)^2$ by (9) and let $m_0 \in (2/5, 9/20)$ be the root of (38) on $(0, 1)$. Then the inequalities

$$\begin{aligned} \psi'(x+1) &< \theta(x, 2) < \theta(x, m_0) < \lim_{m \rightarrow 0} \theta(x, m) \\ &= \frac{1}{2(x+1)} + \frac{1}{2x} \end{aligned} \quad (68)$$

or equivalently

$$\psi'(x) < \frac{1}{2(x+1)} + \frac{1}{2x} + \frac{1}{x^2} \quad (69)$$

hold for all $x > 0$.

4. Remarks

Remark 14. It follows from (67) and the facts that

$$\begin{aligned} \lim_{x \rightarrow \infty} \frac{\psi'(x) - e^{-\psi(x)}}{x^{-3}} &= -\frac{1}{24}, \\ \lim_{x \rightarrow \infty} \frac{\psi'(x+1) - (x + \pi^2 e^{-2\gamma}/6) e^{-2\psi(x+1)}}{x^{-2}} &= \frac{1}{2} - \frac{1}{6} \pi^2 e^{-2\gamma}, \\ \lim_{x \rightarrow \infty} \frac{\psi'(x+1) - (1/2) \left((2/x^2) + 1 - e^{-2/x} + e^{-2\psi(x+1)} \right)}{x^{-6}} &= -\frac{1}{60}, \\ \lim_{x \rightarrow \infty} \frac{\psi'(x) - (e^{1/x} - 1)}{x^{-4}} &= -\frac{1}{24}, \\ \lim_{x \rightarrow \infty} \frac{\psi'(x) - (2e^{2/x}/x^2 (e^{2/x} - 1))}{x^{-2}} &= -\frac{1}{2} \end{aligned} \quad (70)$$

that we clearly see that the upper bound in Theorem 1 for the trigamma function ψ' is better than the upper bounds given in (2), (3), (4), (5), and (6) if x is large enough.

Lemma 15. One has

$$\begin{aligned} e^{1/(x+1)} + e^{-1/x} &> 2 \quad \text{if } x \in (-\infty, -1) \cup (0, \infty), \\ e^{1/(x+1)} + e^{-1/x} &\geq 2e^2 \quad \text{if } x \in (-1, 0). \end{aligned} \quad (71)$$

Proof. Differentiation leads to

$$\begin{aligned} (e^{1/(x+1)} + e^{-1/x})' &= \frac{e^{-1/x}}{x^2} \left(1 - \frac{x^2}{(x+1)^2} \exp\left(\frac{1}{x+1} + \frac{1}{x}\right) \right) \\ &:= \frac{e^{-1/x}}{x^2} V(x), \end{aligned} \quad (72)$$

$$V'(x) = \frac{\exp(1/(x+1) + (1/x))}{(x+1)^4} > 0.$$

If $x \in (0, \infty)$, then, from $V'(x) > 0$, we get $V(x) < \lim_{x \rightarrow \infty} V(x) = 0$, which leads to

$$e^{1/(x+1)} + e^{-1/x} > \lim_{x \rightarrow \infty} (e^{1/(x+1)} + e^{-1/x}) = 2. \quad (73)$$

If $x \in (-\infty, -1)$, then the first inequality in (71) still holds by making a change of variable $y = -(x+1)$. If $x \in (-1, 0)$, since $V(-1/2) = 0$, we see that $V(x) < V(-1/2) = 0$ for $x \in (-1, -1/2)$ and $V(x) > V(-1/2) = 0$ for $x \in (-1/2, 0)$. Hence,

$$e^{1/(x+1)} + e^{-1/x} \geq [e^{1/(x+1)} + e^{-1/x}]_{x=-1/2} = 2e^2. \quad (74)$$

□

Remark 16. Using inequality (71), one has

$$\begin{aligned} \frac{1}{x^2} + \theta(x, 2) - (e^{1/x} - 1) &= \frac{1}{x^2} + \frac{e^{2/(x+1)} - e^{-2/x}}{4} - (e^{1/x} - 1) \\ &< \frac{1}{x^2} + \frac{e^{2/(x+1)} - (2 - e^{1/(x+1)})^2}{4} - (e^{1/x} - 1) \\ &= \frac{1}{x^2} + e^{1/(x+1)} - e^{1/x} = \frac{1}{x^2} + \sum_{n=0}^{\infty} \frac{1}{n!} \left(\frac{1}{(x+1)^n} - \frac{1}{x^n} \right) \\ &< \frac{1}{x^2} + \sum_{n=0}^3 \frac{1}{n!} \left(\frac{1}{(x+1)^n} - \frac{1}{x^n} \right) = -\frac{1}{6x^3(x+1)^3} < 0 \end{aligned} \quad (75)$$

for $x > 0$, which shows that the upper bound $x^{-2} + \theta(x, 2)$ in (8) is better than the upper bound $(e^{1/x} - 1)$ in (6).

Remark 17. The conclusion that the difference $e^{1/x} - \psi'(x)$ is completely monotonic on $(0, \infty)$ given in [6, Lemma 2] implies that

$$e - \psi'(1) > e^{1/(x+1)} - \psi'(x+1) > 1 \quad (76)$$

or

$$e^{1/(x+1)} - e + \frac{\pi^2}{6} < \psi'(x+1) < e^{1/(x+1)} - 1 < \frac{1}{2} \sinh \frac{2}{x}. \quad (77)$$

It is easy to check that the lower bound $e^{1/(x+1)} - e + \pi^2/6$ and $\sinh 1/(x+1)$ for $\psi'(x+1)$ given in (10) are not comparable due to

$$e^{1/(x+1)} - e + \frac{\pi^2}{6} > (<) \sinh \frac{1}{x+1} \quad \text{if } x < (>) 1.62670 \dots \quad (78)$$

Remark 18. Guo et al. [14] proved that

$$\frac{(k-1)!}{x^k} + \frac{k!}{2x^{k+1}} < (-1)^{k+1} \psi^{(k)}(x) < \frac{(k-1)!}{x^k} + \frac{k!}{x^{k+1}} \quad (79)$$

for $x > 0$ if $k \in \mathbb{N}$. In particular, if $k = 1$, one has

$$\frac{1}{x} + \frac{1}{2x^2} < \psi'(x) < \frac{1}{x} + \frac{1}{x^2}. \quad (80)$$

We clearly see that the upper bound given in (69) is better than that in (80) for the trigamma function $\psi'(x)$.

Finally, we give remarks on two mathematical constants π and G (Catalan constant).

Remark 19. It is well known that

$$\sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6}. \quad (81)$$

Let $Q_n = \sum_{k=1}^n (1/k^2)$, and then $\psi'(n+1) = (\pi^2/6) - Q_n$. From Theorem 1, we clearly see that the double inequality

$$\frac{e^{1/(n+1)} - e^{-1/n}}{2} < \frac{\pi^2}{6} - Q_n < \frac{e^{2/(n+1)} - e^{-2/n}}{4} \quad (82)$$

holds for all $n \in \mathbb{N}$.

Let

$$l_n = Q_n + \frac{e^{1/(n+1)} - e^{-1/n}}{2}, \quad L_n = Q_n + \frac{e^{2/(n+1)} - e^{-2/n}}{4}. \quad (83)$$

Then inequalities (82) can be rewritten as

$$l_n < \frac{\pi^2}{6} < L_n \quad (84)$$

or

$$\sqrt{6l_n} < \pi < \sqrt{6L_n}. \quad (85)$$

TABLE 1

n	$ \sqrt{6Q_n} - \pi $	$ \sqrt{6L_n} - \pi $	$ \sqrt{6L_n} - \pi $
1	0.6921	4.3127×10^{-3}	7.663×10^{-4}
2	0.40298	3.7549×10^{-4}	2.8250×10^{-5}
5	0.1782	7.7596×10^{-6}	1.3276×10^{-7}
10	9.2231×10^{-2}	3.1013×10^{-7}	1.4875×10^{-9}
50	1.8966×10^{-2}	1.2112×10^{-10}	2.5320×10^{-14}
100	9.5161×10^{-3}	3.8807×10^{-12}	2.0489×10^{-16}
200	4.7663×10^{-3}	1.2280×10^{-13}	1.6291×10^{-18}
500	1.9085×10^{-3}	1.2669×10^{-15}	2.6973×10^{-21}

It follows from (63) and (67) that

$$\lim_{n \rightarrow \infty} n^5 \left(l_n - \frac{\pi^2}{6} \right) = -\frac{1}{24}, \quad (86)$$

$$\lim_{n \rightarrow \infty} n^7 \left(L_n - \frac{\pi^2}{6} \right) = \frac{1}{45}.$$

Therefore, (85) provides a new approximation algorithm for π . Numerical simulations results carried out with mathematical software show that the given algorithm is more accurate than $\sqrt{6Q_n}$ (see Table 1).

More estimate methods for π can be found in [15–19].

Remark 20. The Catalan constant

$$G = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^2} = 0.9159655941772190 \dots \quad (87)$$

is a mysterious constant in mathematics and physics. From $\psi'(1/4) = \pi^2 + 8G$ and $\psi'(n+(1/4)+1) = \psi'(1/4) - \sum_{k=0}^n (1/(k+1/4)^2)$ (see [17, 20]) together with Theorem 1, we get

$$\frac{e^{1/(n+5/4)} - e^{-1/(n+1/4)}}{16} + \frac{1}{8} \sum_{k=0}^n \frac{1}{(k+1/4)^2} - \frac{\pi^2}{8} < G$$

$$< \frac{e^{2/(n+5/4)} - e^{-2/(n+1/4)}}{32} + \frac{1}{8} \sum_{k=0}^n \frac{1}{(k+1/4)^2} - \frac{\pi^2}{8} \quad (88)$$

for $n \in \mathbb{N}$.

Let

$$u_n = \frac{e^{1/(n+5/4)} - e^{-1/(n+1/4)}}{16} + \frac{1}{8} \sum_{k=0}^n \frac{1}{(k+1/4)^2} - \frac{\pi^2}{8},$$

$$U_n = \frac{e^{2/(n+5/4)} - e^{-2/(n+1/4)}}{32} + \frac{1}{8} \sum_{k=0}^n \frac{1}{(k+1/4)^2} - \frac{\pi^2}{8}. \quad (89)$$

Then inequalities (88) can be rewritten as

$$u_n < G < U_n. \quad (90)$$

It follows from (63) and (67) that we have

$$\lim_{n \rightarrow \infty} n^5 (u_n - G) = -\frac{1}{24},$$

$$\lim_{n \rightarrow \infty} n^7 (U_n - G) = \frac{1}{45}. \quad (91)$$

TABLE 2

n	$ \sum_{k=0}^n (-1)^k/(2k+1)^2 - G $	$ u_n - G $	$ U_n - G $
1	2.7077×10^{-2}	2.7274×10^{-4}	3.7885×10^{-5}
2	1.2923×10^{-2}	3.0912×10^{-5}	1.9591×10^{-6}
5	3.4037×10^{-3}	8.1456×10^{-7}	1.2782×10^{-8}
10	1.0268×10^{-3}	3.6098×10^{-8}	1.6524×10^{-10}
50	4.8045×10^{-5}	1.5468×10^{-11}	3.2017×10^{-15}
100	1.2253×10^{-5}	5.0171×10^{-13}	2.6358×10^{-17}
200	3.0939×10^{-6}	1.5974×10^{-14}	2.1139×10^{-19}
500	4.98×10^{-7}	1.6542×10^{-16}	3.5184×10^{-22}

Therefore, (90) provides a new approximation algorithm for the Catalan constant G . Numerical simulations results carried out with mathematical software show that the given algorithm is more accurate than $\sum_{k=0}^n ((-1)^k/(2k+1)^2)$ (see Table 2).

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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