Research Article

Boundary Stabilization of a Semilinear Wave Equation with Variable Coefficients under the Time-Varying and Nonlinear Feedback

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We study the boundary stabilization of a semilinear wave equation with variable coefficients under the time-varying and nonlinear feedback. By the Riemannian geometry methods, we obtain the stability results of the system under suitable assumptions of the bound of the time-varying term and the nonlinearity of the nonlinear term.

1. Introduction

Many results concerning the boundary stabilization of classical wave equations are available in literatures. See [1–6] for linear cases and [7–14] for nonlinear ones. The stability of a nondissipative system described by partial differential equations (PDEs) has attracted much attention. Reference [15] developed the exponential stability for an abstract nondissipative linear system, and in [16], the Riesz basis property was developed for a beam equation with nondissipativity.

In [17], the following semilinear wave equation was considered:

$$\begin{aligned} u_{tt} - \Delta_g u + h(\nabla u) + f(u) &= 0 & (x, t) \in \Omega \times (0, +\infty), \\ u(x, t)|_{\Gamma_2} &= 0 & t \in (0, +\infty), \\ \frac{\partial u(x, t)}{\partial \mu} + l(u_t) &= 0 & (x, t) \in \Gamma_1 \times (0, +\infty), \\ u(x, 0) = u_0(x), \quad u_t(x, 0) &= u_1(x) & x \in \Omega \end{aligned} \quad (1)$$

and the well-posedness and uniform decay of the energy of the system (1) was also established with linearly bounded $l(u)$ in [17].

Based on [17], we study the system (1) with time-varying and nonlinear feedback:

$$\frac{\partial u(x, t)}{\partial \mu} + \phi(t) l(u) = 0 \quad (x, t) \in \Gamma_1 \times (0, +\infty). \quad (2)$$

The decay rate of the energy (when t goes to infinity) of the wave equation with time-varying feedback was established under the assumption ϕ is decreasing [18–20] or ϕ has an upper bound [21].

In this paper, we consider the decay rate of the energy under suitable assumptions of the bound of the time-varying term $\phi(t)$ and the nonlinearity of the nonlinear term $l(u)$.

2. Some Notation

Let Ω be a bounded domain in $\mathbb{R}^n$ ($n \geq 2$) with smooth boundary Γ . It is assumed that Γ consists of two parts Γ_1 and Γ_2 ($\Gamma = \Gamma_1 \cup \Gamma_2$) with $\Gamma_2 \neq \emptyset$, $\overline{\Gamma_1} \cap \overline{\Gamma_2} = \emptyset$.

Let $A(x) = (a_{ij}(x))$ be symmetric, positively definite matrices for each $x \in \mathbb{R}^n$, and $a_{ij}(x)$ are smooth functions on $\mathbb{R}^n$. As in [22], we define

$$g = A^{-1}(x) \quad \text{for } x \in \mathbb{R}^n \quad (3)$$

as a Riemannian metric on $\mathbb{R}^n$ and consider the couple $(\mathbb{R}^n, g)$ as a Riemannian manifold with an inner product:

$$\langle X, Y \rangle_g = \langle A^{-1}(x) X, Y \rangle, \quad |X|_g^2 = \langle X, X \rangle_g \quad X, Y \in \mathbb{R}_x^n. \quad (4)$$

Denote by D , ∇_g , div_g , and Δ_g the Levi-Civita connection, the gradient operator, the divergence operator, and the Beltrami-Laplace operator in terms of the Riemannian metric g , respectively. It can be easily shown that, under the Euclidean coordinate,

$$\begin{aligned} \nabla_g f &= \sum_{i=1}^n \left(\sum_{j=1}^n a_{ij}(x) \frac{\partial}{\partial x_j} f \right) \frac{\partial}{\partial x_i} = A(x) \nabla f, \\ |\nabla_g u|_g^2 &= \sum_{i,j=1}^n a_{ij}(x) \frac{\partial f}{\partial x_i} \frac{\partial f}{\partial x_j} \quad x \in \mathbb{R}^n, \end{aligned} \quad (5)$$

$$\Delta_g f = \frac{1}{\sqrt{G}} \sum_{i=1}^n \frac{\partial}{\partial x_i} \left(\sqrt{G} \sum_{j=1}^n a_{ij}(x) \frac{\partial f}{\partial x_j} \right), \quad x \in \mathbb{R}^n,$$

where ∇f is the gradient of f in the standard metric and $G = \det(g)$.

Let H be a vector field on $(\mathbb{R}_x^n, g)$. Then for each $x \in \mathbb{R}^n$, the covariant differential DH of H determines a bilinear form on $\mathbb{R}_x^n$:

$$DH(X, Y) = \langle D_Y H, X \rangle_g \quad \forall X, Y \in \mathbb{R}_x^n, \quad (6)$$

where $D_Y H$ stands for the covariant derivative of the vector field H with respect to Y .

3. The Main Results

We consider the semilinear wave equation with variable coefficients under the time-varying and nonlinear boundary feedback:

$$\begin{aligned} u_{tt} - \Delta_g u + f(u) &= 0 \quad (x, t) \in \Omega \times (0, +\infty), \\ u(x, t)|_{\Gamma_2} &= 0 \quad t \in (0, +\infty), \\ \frac{\partial u(x, t)}{\partial \mu} + \phi(t) l(u_t) &= 0 \quad (x, t) \in \Gamma_1 \times (0, +\infty), \\ u(x, 0) &= u_0(x), \quad u_t(x, 0) = u_1(x) \quad x \in \Omega, \end{aligned} \quad (7)$$

where l, f are continuous nonlinear functions and $\mu(x)$ is the outside unit normal vector of the Riemannian manifold (Ω, g) for each $x \in \Gamma$. Different from [18–21], in this paper, we consider a general ϕ ; that is, $\phi \in C^1([0, +\infty))$ satisfies

$$\frac{1}{\Phi(t)} \leq \phi \leq \Phi(t) \quad \forall t \geq 0, \quad (8)$$

where $\Phi(t) \in C([0, +\infty))$ is a positive and nondecreasing function satisfying

$$\lim_{t \rightarrow +\infty} \frac{\Phi(t)}{t} = 0. \quad (9)$$

Let $\Phi'(t) \in C([0, +\infty))$ be a positive and nondecreasing function with 0 as the limit. Then $t\Phi'(t)$ satisfies (9). There are many examples of $\Phi'(t)$ such as $(1+t)^\alpha$ ($\alpha < 0$) and $e^{\beta t}$ ($\beta < 0$).

The main assumptions are listed as follows.

Assumption A. $f \in C^1(\mathbb{R})$, $f(0) = 0$ derives from a potential F :

$$F(s) = \int_0^s f(\tau) d\tau \geq 0 \quad \forall s \in \mathbb{R}, \quad (10)$$

and satisfies

$$|f'(s)| \leq b_1 |s|^\rho + b_2 \quad \forall s \in \mathbb{R}, \quad (11)$$

where b_1, b_2 are positive constants, and the parameter ρ satisfies

$$1 \leq \rho \leq \begin{cases} 2, & n = 2, \\ \frac{n}{n-2}, & n \geq 3. \end{cases} \quad (12)$$

Being different from [17], we assume the nonlinear term $l(u)$ has no growth restriction near zero as in [23, 24].

Assumption B. $l \in C^1(\mathbb{R})$ is a nondecreasing function satisfying

$$l(0) = 0, \quad c_1 |s|^2 \leq sl(s) \leq c_2 |s|^2 \quad \forall |s| \geq 1. \quad (13)$$

Assumption C. There exists a vector field H on $\bar{\Omega}$ such that

$$DH(X, X) = c(x) |X|_g^2 \quad \text{for } X \in \mathbb{R}_x^n, x \in \bar{\Omega}, \quad (14)$$

where $b = \min_{\bar{\Omega}} c(x)$ and $B = \max_{\bar{\Omega}} c(x)$

$$B < \min \left\{ b + \frac{2b}{n}, rb \right\}, \quad (15)$$

where $r > 1$ is a constant. Moreover we assume that

$$\langle H, \mu \rangle_g \leq 0 \quad x \in \Gamma_2, \quad \langle H, \mu \rangle_g \geq 0 \quad x \in \Gamma_1. \quad (16)$$

Condition (14) as a checkable assumption is very useful to study the control and stabilization of the wave equation with variable coefficients and the quasilinear wave equation [22, 25]. For the examples of the condition, see [22, 26].

Based on condition (14), Assumption C was given by [17] to study the stabilization of the wave equation with variable coefficients and nonlinear boundary condition. Being different from [17], the lower bound of $\langle H, \mu \rangle_g$ was relaxed on Γ_1 from a positive constant to zero.

To facilitate the writing, we denote the volume element of (Ω, g) by dx and denote the volume element of (Γ, g) by $d\Gamma$. Define the energy of the system (7) by

$$E(t) = \int_{\Omega} \left(u_t^2 + |\nabla_g u|_g^2 + 2F(u) \right) dx. \quad (17)$$

As in [23, 24], we let $h \in C([0, +\infty))$ be a concave increasing function such that

$$h(0) = 0, \quad s^2 + (g(s))^2 \leq h(g(s)) \quad \text{for } |s| \leq 1. \quad (18)$$

With (18), the stabilization of the wave equation with variable coefficients and time dependent delay was studied by [27].

The main result of this paper is as follows.

Theorem 1. *Let Assumptions A–C hold true. Assume that*

$$2rF(s) \leq sf(s) \quad \forall s \in \mathbb{R}, \quad (19)$$

where r is defined in (15).

(a) *If the function l in (7) satisfies*

$$c_1|s|^2 \leq sl(s) \leq c_2|s|^2 \quad \forall |s| < 1, \quad (20)$$

then there exist constants $C > 0$ such that

$$E(t) \leq \frac{C\Phi(t)}{t} E(0) \quad t > 0. \quad (21)$$

(b) *If the functions $\phi(t), l$ in (7) satisfy*

$$\phi(t) \leq \phi_0 \quad \forall t \geq 0, \quad sl(s) \geq c_1|s|^2 \quad \forall |s| < 1, \quad (22)$$

where ϕ_0 is a positive constant, then there exist constants $C_1, C_2 > 0$ such that

$$C_1 h \left(\frac{C_2 \Phi(T)}{T} E(0) \right) + \frac{C_1 \Phi(T)}{T} E(0) \quad t > 0. \quad (23)$$

(c) *If the function $\Phi(t)$ in (8) is a constant function; that is,*

$$\Phi(t) = \Phi(0) \quad \forall t \geq 0, \quad (24)$$

then there exist constants $C_1, C_2 > 0$ such that

$$C_1 h \left(\frac{C_2 E(0)}{T} \right) + \frac{C_1}{T} E(0) \quad t > 0. \quad (25)$$

4. Well Posedness of the System

Define

$$H_{\Gamma_2}^1(\Omega) = \{u \in H^1(\Omega) \mid u|_{\Gamma_2} = 0\}. \quad (26)$$

By a similar proof as Lemma 7.1 in [17], we have the following result.

Theorem 2. *Let Assumptions A–B hold true. For any initial data $(u_0, u_1) \in H_{\Gamma_2}^1(\Omega) \times L^2(\Omega)$, system (7) admits a unique weak solution u such that $u \in C([0, +\infty), H_{\Gamma_2}^1(\Omega)) \cap C^1([0, +\infty), L^2(\Omega))$.*

To prove Theorem 1, we still need several lemmas further. Define

$$E_0(t) = \int_{\Omega} \left(u_t^2 + |\nabla_g u|_g^2 \right) dx. \quad (27)$$

Then, we have

$$E(t) = E_0(t) + 2 \int_{\Omega} F(u) dx. \quad (28)$$

The following lemma shows the energy of the system (7) is decreasing.

Lemma 3. *Suppose that Assumptions A–B hold true. Let u be the solution of the system (7). Then*

$$E(0) - E(T) = 2 \int_0^T \int_{\Gamma_1} \phi(t) u_t l(u_t) d\Gamma dt. \quad (29)$$

The assertion (29) implies that $E(t)$ is decreasing.

Proof. Differentiating (17), we obtain

$$\begin{aligned} E'(t) &= \int_{\Omega} \left(2u_t u_{tt} + 2 \langle \nabla_g u, \nabla_g u_t \rangle_g + 2f(u) \right) dx \\ &= \int_{\Gamma_1} 2\phi(t) u_t l(u_t) d\Gamma. \end{aligned} \quad (30)$$

Then the inequality (29) follows directly from (30) integrating from 0 to T . $\square$

5. Proofs of Theorem 1

Lemma 4. *Let $u(x, t)$ be the solution of the equation $u_{tt} + \Delta_g u + f(u) = 0$, $(x, t) \in \Omega \times (0, +\infty)$ and that $\mathcal{H}$ is a vector field defined on $\bar{\Omega}$. Then for $T \geq 0$*

$$\begin{aligned} &\int_0^T \int_{\Gamma} \frac{\partial u}{\partial \mu} \mathcal{H}(u) d\Gamma dt + \frac{1}{2} \int_0^T \int_{\Gamma} \left(u_t^2 - |\nabla_g u|_g^2 - 2F(u) \right) \\ &\quad \times \langle \mathcal{H}, \mu \rangle_g d\Gamma dt \\ &= (u_t, \mathcal{H}(u))|_0^T + \int_0^T \int_{\Omega} D\mathcal{H}(\nabla_g u, \nabla_g u) dx dt \\ &\quad + \frac{1}{2} \int_0^T \int_{\Omega} \left(u_t^2 - |\nabla_g u|_g^2 - 2F(u) \right) \operatorname{div}_g \mathcal{H} dx dt. \end{aligned} \quad (31)$$

Moreover, assume that $P \in C^1(\bar{\Omega})$. Then

$$\begin{aligned} &\int_0^T \int_{\Omega} \left(u_t^2 - |\nabla_g u|_g^2 - uf(u) \right) P dx dt \\ &= (u_t, uP)|_0^T + \frac{1}{2} \int_0^T \int_{\Omega} \nabla_g P(u^2) dx dt \\ &\quad - \int_0^T \int_{\Gamma} Pu \frac{\partial u}{\partial \mu} d\Gamma dt. \end{aligned} \quad (32)$$

Proof. Note that

$$\mathcal{H}(u) f(u) = \mathcal{H}(F(u)) = \operatorname{div}_g(F(u) \mathcal{H}) - F(u) \operatorname{div}_g \mathcal{H}. \quad (33)$$

The equality (31) and the equality (32) follow from Proposition 2.1 in [22]. $\square$

Lemma 5. *Suppose that all assumptions in Theorem 1 hold true. Let u solve the system (7). Then there exist positive constants $\bar{T}, C$ for which*

$$E(T) \leq \frac{C}{T} \int_0^T \int_{\Gamma_1} \left(u_t^2 + \left(\frac{\partial u}{\partial \mu} \right)^2 \right) d\Gamma dt, \quad (34)$$

where $T \geq \bar{T}$.

Proof. From (15), we choose a positive constant θ satisfying

$$\theta < \frac{nb}{2}, \quad b + \theta - \frac{nB}{2} > 0, \quad 2r\theta > nB. \quad (35)$$

Set

$$\mathcal{H} = H, \quad P = \theta. \quad (36)$$

We substitute the formula (32) into the formula (31), and we have

$$\begin{aligned} \Pi_\Gamma &= (u_t, H(u) + Pu)|_0^T \\ &+ \int_0^T \int_\Omega (DH(\nabla_g u, \nabla_g u) - b|\nabla_g u|_g^2) dx dt \\ &+ \int_0^T \int_\Omega \left(\left(\frac{1}{2} \operatorname{div} H - \theta \right) u_t^2 \right. \\ &\quad \left. + \left(b + \theta - \frac{1}{2} \operatorname{div} H \right) |\nabla_g u|_g^2 \right) dx dt \\ &+ \int_0^T \int_\Omega [\theta (uf(u) - 2rF(u)) \\ &\quad + (2r\theta - \operatorname{div} H) F(u)] dx dt, \end{aligned} \quad (37)$$

where

$$\begin{aligned} \Pi_\Gamma &= \int_0^T \int_\Gamma \frac{\partial u}{\partial \mu} (H(u) + uP) d\Gamma dt \\ &+ \frac{1}{2} \int_0^T \int_\Gamma (u_t^2 - |\nabla_g u|_g^2 - 2F(u)) \langle H, \mu \rangle_g d\Gamma dt. \end{aligned} \quad (38)$$

Decompose Π_Γ as

$$\Pi_\Gamma = \Pi_{\Gamma_1} + \Pi_{\Gamma_2}, \quad (39)$$

where $\Pi_{\Gamma_1}(\Pi_{\Gamma_2})$ stands by the value of the terms on the right side of (38) integrating on $\Gamma_1(\Gamma_2)$.

Similar to [5, 22], we deal with Π_{Γ_2} as follows.

Since $u|_{\Gamma_2} = 0$, we have $\nabla_\Gamma u|_{\Gamma_2} = 0$; that is,

$$\nabla_g u = \frac{\partial u}{\partial \mu} \mu \quad \text{for } x \in \Gamma_2. \quad (40)$$

Similarly, we obtain

$$H(u) = \langle H, \nabla_g u \rangle_g = \frac{\partial u}{\partial \mu} \langle H, \mu \rangle_g \quad \text{for } x \in \Gamma_2. \quad (41)$$

Using the equality (40) and (41) in the equality (38) on the portion Γ_2 , with (16) we obtain

$$\Pi_{\Gamma_2} = \frac{1}{2} \int_0^T \int_{\Gamma_2} \left(\frac{\partial u}{\partial \mu} \right)^2 \langle H, \mu \rangle_g d\Gamma dt \leq 0. \quad (42)$$

Let H_1 be a vector field on $\overline{\Omega}$ such that

$$\begin{aligned} H_1 &= \mu & x \in \Gamma_1, \\ H_1 &= 0 & x \in \Gamma_2. \end{aligned} \quad (43)$$

Set $\mathcal{H} = H_1$; it follows from (31) that

$$\begin{aligned} &\int_0^T \int_{\Gamma_1} \left(\frac{\partial u}{\partial \mu} \right)^2 d\Gamma dt + \frac{1}{2} \int_0^T \int_{\Gamma_1} (u_t^2 - |\nabla_g u|_g^2) d\Gamma dt \\ &= (u_t, H_1(u))|_0^T + \int_0^T \int_\Omega DH_1(\nabla_g u, \nabla_g u) dx \\ &\quad + \frac{1}{2} \int_0^T \int_\Omega (u_t^2 - |\nabla_g u|_g^2 - 2F(u)) \operatorname{div}_g H_1 dx. \end{aligned} \quad (44)$$

Then we obtain that

$$\begin{aligned} &\int_0^T \int_{\Gamma_1} |\nabla_g u|_g^2 d\Gamma dt \\ &\leq C \int_0^T \int_{\Gamma_1} \left(u_t^2 + \left(\frac{\partial u}{\partial \mu} \right)^2 \right) d\Gamma dt + C(E_0(0) + E_0(T)) \\ &\quad + C \int_0^T \int_\Omega (u_t^2 + |\nabla_g u|_g^2 + 2F(u)) dx dt. \end{aligned} \quad (45)$$

With (16) and (45), we have

$$\begin{aligned} \Pi_{\Gamma_1} &= \int_0^T \int_{\Gamma_1} \frac{\partial u}{\partial \mu} (H(u) + uP) d\Gamma dt \\ &+ \frac{1}{2} \int_0^T \int_{\Gamma_1} (u_t^2 - |\nabla_g u|_g^2 - F(u)) \langle H, \mu \rangle_g d\Gamma dt \\ &\leq C_\varepsilon \int_0^T \int_{\Gamma_1} \left(\frac{\partial u}{\partial \mu} \right)^2 d\Gamma dt + \varepsilon \int_0^T \int_{\Gamma_1} (u^2 + |\nabla_g u|_g^2) d\Gamma dt \\ &\quad + C \int_0^T \int_{\Gamma_1} u_t^2 d\Gamma dt \\ &\leq C \int_0^T \int_{\Gamma_1} \left(\frac{\partial u}{\partial \mu} \right)^2 d\Gamma dt \\ &\quad + \varepsilon \left(E_0(0) + E_0(T) + \int_0^T E(t) dt \right) + C \int_0^T \int_{\Gamma_1} u_t^2 d\Gamma dt. \end{aligned} \quad (46)$$

Note that

$$nb \leq \operatorname{div}_g H \leq nB \quad \forall x \in \overline{\Omega}. \quad (47)$$

Substituting the formulas (42) and (46) into the formula (37), with (19) and (35), we obtain

$$\begin{aligned} &\int_0^T E(t) dt \leq C(E_0(0) + E_0(T)) \\ &\quad + C \int_0^T \int_{\Gamma_1} \left(u_t^2 + \left(\frac{\partial u}{\partial \mu} \right)^2 \right) d\Gamma dt. \end{aligned} \quad (48)$$

Since

$$\begin{aligned} E_0(0) &= E_0(T) - \int_0^T \int_{\Gamma_1} u_t \frac{\partial u}{\partial \mu} d\Gamma dt \\ &\leq E_0(T) + \frac{1}{2} \int_0^T \int_{\Gamma_1} \left(u_t^2 + \left(\frac{\partial u}{\partial \mu} \right)^2 \right) d\Gamma dt, \end{aligned} \quad (49)$$

from (48), we have

$$\int_0^T E(t) dt \leq CE(T) + C \int_0^T \int_{\Gamma_1} \left(u_t^2 + \left(\frac{\partial u}{\partial \mu} \right)^2 \right) d\Gamma dt. \quad (50)$$

Since $E(t)$ is decreasing, we deduce that

$$\int_0^T E(t) dt \geq TE(T). \quad (51)$$

Substituting the formulas (51) into the formula (50), for sufficiently large T , we have

$$E(T) \leq \frac{C}{T} \int_0^T \int_{\Gamma_1} \left(u_t^2 + \left(\frac{\partial u}{\partial \mu} \right)^2 \right) d\Gamma dt. \quad (52)$$

The inequality (34) holds. $\square$

Proof of Theorem 1. (a) From (8), (13), (20), (29), and (34), for $T \geq \bar{T}$ we deduce that

$$\begin{aligned} E(T) &\leq \frac{C}{T} \int_0^T \int_{\Gamma_1} (\phi^2(t) + 1) u_t^2 d\Gamma dt \\ &\leq \frac{C}{T} \left(\sup \{ \phi(t) \mid 0 \leq t \leq T \} \right. \\ &\quad \left. + \sup \left\{ \frac{1}{\phi(t)} \mid 0 \leq t \leq T \right\} \right) \\ &\quad \times \int_0^T \int_{\Gamma_1} \phi(t) u_t^2 d\Gamma dt \leq \frac{C\Phi(T)}{T} E(0). \end{aligned} \quad (53)$$

Note that $E(t)$ is decreasing, and the estimate (21) holds.

(b) From (8), (13), (22), (29), and (34), for $T \geq \bar{T}$ we deduce that

$$\begin{aligned} E(T) &\leq \frac{C}{T} \int_0^T \int_{\Gamma_1} (\phi^2(t) g^2(u_t) + u_t^2) d\Gamma dt \\ &\leq \frac{C}{T} \left\{ \int_0^T \int_{\Gamma_1} \phi(t) g^2(u_t) d\Gamma dt \right. \\ &\quad \left. + \Phi(T) \int_0^T \int_{\Gamma_1} \phi(t) u_t^2 d\Gamma dt \right\} \end{aligned}$$

$$\begin{aligned} &\leq \frac{C}{T} \left\{ \int_0^T \int_{\{x \in \Gamma_1, |u_t| \leq 1\}} \phi(t) g^2(u_t) d\Gamma dt \right. \\ &\quad \left. + \Phi(T) \int_0^T \int_{\Gamma_1} \phi(t) u_t g(u_t) d\Gamma dt \right\} \\ &\leq \frac{C}{T} \int_0^T \int_{\{x \in \Gamma_1, |u_t| \leq 1\}} \phi(t) h(u_t g(u_t)) d\Gamma dt \\ &\quad + \frac{C\Phi(T)}{T} E(0) \\ &\leq \frac{C}{T} \int_0^T \int_{\Gamma_1} \phi(t) h(u_t g(u_t)) d\Gamma dt + \frac{C\Phi(T)}{T} E(0) \\ &\leq \frac{C \int_0^T \phi(t) dt \cdot \text{meas}(\Gamma_1)}{T} h \\ &\quad \times \left(\frac{\int_0^T \int_{\Gamma_1} \phi(t) u_t g(u_t) d\Gamma dt}{\int_0^T \phi(t) dt \cdot \text{meas}(\Gamma_1)} \right) + \frac{C\Phi(T)}{T} E(0) \\ &\leq C_1 h \left(\frac{C_2 \Phi(T)}{T} E(0) \right) + \frac{C_1 \Phi(T)}{T} E(0). \end{aligned} \quad (54)$$

Note that $E(t)$ is decreasing, and the estimate (23) holds.

(c) From (8), (13), (24), (29), and (34), for $T \geq \bar{T}$ we deduce that

$$\begin{aligned} E(T) &\leq \frac{C}{T} \int_0^T \int_{\Gamma_1} (\phi^2(t) g^2(u_t) + u_t^2) d\Gamma dt \\ &\leq \frac{C}{T} \int_0^T \int_{\Gamma_1} \phi(t) (g^2(u_t) + u_t^2) d\Gamma dt \\ &\leq \frac{C}{T} \int_0^T \int_{\{x \in \Gamma_1, |u_t| \leq 1\}} \phi(t) h(u_t g(u_t)) d\Gamma dt \\ &\quad + \frac{C}{T} \int_0^T \int_{\{x \in \Gamma_1, |u_t| > 1\}} \phi(t) u_t^2 d\Gamma dt \\ &\leq \frac{C}{T} \int_0^T \int_{\Gamma_1} \phi(t) h(u_t g(u_t)) d\Gamma dt \\ &\quad + \frac{C}{T} \int_0^T \int_{\Gamma_1} \phi(t) u_t g(u_t) d\Gamma dt \\ &\leq \frac{C \int_0^T \phi(t) dt \cdot \text{meas}(\Gamma_1)}{T} h \\ &\quad \times \left(\frac{\int_0^T \int_{\Gamma_1} \phi(t) u_t g(u_t) d\Gamma dt}{\int_0^T \phi(t) dt \cdot \text{meas}(\Gamma_1)} \right) + \frac{C}{T} E(0) \\ &\leq C_1 h \left(\frac{C_2 E(0)}{T} \right) + \frac{C_1}{T} E(0). \end{aligned} \quad (55)$$

Note that $E(t)$ is decreasing, and the estimate (25) holds. $\square$

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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