Research Article

New Hermite-Hadamard Type Inequalities for n -Times Differentiable and s -Logarithmically Preinvex Functions

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The concept of s -logarithmically preinvex function is introduced, and by creating an integral identity involving an n -times differentiable function, some new Hermite-Hadamard type inequalities for s -logarithmically preinvex functions are established.

1. Introduction

Throughout this paper, let $\mathbb{R} = (-\infty, \infty)$, $\mathbb{R}_+ = (0, \infty)$, $\mathbb{N}$ denote the set of all positive integers, I denote the interval in $\mathbb{R}$, and A denote the set in $\mathbb{R}^n$, $n \in \mathbb{N}$.

Let us recall some definitions of various convex functions.

Definition 1. A function $f : I \subseteq \mathbb{R} \rightarrow \mathbb{R}$ is said to be convex if

$$f(\lambda x + (1 - \lambda)y) \leq \lambda f(x) + (1 - \lambda)f(y) \quad (1)$$

holds for all $x, y \in I$ and $\lambda \in [0, 1]$. If inequality (1) reverses, then f is said to be concave on I .

Definition 2 (see [1]). A set $A \subseteq \mathbb{R}^n$ is said to be invex with respect to the map $\eta : A \times A \rightarrow \mathbb{R}^n$, if for every $x, y \in A$ and $t \in [0, 1]$

$$y + t\eta(x, y) \in A. \quad (2)$$

The invex set A is also called a η -connected set.

It is obvious that every convex set is invex with respect to the map $\eta(x, y) = x - y$, but there exist invex sets which are not convex (see [1], e.g.).

Definition 3 (see [1]). Let $A \subseteq \mathbb{R}^n$ be an invex set with respect to $\eta : A \times A \rightarrow \mathbb{R}^n$. For every $x, y \in A$, the η -path P_{xy} joining the points x and $y = x + \eta(y, x)$ is defined by

$$P_{xy} = \{z \mid z = x + t\eta(y, x), t \in [0, 1]\}. \quad (3)$$

Definition 4 (see [2]). Let $A \subseteq \mathbb{R}^n$ be an invex set with respect to $\eta : A \times A \rightarrow \mathbb{R}^n$. A function $f : A \rightarrow \mathbb{R}$ is said to be preinvex with respect to η , if for every $x, y \in A$ and $t \in [0, 1]$

$$f(y + t\eta(x, y)) \leq tf(x) + (1 - t)f(y). \quad (4)$$

The function f is said to be preincave if and only if $-f$ is preinvex.

Every convex function is preinvex with respect to the map $\eta(x, y) = x - y$, but not conversely (see [2], e.g.).

Definition 5 (see [3]). Let $A \subseteq \mathbb{R}^n$ be an invex set with respect to $\eta : A \times A \rightarrow \mathbb{R}^n$. The function $f : A \rightarrow \mathbb{R}_+$ on the set A is said to be logarithmically preinvex with respect to η , if for every $x, y \in A$ and $t \in [0, 1]$

$$f(y + t\eta(x, y)) \leq [f(x)]^t [f(y)]^{1-t}. \quad (5)$$

For properties and applications of preinvex and logarithmically preinvex functions, please refer to [1–8] and closely related references therein.

The most important inequality in the theory of convex functions, the well known Hermite-Hadamard's integral inequality, may be stated as follows. Let $I \subseteq \mathbb{R}$ and $a, b \in I$ with $a < b$. If $f : [a, b] \subseteq I \rightarrow \mathbb{R}$ is a convex function on $[a, b]$, then

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b f(x) dx \leq \frac{f(a) + f(b)}{2}. \quad (6)$$

If f is concave on $[a, b]$, then inequality (6) is reversed.

Inequality (6) has been generalized by many mathematicians. Some of them may be recited as follows.

Theorem 6 (see [9, Theorem 2.2]). Let $f : I \subseteq \mathbb{R} \rightarrow \mathbb{R}$ be a differentiable mapping on I and $a, b \in I$ with $a < b$. If $|f'(x)|$ is convex on $[a, b]$, then

$$\left| \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(x) dx \right| \leq \frac{(b-a) \left[|f'(a)| + |f'(b)| \right]}{8}. \quad (7)$$

Theorem 7 (see [10, Theorem 1]). Let $I \subseteq \mathbb{R}$ and $a, b \in I$ with $a < b$. If $f : [a, b] \subseteq I \rightarrow \mathbb{R}$ is differentiable on $[a, b]$ such that $|f'(x)|^q$ is a convex function on $[a, b]$ for $q \geq 1$, then

$$\left| \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(x) dx \right| \leq \frac{b-a}{4} \left[\frac{|f'(a)|^q + |f'(b)|^q}{2} \right]^{1/q}. \quad (8)$$

Theorem 8 (see [11, Theorem 2.3]). Let $f : I \subseteq \mathbb{R} \rightarrow \mathbb{R}$ be differentiable on I , $a, b \in I$ with $a < b$ and $p > 1$. If $|f'(x)|^{p/(p-1)}$ is convex on $[a, b]$, then

$$\begin{aligned} & \left| f\left(\frac{a+b}{2}\right) - \frac{1}{b-a} \int_a^b f(x) dx \right| \\ & \leq \frac{b-a}{16} \left(\frac{4}{p+1} \right)^{1/p} \\ & \times \left\{ \left[|f'(a)|^{p/(p-1)} + 3|f'(b)|^{p/(p-1)} \right]^{1-1/p} \right. \\ & \left. + \left[3|f'(a)|^{p/(p-1)} + |f'(b)|^{p/(p-1)} \right]^{1-1/p} \right\}. \end{aligned} \quad (9)$$

Theorem 9 (see [6]). Let $A \subseteq \mathbb{R}$ be an open invex set with respect to $\eta : A \times A \rightarrow \mathbb{R}$ and $a, b \in A$ with $\eta(a, b) > 0$ for all $a \neq b$. If $f : A \rightarrow \mathbb{R}_+$ is a preinvex function on A , then the following inequality holds:

$$\begin{aligned} f\left(\frac{2b + \eta(a, b)}{2}\right) & \leq \frac{1}{\eta(a, b)} \int_b^{b+\eta(a, b)} f(x) dx \\ & \leq \frac{f(a) + f(b)}{2}. \end{aligned} \quad (10)$$

Theorem 10 (see [4, Theorem 4.3]). Let $A \subseteq \mathbb{R}$ be an open invex set with respect to $\eta : A \times A \rightarrow \mathbb{R}$ and $a, b \in A$ with $\eta(a, b) > 0$ for all $a \neq b$. Suppose that $f : A \rightarrow \mathbb{R}$ is a twice differentiable function on A and $|f''(x)|$ is preinvex on A . If $q > 1$ and f'' is integrable on the η -path P_{bc} for $c = b + \eta(a, b)$, then

$$\begin{aligned} & \left| \frac{f(b) + f(b + \eta(a, b))}{2} - \frac{1}{\eta(a, b)} \int_b^{b+\eta(a, b)} f(x) dx \right| \\ & \leq \frac{[\eta(a, b)]^2}{12} \left(\frac{1}{2} \right)^{1/q} \left[|f''(a)|^q + |f''(b)|^q \right]^{1/q}. \end{aligned} \quad (11)$$

Theorem 11 (see [12, Theorem 3.1]). For $n \in \mathbb{N}$ and $n \geq 2$, let $A \subseteq \mathbb{R}$ be an open invex set with respect to $\eta : A \times A \rightarrow \mathbb{R}$ and $a, b \in A$ with $\eta(a, b) > 0$ for all $a \neq b$. Suppose that $f : A \rightarrow \mathbb{R}$ is an n -times differentiable function on A and $f^{(n)}$ is integrable on the η -path P_{bc} for $c = b + \eta(a, b)$. If $|f^{(n)}|^q$ is preinvex on A for $q \geq 1$, then

$$\begin{aligned} & \left| \frac{f(b) + f(b + \eta(a, b))}{2} - \frac{1}{\eta(a, b)} \int_b^{b+\eta(a, b)} f(x) dx \right. \\ & \quad \left. + \sum_{k=1}^{n-1} \frac{[\eta(a, b)]^k (1-k)}{4[(k+1)!]} \right. \\ & \quad \left. \times [f^{(k)}(b) + (-1)^k f^{(k)}(b + \eta(a, b))] \right| \\ & \leq \frac{[\eta(a, b)]^n (n-1)^{1-1/q}}{4[(n+1)!] (n+2)^{1/q}} \\ & \quad \times \left\{ [n|f^{(n)}(a)|^q + (n^2-2)|f^{(n)}(b)|^q]^{1/q} \right. \\ & \quad \left. + [(n^2-2)|f^{(n)}(a)|^q + n|f^{(n)}(b)|^q]^{1/q} \right\}. \end{aligned} \quad (12)$$

Theorem 12 (see [13, Theorem 5]). For $n \in \mathbb{N}$ and $n \geq 2$, let $A \subseteq \mathbb{R}$ be an open invex set with respect to $\eta : A \times A \rightarrow \mathbb{R}$ and $a, b \in A$ with $\eta(a, b) > 0$ for all $a \neq b$. Suppose that $f : A \rightarrow \mathbb{R}$ is a function such that $f^{(n)}$ exists on A and $f^{(n)}$ is integrable on $[a, a + \eta(b, a)]$. If $|f^{(n)}|^q$ is logarithmically preinvex on A for $q \geq 1$, then we have the inequality

$$\begin{aligned} & \left| \frac{f(a) + f(a + \eta(b, a))}{2} - \frac{1}{\eta(b, a)} \int_a^{a+\eta(b, a)} f(x) dx \right. \\ & \quad \left. - \sum_{k=2}^{n-1} \frac{(-1)^k (k-1) [\eta(b, a)]^k}{2(k+1)!} f^{(k)}(a + \eta(b, a)) \right| \\ & \leq \frac{[\eta(b, a)]^n}{2n!} \left(\frac{n-1}{n+1} \right)^{1-1/q} [E_1(n, q)]^{1/q}, \end{aligned} \quad (13)$$

where

$$E_1(n, q) = \frac{(-1)^n n! \{q [\ln(|f^{(n)}(b)|) - \ln(|f^{(n)}(a)|)] + 2\} |f^{(n)}(a)|^q}{q^{n+1} [\ln(|f^{(n)}(b)|) - \ln(|f^{(n)}(a)|)]^{n+1}} - \frac{2|f^{(n)}(b)|^q}{q [\ln(|f^{(n)}(b)|) - \ln(|f^{(n)}(a)|)]} - n! |f^{(n)}(b)|^q \times \sum_{k=1}^n \frac{(-1)^k \{q [\ln(|f^{(n)}(b)|) - \ln(|f^{(n)}(a)|)] + 2\}}{(n-k)! q^{k+1} [\ln(|f^{(n)}(b)|) - \ln(|f^{(n)}(a)|)]^{k+1}}. \quad (14)$$

Recently, some related inequalities for preinvex functions were also obtained in [14, 15].

In the paper, the concept of s -logarithmically preinvex function is introduced, and by creating an integral identity involving an n -times differentiable function, some new Hermite-Hadamard type inequalities for s -logarithmically preinvex functions are established which generalize some known results.

2. New Definition and Lemmas

Now we introduce concepts of s -logarithmically preinvex functions.

Definition 13. Let $A \subseteq \mathbb{R}^n$ be an invex set with respect to $\eta : A \times A \rightarrow \mathbb{R}^n$. The function $f : A \rightarrow \mathbb{R}_+$ on the set A is said to be s -logarithmically preinvex with respect to η , if for every $x, y \in A, t \in [0, 1]$, and some $s \in (0, 1]$

$$f(y + t\eta(x, y)) \leq [f(x)]^t [f(y)]^{(1-t)^s}. \quad (15)$$

Clearly, when taking $s = 1$ in (15), then f becomes the standard logarithmically convex function on A .

In order to obtain our main results, we need the following lemmas.

Lemma 14. For $n \in \mathbb{N}$, let $A \subseteq \mathbb{R}$ be an open invex set with respect to $\eta : A \times A \rightarrow \mathbb{R}$ and $a, b \in A$ with $\eta(a, b) > 0$ for all $a \neq b$. If $f : A \rightarrow \mathbb{R}$ is an n -times differentiable function on A and $f^{(n)}$ is integrable on the η -path P_{bc} for $c = b + \eta(a, b)$, then

$$\begin{aligned} & \frac{f(b) + f(b + \eta(a, b))}{2} - \frac{1}{\eta(a, b)} \int_b^{b+\eta(a, b)} f(x) dx \\ & + \sum_{k=2}^n \frac{1}{2k! \eta(a, b)} \\ & \times \left[(b-x)^{k-1} (2x - 2b - k\eta(a, b)) + (b + \eta(a, b) - x)^{k-1} \right. \\ & \quad \left. \times (2b + 2\eta(a, b) - 2x - k\eta(a, b)) \right] f^{(k-1)}(x) \\ & = \frac{[\eta(a, b)]^n}{2n!} \end{aligned}$$

$$\begin{aligned} & \times \left[\int_0^{(x-b)/(\eta(a, b))} (-t)^{n-1} (2t - n) f^{(n)}(b + t\eta(a, b)) dt \right. \\ & \quad \left. + \int_{(x-b)/(\eta(a, b))}^1 (1-t)^{n-1} (2t + n - 2) \right. \\ & \quad \left. \times f^{(n)}(b + t\eta(a, b)) dt \right], \quad (16) \end{aligned}$$

where $x \in [b, b + \eta(a, b)]$ and the above summation is zero for $n = 1$.

Proof. Since $a, b \in A$ and A is an invex set with respect to η , for every $t \in [0, 1]$, we have $b + t\eta(a, b) \in A$. When $n = 1$, by integrating by part in the right-hand side of (16), one gives

$$\begin{aligned} & \frac{\eta(a, b)}{2} \left[\int_0^1 (2t - 1) f'(b + t\eta(a, b)) dt \right] \\ & = \frac{f(b) + f(b + \eta(a, b))}{2} - \frac{1}{\eta(a, b)} \int_b^{b+\eta(a, b)} f(x) dx. \quad (17) \end{aligned}$$

Hence, the identity (16) holds for $n = 1$.

When $n = m$ and $m \geq 1$, suppose that the identity (16) is valid.

When $n = m + 1$, by the hypothesis, we have

$$\begin{aligned} & \frac{[\eta(a, b)]^{m+1}}{2(m+1)!} \\ & \times \left[\int_0^{(x-b)/(\eta(a, b))} (-t)^m (2t - m - 1) \right. \\ & \quad \times f^{(m+1)}(b + t\eta(a, b)) dt \\ & \quad \left. + \int_{(x-b)/(\eta(a, b))}^1 (1-t)^m (2t + m - 1) \right. \\ & \quad \left. \times f^{(m+1)}(b + t\eta(a, b)) dt \right] \\ & = \frac{[\eta(a, b)]^m}{2(m+1)!} \\ & \times \left[\left(\frac{b-x}{\eta(a, b)} \right)^m \left(\frac{2x-2b}{\eta(a, b)} - m - 1 \right) f^{(m)}(x) \right. \\ & \quad \left. - \int_0^{(x-b)/(\eta(a, b))} (-t)^{m-1} (m+1)(m-2t) \right. \\ & \quad \left. \times f^{(m)}(b + t\eta(a, b)) dt \right] \end{aligned}$$

$$\begin{aligned}
& - \left(\frac{b + \eta(a, b) - x}{\eta(a, b)} \right)^m \left(\frac{2x - 2b}{\eta(a, b)} + m - 1 \right) f^{(m)}(x) \\
& - \int_{(x-b)/(\eta(a, b))}^1 (1-t)^{m-1} (m+1) \\
& \quad \times (2-2t-m) f^{(m)}(b+t\eta(a, b)) dt \Big] \\
& = \frac{1}{2(m+1)!\eta(a, b)} \\
& \quad \times \left\{ (b-x)^m [2x-2b-(m+1)\eta(a, b)] \right. \\
& \quad \left. - (b+\eta(a, b)-x)^m [2x-2b+(m-1)\eta(a, b)] \right\} \\
& \quad \times f^{(m)}(x) + \frac{[\eta(a, b)]^m}{2m!} \\
& \quad \times \left[\int_0^{(x-b)/(\eta(a, b))} (-t)^{m-1} (2t-m) f^{(m)}(b+t\eta(a, b)) dt \right. \\
& \quad \left. + \int_{(x-b)/(\eta(a, b))}^1 (1-t)^{m-1} (2t+m-2) \right. \\
& \quad \left. \times f^{(m)}(b+t\eta(a, b)) dt \right] \\
& = \frac{f(b) + f(b+\eta(a, b))}{2} - \frac{1}{\eta(a, b)} \int_b^{b+\eta(a, b)} f(x) dx \\
& \quad + \sum_{k=2}^{m+1} \frac{1}{2k!\eta(a, b)} \\
& \quad \times \left[(b-x)^{k-1} (2x-2b-k\eta(a, b)) \right. \\
& \quad \left. + (b+\eta(a, b)-x)^{k-1} (2b+2\eta(a, b)-2x-k\eta(a, b)) \right] \\
& \quad \times f^{(k-1)}(x). \tag{18}
\end{aligned}$$

Therefore, when $n = m + 1$, the identity (16) holds. By induction, the proof of Lemma 14 is complete. $\square$

Remark 15. Under the conditions of Lemma 14, we have

$$\begin{aligned}
& \frac{f(b) + f(b+\eta(a, b))}{2} - \frac{1}{\eta(a, b)} \int_b^{b+\eta(a, b)} f(x) dx \\
& + \sum_{k=2}^n \frac{[\eta(a, b)]^{k-1} (2-k)}{2k!} f^{(k-1)}(b) \\
& = \frac{[\eta(a, b)]^n}{2n!} \int_0^1 (1-t)^{n-1} (2t+n-2) \\
& \quad \times f^{(n)}(b+t\eta(a, b)) dt, \\
& \frac{f(b) + f(b+\eta(a, b))}{2} - \frac{1}{\eta(a, b)} \int_b^{b+\eta(a, b)} f(x) dx
\end{aligned} \tag{19}$$

$$\begin{aligned}
& + \sum_{k=2}^n \frac{[\eta(a, b)]^{k-1} (1-k) [1+(-1)^{k-1}]}{2^k k!} \\
& \quad \times f^{(k-1)} \left(b + \frac{\eta(a, b)}{2} \right) \\
& = \frac{[\eta(a, b)]^n}{2n!} \\
& \quad \times \left[\int_0^{1/2} (-t)^{n-1} (2t-n) f^{(n)}(b+t\eta(a, b)) dt \right. \\
& \quad \left. + \int_{1/2}^1 (1-t)^{n-1} (2t+n-2) f^{(n)}(b+t\eta(a, b)) dt \right], \tag{20}
\end{aligned}$$

$$\begin{aligned}
& \frac{f(b) + f(b+\eta(a, b))}{2} - \frac{1}{\eta(a, b)} \int_b^{b+\eta(a, b)} f(x) dx \\
& + \sum_{k=2}^n \frac{[-\eta(a, b)]^{k-1} (2-k)}{2k!} f^{(k-1)}(b+\eta(a, b)) \\
& = \frac{[\eta(a, b)]^n}{2n!} \int_0^1 (-t)^{n-1} (2t-n) f^{(n)}(b+t\eta(a, b)) dt. \tag{21}
\end{aligned}$$

Proof. These are special cases of Lemma 14 for $x = b, b + \eta(a, b)/2, b + \eta(a, b)$, respectively. $\square$

Remark 16. Adding the identities (19) and (21) and then dividing by 2 result in Lemma 14 from [12].

Lemma 17. Let $\mu > 0$ and $x \geq 0$. Then

$$\begin{aligned}
E(n; \mu, x) & \triangleq \int_0^x t^n \mu^t dt \\
& = \begin{cases} \frac{(-1)^{n+1} n!}{(\ln \mu)^{n+1}} + n! \mu^x \sum_{k=0}^n \frac{(-1)^k x^{n-k}}{(n-k)! (\ln \mu)^{k+1}}, & \mu \neq 1, \\ \frac{x^{n+1}}{n+1}, & \mu = 1 \end{cases} \tag{22}
\end{aligned}$$

for $n \geq 0, n \in \mathbb{N}$.

Proof. When $\mu = 1$, the proof is straightforward.

When $\mu \neq 1$, for $n = 0$, we have

$$\int_0^x \mu^t dt = \frac{\mu^x - 1}{\ln \mu}, \tag{23}$$

which coincides with the right-hand side of (22) for $n = 0$.

For $n = 1$, we get

$$\begin{aligned}
\int_0^x t \mu^t dt & = \frac{x \mu^x}{\ln \mu} - \frac{1}{\ln \mu} \int_0^x \mu^t dt \\
& = \frac{1}{(\ln \mu)^2} + \mu^x \left[\frac{x}{\ln \mu} - \frac{1}{(\ln \mu)^2} \right], \tag{24}
\end{aligned}$$

which coincides with the right-hand side of (22) for $n = 1$.

Suppose that (22) is true for $n = m$, $m \geq 0$, then, for $n = m + 1$, it follows that

$$\begin{aligned} \int_0^x t^{m+1} \mu^t dt &= \frac{x^{m+1} \mu^x}{\ln \mu} - \frac{m+1}{\ln \mu} \int_0^x t^m \mu^t dt \\ &= \frac{x^{m+1} \mu^x}{\ln \mu} - \frac{m+1}{\ln \mu} \left[\frac{(-1)^{m+1} m!}{(\ln \mu)^{m+1}} \right. \\ &\quad \left. + m! \mu^x \sum_{k=0}^m \frac{(-1)^k x^{m-k}}{(m-k)! (\ln \mu)^{k+1}} \right] \\ &= \frac{(-1)^{m+2} (m+1)!}{(\ln \mu)^{m+2}} + \frac{x^{m+1} \mu^x}{\ln \mu} \\ &\quad + (m+1)! \mu^x \sum_{k=1}^{m+1} \frac{(-1)^k x^{m-k+1}}{(m-k+1)! (\ln \mu)^{k+1}} \\ &= \frac{(-1)^{m+2} (m+1)!}{(\ln \mu)^{m+2}} + (m+1)! \mu^x \sum_{k=0}^{m+1} \frac{(-1)^k x^{m-k+1}}{(m-k+1)! (\ln \mu)^{k+1}}. \end{aligned} \quad (25)$$

Therefore, when $n = m + 1$, the identity (22) holds. By induction, the proof of Lemma 17 is complete. $\square$

Lemma 18 (see [16]). Let $\gamma > 0$, $\mu > 0$, and $x \geq 0$. Then

$$G(\gamma; \mu, x) \triangleq \int_0^x t^{\gamma-1} \mu^t dt = x^\gamma \mu^x \sum_{k=1}^{\infty} \frac{(-x \ln \mu)^{k-1}}{(\gamma)_k} < \infty, \quad (26)$$

where $(\gamma)_k = \gamma(\gamma+1)(\gamma+2) \cdots (\gamma+k-1)$.

By Lemmas 17 and 18, a straightforward computation gives the following lemmas.

Lemma 19. Let $\mu > 0$ and $x \geq 0$. Then

$$\begin{aligned} F(n; \mu, x) &\triangleq nE(n-1; \mu, x) - 2E(n; \mu, x) \\ &= \begin{cases} \frac{2\mu^x x^n}{\ln \mu} + n! (\ln \mu + 2) \\ \quad \times \left[\frac{(-1)^n}{(\ln \mu)^{n+1}} + \mu^x \sum_{k=1}^n \frac{(-1)^{k-1} x^{n-k}}{(n-k)! (\ln \mu)^{k+1}} \right], & \mu \neq 1, \\ \frac{x^n (n+1-2x)}{n+1}, & \mu = 1 \end{cases} \end{aligned} \quad (27)$$

for $n \geq 0$, $n \in \mathbb{N}$.

Lemma 20. Let $\gamma > 0$, $\mu > 0$, and $x \geq 0$. Then

$$\begin{aligned} H(\gamma; \mu, x) &\triangleq nG(\gamma; \mu, x) - 2G(\gamma+1; \mu, x) \\ &= x^\gamma \mu^x \sum_{k=1}^{\infty} \frac{(-x \ln \mu)^{k-1} [n(\gamma+k) - 2x\gamma]}{(\gamma)_{k+1}}, \end{aligned} \quad (28)$$

where $(\gamma)_{k+1} = \gamma(\gamma+1)(\gamma+2) \cdots (\gamma+k)$.

Lemma 21 (see [17]). Let $0 < \phi \leq 1 \leq \psi$ and $0 < t, s \leq 1$. Then

$$\begin{aligned} \phi^{t^s} &\leq \phi^{st}, \\ \psi^{t^s} &\leq \psi^{st+1-s}. \end{aligned} \quad (29)$$

3. Hermite-Hadamard Type Inequalities

Now we start out to establish some new Hermite-Hadamard type inequalities for n -times differentiable and s -logarithmically preinvex functions.

Theorem 22. For $n \in \mathbb{N}$ and $n \geq 2$, let $A \subseteq \mathbb{R}$ be an open invex set with respect to $\eta : A \times A \rightarrow \mathbb{R}$ and $a, b \in A$ with $\eta(a, b) > 0$ for all $a \neq b$. Suppose that $f : A \rightarrow \mathbb{R}$ is an n -times differentiable function on A and $f^{(n)}$ is integrable on the η -path P_{bc} for $c = b + \eta(a, b)$. If $|f^{(n)}|^q$ is s -logarithmically preinvex on A for $q \geq 1$, then for $x \in [b, b + \eta(a, b)]$ and some $s \in (0, 1]$

$$\begin{aligned} &\left| \frac{f(b) + f(b + \eta(a, b))}{2} - \frac{1}{\eta(a, b)} \int_b^{b+\eta(a, b)} f(x) dx \right. \\ &\quad + \sum_{k=2}^n \frac{1}{2k! \eta(a, b)} \\ &\quad \times \left[(b-x)^{k-1} (2x - 2b - k\eta(a, b)) \right. \\ &\quad \left. + (b + \eta(a, b) - x)^{k-1} (2b + 2\eta(a, b) - 2x - k\eta(a, b)) \right] \\ &\quad \times f^{(k-1)}(x) \Big| \\ &\leq \frac{[\eta(a, b)]^{(n+1-q)/q}}{2n!(n+1)^{1-1/q}} \\ &\quad \times \left\{ |f^{(n)}(a)|^\alpha |f^{(n)}(b)|^\beta \right. \\ &\quad \times [(x-b)^n [(n+1)\eta(a, b) - 2x + 2b]]^{1-1/q} \\ &\quad \times \left[F\left(n; \xi^{sq}, \frac{x-b}{\eta(a, b)}\right) \right]^{1/q} + |f^{(n)}(a)|^\delta |f^{(n)}(b)|^\theta \\ &\quad \times [(b + \eta(a, b) - x)^n [(n-1)\eta(a, b) + 2x - 2b]]^{1-1/q} \\ &\quad \times \left[F\left(n; \xi^{-sq}, \frac{b + \eta(a, b) - x}{\eta(a, b)}\right) \right]^{1/q} \Big\}, \end{aligned} \quad (30)$$

where $\xi = |f^{(n)}(a)/f^{(n)}(b)|$, $F(n; \mu, x)$ is defined in Lemma 19, and

$$\begin{aligned} & \{\alpha, \beta\} \\ &= \begin{cases} \{0, s\}, & \text{if } 0 < |f^{(n)}(a)|, |f^{(n)}(b)| \leq 1, \\ \{0, 1\}, & \text{if } 0 < |f^{(n)}(a)| < 1 < |f^{(n)}(b)|, \\ \{1-s, s\}, & \text{if } 0 < |f^{(n)}(b)| < 1 < |f^{(n)}(a)|, \\ \{1-s, 1\}, & \text{if } |f^{(n)}(a)|, |f^{(n)}(b)| > 1, \end{cases} \\ & \{\delta, \theta\} \\ &= \begin{cases} \{s, 0\}, & \text{if } 0 < |f^{(n)}(a)|, |f^{(n)}(b)| \leq 1, \\ \{s, 1-s\}, & \text{if } 0 < |f^{(n)}(a)| < 1 < |f^{(n)}(b)|, \\ \{1, 0\}, & \text{if } 0 < |f^{(n)}(b)| < 1 < |f^{(n)}(a)|, \\ \{1, 1-s\}, & \text{if } |f^{(n)}(a)|, |f^{(n)}(b)| > 1. \end{cases} \end{aligned} \quad (31)$$

Proof. Since $a, b \in A$ and A is an invex set with respect to η , for every $t \in [0, 1]$, we have $b + t\eta(a, b) \in A$. Using Lemma 14, Hölder's inequality, and s -logarithmically preinvexity of $|f^{(n)}|^q$, it yields that

$$\begin{aligned} & \left| \frac{f(b) + f(b + \eta(a, b))}{2} - \frac{1}{\eta(a, b)} \int_b^{b+\eta(a, b)} f(x) dx \right. \\ &+ \sum_{k=2}^n \frac{1}{2k! \eta(a, b)} \\ & \times \left[(b-x)^{k-1} (2x - 2b - k\eta(a, b)) \right. \\ & \left. + (b + \eta(a, b) - x)^{k-1} (2b + 2\eta(a, b) - 2x - k\eta(a, b)) \right] \\ & \times f^{(k-1)}(x) \Big| \\ & \leq \frac{[\eta(a, b)]^n}{2n!} \\ & \times \left[\int_0^{(x-b)/(\eta(a, b))} t^{n-1} (n-2t) |f^{(n)}(b + t\eta(a, b))| dt \right. \\ & \left. + \int_{(x-b)/(\eta(a, b))}^1 (1-t)^{n-1} (2t + n-2) \right. \\ & \left. \times |f^{(n)}(b + t\eta(a, b))| dt \right] \end{aligned}$$

$$\begin{aligned} & \leq \frac{[\eta(a, b)]^n}{2n!} \\ & \times \left\{ \left[\int_0^{(x-b)/(\eta(a, b))} t^{n-1} (n-2t) dt \right]^{1-1/q} \right. \\ & \times \left[\int_0^{(x-b)/(\eta(a, b))} t^{n-1} (n-2t) |f^{(n)}(b)|^{q(1-t)^s} \right. \\ & \times |f^{(n)}(a)|^{qt^s} dt \Big]^{1/q} \\ & + \left[\int_{(x-b)/(\eta(a, b))}^1 (1-t)^{n-1} (2t + n-2) dt \right]^{1-1/q} \\ & \times \left[\int_{(x-b)/(\eta(a, b))}^1 (1-t)^{n-1} (2t + n-2) \right. \\ & \times |f^{(n)}(b)|^{q(1-t)^s} |f^{(n)}(a)|^{qt^s} dt \Big]^{1/q} \Big\}. \end{aligned} \quad (32)$$

By Lemmas 17 and 21, for $0 < |f^{(n)}(a)|, |f^{(n)}(b)| \leq 1$, we give

$$\begin{aligned} & \int_0^{(x-b)/(\eta(a, b))} t^{n-1} (n-2t) |f^{(n)}(b)|^{q(1-t)^s} |f^{(n)}(a)|^{qt^s} dt \\ & \leq \int_0^{(x-b)/(\eta(a, b))} t^{n-1} (n-2t) |f^{(n)}(b)|^{sq(1-t)} |f^{(n)}(a)|^{sq t} dt \\ & = |f^{(n)}(b)|^{sq} \left[nE \left(n-1; \xi^{sq}, \frac{x-b}{\eta(a, b)} \right) \right. \\ & \quad \left. - 2E \left(n; \xi^{sq}, \frac{x-b}{\eta(a, b)} \right) \right], \end{aligned} \quad (33)$$

$$\begin{aligned} & \int_{(x-b)/(\eta(a, b))}^1 (1-t)^{n-1} (2t + n-2) |f^{(n)}(b)|^{q(1-t)^s} \\ & \times |f^{(n)}(a)|^{qt^s} dt \\ & \leq \int_{(x-b)/(\eta(a, b))}^1 (1-t)^{n-1} (2t + n-2) |f^{(n)}(b)|^{sq(1-t)} \\ & \times |f^{(n)}(a)|^{sq t} dt \\ & = \int_0^{(b+\eta(a, b)-x)/(\eta(a, b))} t^{n-1} (n-2t) |f^{(n)}(b)|^{sq t} |f^{(n)}(a)|^{sq(1-t)} dt \end{aligned}$$

$$= \left| f^{(n)}(a) \right|^{sq} \left[nE \left(n-1; \xi^{-sq}, \frac{b+\eta(a,b)-x}{\eta(a,b)} \right) - 2E \left(n; \xi^{-sq}, \frac{b+\eta(a,b)-x}{\eta(a,b)} \right) \right]. \quad (34)$$

For $0 < |f^{(n)}(a)| < 1 < |f^{(n)}(b)|$, we get

$$\begin{aligned} & \int_0^{(x-b)/(\eta(a,b))} t^{n-1} (n-2t) \left| f^{(n)}(b) \right|^{q(1-t)^s} \left| f^{(n)}(a) \right|^{qt^s} dt \\ & \leq \int_0^{(x-b)/(\eta(a,b))} t^{n-1} (n-2t) \left| f^{(n)}(b) \right|^{q[s(1-t)+1-s]} \left| f^{(n)}(a) \right|^{sq t} dt \\ & = \left| f^{(n)}(b) \right|^q \left[nE \left(n-1; \xi^{sq}, \frac{x-b}{\eta(a,b)} \right) - 2E \left(n; \xi^{sq}, \frac{x-b}{\eta(a,b)} \right) \right], \end{aligned} \quad (35)$$

$$\begin{aligned} & \int_{(x-b)/(\eta(a,b))}^1 (1-t)^{n-1} (2t+n-2) \left| f^{(n)}(b) \right|^{q(1-t)^s} \left| f^{(n)}(a) \right|^{qt^s} dt \\ & \leq \int_{(x-b)/(\eta(a,b))}^1 (1-t)^{n-1} (2t+n-2) \left| f^{(n)}(b) \right|^{q[s(1-t)+1-s]} \\ & \quad \times \left| f^{(n)}(a) \right|^{sq t} dt \\ & = \int_0^{(b+\eta(a,b)-x)/(\eta(a,b))} t^{n-1} (n-2t) \left| f^{(n)}(b) \right|^{q(st+1-s)} \\ & \quad \times \left| f^{(n)}(a) \right|^{sq(1-t)} dt \\ & = \left| f^{(n)}(a) \right|^{sq} \left| f^{(n)}(b) \right|^{q(1-s)} \left[nE \left(n-1; \xi^{-sq}, \frac{b+\eta(a,b)-x}{\eta(a,b)} \right) - 2E \left(n; \xi^{-sq}, \frac{b+\eta(a,b)-x}{\eta(a,b)} \right) \right]. \end{aligned} \quad (36)$$

For $0 < |f^{(n)}(b)| < 1 < |f^{(n)}(a)|$, we obtain

$$\begin{aligned} & \int_0^{(x-b)/(\eta(a,b))} t^{n-1} (n-2t) \left| f^{(n)}(b) \right|^{q(1-t)^s} \left| f^{(n)}(a) \right|^{qt^s} dt \\ & \leq \int_0^{(x-b)/(\eta(a,b))} t^{n-1} (n-2t) \left| f^{(n)}(b) \right|^{qs(1-t)} \left| f^{(n)}(a) \right|^{q(st+1-s)} dt \\ & = \left| f^{(n)}(b) \right|^{qs} \left| f^{(n)}(a) \right|^{q(1-s)} \left[nE \left(n-1; \xi^{sq}, \frac{x-b}{\eta(a,b)} \right) - 2E \left(n; \xi^{sq}, \frac{x-b}{\eta(a,b)} \right) \right], \end{aligned}$$

$$\begin{aligned} & \int_{(x-b)/(\eta(a,b))}^1 (1-t)^{n-1} (2t+n-2) \left| f^{(n)}(b) \right|^{q(1-t)^s} \left| f^{(n)}(a) \right|^{qt^s} dt \\ & \leq \int_{(x-b)/(\eta(a,b))}^1 (1-t)^{n-1} (2t+n-2) \left| f^{(n)}(b) \right|^{qs(1-t)} \\ & \quad \times \left| f^{(n)}(a) \right|^{q(st+1-s)} dt \\ & = \int_0^{(b+\eta(a,b)-x)/(\eta(a,b))} t^{n-1} (n-2t) \left| f^{(n)}(b) \right|^{qst} \\ & \quad \times \left| f^{(n)}(a) \right|^{q[s(1-t)+1-s]} dt \\ & = \left| f^{(n)}(a) \right|^q \left[nE \left(n-1; \xi^{-sq}, \frac{b+\eta(a,b)-x}{\eta(a,b)} \right) - 2E \left(n; \xi^{-sq}, \frac{b+\eta(a,b)-x}{\eta(a,b)} \right) \right]. \end{aligned} \quad (37)$$

For $|f^{(n)}(a)|, |f^{(n)}(b)| > 1$, we have

$$\begin{aligned} & \int_0^{(x-b)/(\eta(a,b))} t^{n-1} (n-2t) \left| f^{(n)}(b) \right|^{q(1-t)^s} \left| f^{(n)}(a) \right|^{qt^s} dt \\ & \leq \int_0^{(x-b)/(\eta(a,b))} t^{n-1} (n-2t) \left| f^{(n)}(b) \right|^{q[s(1-t)+1-s]} \\ & \quad \times \left| f^{(n)}(a) \right|^{q(st+1-s)} dt \\ & = \left| f^{(n)}(b) \right|^q \left| f^{(n)}(a) \right|^{q(1-s)} \left[nE \left(n-1; \xi^{sq}, \frac{x-b}{\eta(a,b)} \right) - 2E \left(n; \xi^{sq}, \frac{x-b}{\eta(a,b)} \right) \right], \end{aligned} \quad (38)$$

$$\begin{aligned} & \int_{(x-b)/(\eta(a,b))}^1 (1-t)^{n-1} (2t+n-2) \left| f^{(n)}(b) \right|^{q(1-t)^s} \left| f^{(n)}(a) \right|^{qt^s} dt \\ & \leq \int_{(x-b)/(\eta(a,b))}^1 (1-t)^{n-1} (2t+n-2) \\ & \quad \times \left| f^{(n)}(b) \right|^{q[s(1-t)+1-s]} \left| f^{(n)}(a) \right|^{q(st+1-s)} dt \\ & = \int_0^{(b+\eta(a,b)-x)/(\eta(a,b))} t^{n-1} (n-2t) \\ & \quad \times \left| f^{(n)}(b) \right|^{q(st+1-s)} \left| f^{(n)}(a) \right|^{q[s(1-t)+1-s]} dt \\ & = \left| f^{(n)}(a) \right|^q \left| f^{(n)}(b) \right|^{q(1-s)} \left[nE \left(n-1; \xi^{-sq}, \frac{b+\eta(a,b)-x}{\eta(a,b)} \right) - 2E \left(n; \xi^{-sq}, \frac{b+\eta(a,b)-x}{\eta(a,b)} \right) \right]. \end{aligned} \quad (39)$$

Using Lemma 19 and substituting (33) to (39) into (32), we get inequality (30).

Theorem 22 is thus proved. $\square$

Corollary 23. Under the assumptions of Theorem 22,

(1) if $q = 1$, then

$$\begin{aligned} & \left| \frac{f(b) + f(b + \eta(a, b))}{2} - \frac{1}{\eta(a, b)} \int_b^{b+\eta(a, b)} f(x) dx \right. \\ & \quad \left. + \sum_{k=2}^n \frac{1}{2k! \eta(a, b)} \right. \\ & \quad \times \left[(b-x)^{k-1} (2x - 2b - k\eta(a, b)) + (b + \eta(a, b) - x)^{k-1} \right. \\ & \quad \left. \times (2b + 2\eta(a, b) - 2x - k\eta(a, b)) \right] f^{(k-1)}(x) \Big| \\ & \leq \frac{[\eta(a, b)]^n}{2n!} \left\{ |f^{(n)}(a)|^\alpha |f^{(n)}(b)|^\beta \left[F\left(n; \xi^s, \frac{x-b}{\eta(a, b)}\right) \right] \right. \\ & \quad \left. + |f^{(n)}(a)|^\delta |f^{(n)}(b)|^\theta \right. \\ & \quad \left. \times \left[F\left(n; \xi^{-s}, \frac{b + \eta(a, b) - x}{\eta(a, b)}\right) \right] \right\}; \end{aligned} \quad (40)$$

(2) if $s = 1$, then

$$\begin{aligned} & \left| \frac{f(b) + f(b + \eta(a, b))}{2} - \frac{1}{\eta(a, b)} \int_b^{b+\eta(a, b)} f(x) dx \right. \\ & \quad \left. + \sum_{k=2}^n \frac{1}{2k! \eta(a, b)} \right. \\ & \quad \times \left[(b-x)^{k-1} (2x - 2b - k\eta(a, b)) + (b + \eta(a, b) - x)^{k-1} \right. \\ & \quad \left. \times (2b + 2\eta(a, b) - 2x - k\eta(a, b)) \right] f^{(k-1)}(x) \Big| \\ & \leq \frac{[\eta(a, b)]^{(n+1-q)/q}}{2n!(n+1)^{1-1/q}} \\ & \quad \times \left\{ |f^{(n)}(b)| \left[(x-b)^n [(n+1)\eta(a, b) - 2x + 2b] \right]^{1-1/q} \right. \\ & \quad \left. \times \left[F\left(n; \xi^q, \frac{x-b}{\eta(a, b)}\right) \right]^{1/q} + |f^{(n)}(a)| \right. \end{aligned}$$

$$\begin{aligned} & \times \left[(b + \eta(a, b) - x)^n [(n-1)\eta(a, b) + 2x - 2b] \right]^{1-1/q} \\ & \times \left[F\left(n; \xi^{-q}, \frac{b + \eta(a, b) - x}{\eta(a, b)}\right) \right]^{1/q} \Big\}. \end{aligned} \quad (41)$$

Theorem 24. For $n \in \mathbb{N}$ and $n \geq 2$, let $A \subseteq \mathbb{R}$ be an open invex set with respect to $\eta : A \times A \rightarrow \mathbb{R}$ and $a, b \in A$ with $\eta(a, b) > 0$ for all $a \neq b$. Suppose that $f : A \rightarrow \mathbb{R}$ is an n -times differentiable function on A and $f^{(n)}$ is integrable on the η -path P_{bc} for $c = b + \eta(a, b)$. If $|f^{(n)}|^q$ is s -logarithmically preinvex on A for $q > 1$, then for $x \in [b, b + \eta(a, b)]$ and some $s \in (0, 1]$

$$\begin{aligned} & \left| \frac{f(b) + f(b + \eta(a, b))}{2} - \frac{1}{\eta(a, b)} \int_b^{b+\eta(a, b)} f(x) dx \right. \\ & \quad \left. + \sum_{k=2}^n \frac{1}{2k! \eta(a, b)} \right. \\ & \quad \times \left[(b-x)^{k-1} (2x - 2b - k\eta(a, b)) + (b + \eta(a, b) - x)^{k-1} \right. \\ & \quad \left. \times (2b + 2\eta(a, b) - 2x - k\eta(a, b)) \right] f^{(k-1)}(x) \Big| \\ & \leq \frac{[\eta(a, b)]^{n-2+1/q}}{2^{2-1/q} n!} \left(\frac{1-q}{2q-1} \right)^{1-1/q} \\ & \quad \times \left\{ \left[(n\eta(a, b) - 2x + 2b)^{(2q-1)/(q-1)} \right. \right. \\ & \quad \left. \left. - (n\eta(a, b))^{(2q-1)/(q-1)} \right]^{1-1/q} |f^{(n)}(a)|^\alpha |f^{(n)}(b)|^\beta \right. \\ & \quad \times \left[G\left(nq - q + 1; \xi^{sq}, \frac{x-b}{\eta(a, b)}\right) \right]^{1/q} \\ & \quad + \left[((n-2)\eta(a, b) + 2x - 2b)^{(2q-1)/(q-1)} \right. \\ & \quad \left. - (n\eta(a, b))^{(2q-1)/(q-1)} \right]^{1-1/q} \\ & \quad \times |f^{(n)}(a)|^\delta |f^{(n)}(b)|^\theta \\ & \quad \times \left[G\left(nq - q + 1; \xi^{-sq}, \frac{b + \eta(a, b) - x}{\eta(a, b)}\right) \right]^{1/q} \Big\}, \end{aligned} \quad (42)$$

where ξ , $\{\alpha, \beta\}$ and $\{\delta, \theta\}$ are given in Theorem 22 and $G(\gamma; \mu, x)$ is defined by (26).

Proof. Since $a, b \in A$ and A is an invex set with respect to η , for every $t \in [0, 1]$, we have $b + t\eta(a, b) \in A$.

Using Lemma 14, Hölder's inequality, and s -logarithmically preinvexity of $|f^{(n)}|^q$, it follows that

$$\begin{aligned} & \left| \frac{f(b) + f(b + \eta(a, b))}{2} - \frac{1}{\eta(a, b)} \int_b^{b+\eta(a, b)} f(x) dx \right. \\ & + \sum_{k=2}^n \frac{1}{2k! \eta(a, b)} \\ & \times \left[(b-x)^{k-1} (2x - 2b - k\eta(a, b)) + (b + \eta(a, b) - x)^{k-1} \right. \\ & \left. \left. \times (2b + 2\eta(a, b) - 2x - k\eta(a, b)) \right] f^{(k-1)}(x) \right| \\ & \leq \frac{[\eta(a, b)]^n}{2n!} \\ & \times \left\{ \left[\int_0^{(x-b)/(\eta(a, b))} (n-2t)^{q/(q-1)} dt \right]^{1-1/q} \right. \\ & \times \left[\int_0^{(x-b)/(\eta(a, b))} t^{(n-1)q} |f^{(n)}(b)|^{q(1-t)^s} |f^{(n)}(a)|^{qt^s} dt \right]^{1/q} \\ & + \left[\int_{(x-b)/(\eta(a, b))}^1 (2t + n - 2)^{q/(q-1)} dt \right]^{1-1/q} \\ & \times \left[\int_{(x-b)/(\eta(a, b))}^1 (1-t)^{(n-1)q} |f^{(n)}(b)|^{q(1-t)^s} \right. \\ & \left. \left. \times |f^{(n)}(a)|^{qt^s} dt \right]^{1/q} \right\}. \end{aligned} \quad (43)$$

The rest is the same as the proof of Theorem 22. $\square$

Corollary 25. Under the assumptions of Theorem 24, if $s = 1$, then

$$\begin{aligned} & \left| \frac{f(b) + f(b + \eta(a, b))}{2} - \frac{1}{\eta(a, b)} \int_b^{b+\eta(a, b)} f(x) dx \right. \\ & + \sum_{k=2}^n \frac{1}{2k! \eta(a, b)} \\ & \times \left[(b-x)^{k-1} (2x - 2b - k\eta(a, b)) + (b + \eta(a, b) - x)^{k-1} \right. \\ & \left. \left. \times (2b + 2\eta(a, b) - 2x - k\eta(a, b)) \right] f^{(k-1)}(x) \right| \\ & \leq \frac{[\eta(a, b)]^{n-2+1/q}}{2^{2-1/q} n!} \left(\frac{1-q}{2q-1} \right)^{1-1/q} \\ & \times \left\{ \left[(n\eta(a, b) - 2x + 2b)^{(2q-1)/(q-1)} \right. \right. \\ & \left. \left. - (n\eta(a, b))^{(2q-1)/(q-1)} \right]^{1-1/q} |f^{(n)}(b)| \right. \end{aligned}$$

$$\begin{aligned} & \times \left[G\left(nq - q + 1; \xi^q, \frac{x-b}{\eta(a, b)}\right) \right]^{1/q} \\ & + \left[((n-2)\eta(a, b) + 2x - 2b)^{(2q-1)/(q-1)} \right. \\ & \left. - (n\eta(a, b))^{(2q-1)/(q-1)} \right]^{1-1/q} |f^{(n)}(a)| \\ & \times \left[G\left(nq - q + 1; \xi^{-q}, \frac{b + \eta(a, b) - x}{\eta(a, b)}\right) \right]^{1/q} \Big\}. \end{aligned} \quad (44)$$

Theorem 26. For $n \in \mathbb{N}$ and $n \geq 2$, let $A \subseteq \mathbb{R}$ be an open invex set with respect to $\eta : A \times A \rightarrow \mathbb{R}$ and $a, b \in A$ with $\eta(a, b) > 0$ for all $a \neq b$. Suppose that $f : A \rightarrow \mathbb{R}$ is an n -times differentiable function on A and $f^{(n)}$ is integrable on the η -path P_{bc} for $c = b + \eta(a, b)$. If $|f^{(n)}|^q$ is s -logarithmically preinvex on A for $q > 1$, then for $x \in [b, b + \eta(a, b)]$, $0 \leq r \leq (n-1)q$, and some $s \in (0, 1)$

$$\begin{aligned} & \left| \frac{f(b) + f(b + \eta(a, b))}{2} - \frac{1}{\eta(a, b)} \int_b^{b+\eta(a, b)} f(x) dx \right. \\ & + \sum_{k=2}^n \frac{1}{2k! \eta(a, b)} \\ & \times \left[(b-x)^{k-1} (2x - 2b - k\eta(a, b)) + (b + \eta(a, b) - x)^{k-1} \right. \\ & \left. \left. \times (2b + 2\eta(a, b) - 2x - k\eta(a, b)) \right] f^{(k-1)}(x) \right| \\ & \leq \frac{[\eta(a, b)]^{(r+2-q)/q}}{2n!} \left[\frac{q-1}{(nq-r-1)(nq+q-r-2)} \right]^{1-1/q} \\ & \times \left\{ |f^{(n)}(a)|^\alpha |f^{(n)}(b)|^\beta (x-b)^{(nq-r-1)/q} \right. \\ & \times [n(nq+q-r-2)\eta(a, b) \\ & \quad \left. - 2(nq-r-1)(x-b)]^{1-1/q} \right. \\ & \times \left[H\left(r+1; \xi^{sq}, \frac{x-b}{\eta(a, b)}\right) \right]^{1/q} + |f^{(n)}(a)|^\delta |f^{(n)}(b)|^\theta \\ & \times [n(nq+q-r-2)\eta(a, b) \\ & \quad \left. - 2(nq-r-1)(b + \eta(a, b) - x)]^{1-1/q} \right. \\ & \times (b + \eta(a, b) - x)^{(nq-r-1)/q} \\ & \left. \times \left[H\left(r+1; \xi^{-sq}, \frac{b + \eta(a, b) - x}{\eta(a, b)}\right) \right]^{1/q} \right\}, \end{aligned} \quad (45)$$

where $\xi, \{\alpha, \beta\}$ and $\{\delta, \theta\}$ are given in Theorem 22 and $H(\gamma; \mu, x)$ is defined by (28).

Proof. Since $a, b \in A$ and A is an invex set with respect to η , for every $t \in [0, 1]$, we have $b + t\eta(a, b) \in A$.

Using Lemma 14, Hölder's inequality, and s -logarithmically preinvexity of $|f^{(n)}|^q$, it turns out that

$$\begin{aligned} & \left| \frac{f(b) + f(b + \eta(a, b))}{2} - \frac{1}{\eta(a, b)} \int_b^{b+\eta(a, b)} f(x) dx \right. \\ & \quad + \sum_{k=2}^n \frac{1}{2k! \eta(a, b)} \\ & \quad \times \left[(b-x)^{k-1} (2x - 2b - k\eta(a, b)) + (b + \eta(a, b) - x)^{k-1} \right. \\ & \quad \left. \left. \times (2b + 2\eta(a, b) - 2x - k\eta(a, b)) \right] f^{(k-1)}(x) \right| \\ & \leq \frac{[\eta(a, b)]^n}{2n!} \\ & \quad \times \left\{ \left[\int_0^{(x-b)/(\eta(a, b))} t^{(nq-q-r)/(q-1)} (n-2t) dt \right]^{1-1/q} \right. \\ & \quad \times \left[\int_0^{(x-b)/(\eta(a, b))} t^r (n-2t) |f^{(n)}(b)|^{q(1-t)^s} \right. \\ & \quad \left. \left. \times |f^{(n)}(a)|^{qt^s} dt \right]^{1/q} \right. \\ & \quad + \left[\int_{(x-b)/(\eta(a, b))}^1 (1-t)^{(nq-q-r)/(q-1)} (2t+n-2) dt \right]^{1-1/q} \\ & \quad \times \left[\int_{(x-b)/(\eta(a, b))}^1 (1-t)^r (2t+n-2) \right. \\ & \quad \left. \times |f^{(n)}(b)|^{q(1-t)^s} |f^{(n)}(a)|^{qt^s} dt \right]^{1/q} \Bigg\}. \end{aligned} \quad (46)$$

The rest is also similar to the proof of Theorem 22. $\square$

Corollary 27. Under the assumptions of Theorem 26,

(1) if $r = 0$, then

$$\begin{aligned} & \left| \frac{f(b) + f(b + \eta(a, b))}{2} - \frac{1}{\eta(a, b)} \int_b^{b+\eta(a, b)} f(x) dx \right. \\ & \quad + \sum_{k=2}^n \frac{1}{2k! \eta(a, b)} \\ & \quad \times \left[(b-x)^{k-1} (2x - 2b - k\eta(a, b)) + (b + \eta(a, b) - x)^{k-1} \right. \\ & \quad \left. \left. \times (2b + 2\eta(a, b) - 2x - k\eta(a, b)) \right] f^{(k-1)}(x) \right| \\ & \leq \frac{[\eta(a, b)]^{(2-q)/q}}{2n!} \left[\frac{q-1}{(nq-1)(nq+q-2)} \right]^{1-1/q} \\ & \quad \times \left\{ |f^{(n)}(a)|^\alpha |f^{(n)}(b)|^\beta (x-b)^{(nq-1)/q} \right. \end{aligned}$$

$$\begin{aligned} & \times [n(nq+q-2)\eta(a, b) - 2(nq-1)(x-b)]^{1-1/q} \\ & \times \left[H\left(1; \xi^{sq}, \frac{x-b}{\eta(a, b)}\right) \right]^{1/q} + |f^{(n)}(a)|^\delta |f^{(n)}(b)|^\theta \\ & \times [(n-1)(nq-2)\eta(a, b) - 2(nq-1)(b-x)]^{1-1/q} \\ & \times (b + \eta(a, b) - x)^{nq-1/q} \\ & \times \left[H\left(1; \xi^{-sq}, \frac{b + \eta(a, b) - x}{\eta(a, b)}\right) \right]^{1/q} \Bigg\}; \end{aligned} \quad (47)$$

(2) if $r = (n-1)q$, then

$$\begin{aligned} & \left| \frac{f(b) + f(b + \eta(a, b))}{2} - \frac{1}{\eta(a, b)} \int_b^{b+\eta(a, b)} f(x) dx \right. \\ & \quad + \sum_{k=2}^n \frac{1}{2k! \eta(a, b)} \\ & \quad \times \left[(b-x)^{k-1} (2x - 2b - k\eta(a, b)) + (b + \eta(a, b) - x)^{k-1} \right. \\ & \quad \left. \left. \times (2b + 2\eta(a, b) - 2x - k\eta(a, b)) \right] f^{(k-1)}(x) \right| \\ & \leq \frac{[\eta(a, b)]^{n-2+2/q}}{2n!} \\ & \quad \times \left\{ |f^{(n)}(a)|^\alpha |f^{(n)}(b)|^\beta [(x-b)(n\eta(a, b) - x + b)]^{1-1/q} \right. \\ & \quad \times \left[H\left((n-1)q+1; \xi^{sq}, \frac{x-b}{\eta(a, b)}\right) \right]^{1/q} + |f^{(n)}(a)|^\delta \\ & \quad \times |f^{(n)}(b)|^\theta [(b + \eta(a, b) - x) \\ & \quad \times ((n-1)\eta(a, b) + x - b)]^{1-1/q} \\ & \quad \times \left[H\left((n-1)q+1; \xi^{-sq}, \frac{b + \eta(a, b) - x}{\eta(a, b)}\right) \right]^{1/q} \Bigg\}; \end{aligned} \quad (48)$$

(3) if $s = 1$, then

$$\begin{aligned} & \left| \frac{f(b) + f(b + \eta(a, b))}{2} - \frac{1}{\eta(a, b)} \int_b^{b+\eta(a, b)} f(x) dx \right. \\ & \quad + \sum_{k=2}^n \frac{1}{2k! \eta(a, b)} \\ & \quad \times \left[(b-x)^{k-1} (2x - 2b - k\eta(a, b)) + (b + \eta(a, b) - x)^{k-1} \right. \\ & \quad \left. \left. \times (2b + 2\eta(a, b) - 2x - k\eta(a, b)) \right] f^{(k-1)}(x) \right| \end{aligned}$$

$$\begin{aligned}
&\leq \frac{[\eta(a, b)]^{(r+2-q)/q}}{2n!} \left[\frac{q-1}{(nq-r-1)(nq+q-r-2)} \right]^{1-1/q} \\
&\times \left\{ |f^{(n)}(b)| (x-b)^{(nq-r-1)/q} \right. \\
&\quad \times [n(nq+q-r-2)\eta(a, b) \\
&\quad \left. - 2(nq-r-1)(x-b)]^{1-1/q} \right. \\
&\quad \times \left[H\left(r+1; \xi^q, \frac{x-b}{\eta(a, b)}\right) \right]^{1/q} + |f^{(n)}(a)| \\
&\quad \times [n(nq+q-r-2)\eta(a, b) \\
&\quad \left. - 2(nq-r-1)(b+\eta(a, b)-x)]^{1-1/q} \right. \\
&\quad \times (b+\eta(a, b)-x)^{(nq-r-1)/q} \\
&\quad \left. \times \left[H\left(r+1; \xi^{-q}, \frac{b+\eta(a, b)-x}{\eta(a, b)}\right) \right]^{1/q} \right\}.
\end{aligned} \tag{49}$$

Conflict of Interests

The authors declare that they have no conflict of interests.

Authors' Contribution

All authors contributed equally to the paper and read and approved the final paper.

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