Research Article
Convergence Axioms on Dislocated Symmetric Spaces

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Dislocated symmetric spaces are introduced, and implications and nonimplications among various kinds of convergence axioms are derived.

1. Introduction

A metric space is a special kind of topological space. In a metric space, topological properties are characterized by means of sequences. Sequences are not sufficient in topological spaces for such purposes. It is natural to try to find classes intermediate between those of topological spaces and those of metric spaces in which members sequences play a predominant part in deciding their topological properties. A galaxy of mathematicians consisting of such luminaries as Frechet [1], Chittenden [2], Frink [3], Wilson [4], Niemytzki [5], and Aranđelović and Kečkić [6] have made important contributions in this area. The basic definition needed by most of these studies is that of a symmetric space. If \( X \) is a nonempty set, a function \( \delta : X \times X \to \mathbb{R}^+ \) is called a dislocated symmetric on \( X \) if \( \delta(x, y) = 0 \) implies that \( x = y \) and \( \delta(x, y) = \delta(y, x) \) for all \( x, y \in X \). A dislocated symmetric (simply \( d \)-symmetric) on \( X \) is called symmetric on \( X \) if \( d(x, x) = 0 \) for all \( x \) in \( X \). The names dislocated symmetric space and symmetric space have expected meanings. Obviously, a symmetric space that satisfies the triangle inequality is a metric space. Since the aim of our study is to find how sequential properties and topological properties influence each other, we collect various properties of sequences that have been shown in the literature to have a bearing on the problem under study. In what follows "\( d \)" denotes a dislocated distance on a nonempty set \( X \). \( x_n, y_n, x, y \), and so forth are elements of \( X \) and \( C_i \) for \( 1 \leq i \leq 5 \) and \( W_i \) for \( 1 \leq i \leq 3 \) indicate properties of sequences in \( (X, d) \). Consider

\[
C_1: \lim d(x_n, y_n) = 0 = \lim d(x_n, x) \Rightarrow \lim d(y_n, x) = 0,
\]

\[
C_2: \lim d(x_n, x) = 0 = \lim d(y_n, x) \Rightarrow \lim d(x_n, y_n) = 0,
\]

\[
C_3: \lim d(x_n, y_n) = 0 = \lim d(y_n, z_n) \Rightarrow \lim d(x_n, z_n) = 0.
\]

A space in which \( C_1 \) is satisfied is called coherent by Pitcher and Chittenden [7]. Niemytzki [5] proved that a coherent symmetric space \( (X, d) \) is metrizable, and in fact there is a metric \( \rho \) on \( X \) such that \( (X, d) \) and \( (X, \rho) \) have identical topologies and also that \( \lim d(x_n, x) = 0 \) if and only if \( \lim \rho(x_n, x) = 0 \).

Cho et al. [8] have introduced

\[
W_1: \text{for each pair of distinct points } a, b \text{ in } X \text{ there corresponds a positive number } r = r(a, b) \text{ such that } r < \inf_{c \in X} d(a, c) + d(b, c),
\]

\[
W_2: \text{for each pair of distinct points } a, b \text{ in } X \text{ there corresponds a positive number } r = r(a, b) \text{ such that } r < \inf_{c \in X} d(a, c) + d(b, c),
\]

\[
W_3: \text{for each pair of distinct points } a, b \text{ in } X \text{ there corresponds a positive number } r = r(a, b) \text{ such that } r < \inf_{c \in X} d(a, c) + d(b, c),
\]

The following properties were introduced by Wilson [4]:

\[
W_4: \text{for each pair of distinct points } a, b \text{ in } X \text{ there corresponds a positive number } r = r(a, b) \text{ such that } r < \inf_{c \in X} d(a, c) + d(b, c),
\]

\[
W_5: \text{for each pair of distinct points } a, b \text{ in } X \text{ there corresponds a positive number } r = r(a, b) \text{ such that } r < \inf_{c \in X} d(a, c) + d(b, c),
\]
W₂: for each \( a \in X \), for each \( k > 0 \), there corresponds a positive number \( r = r(a,k) \) such that if \( b \) is a point of \( X \) such that \( d(a,b) \geq k \) and \( c \) is any point of \( X \) then \( d(a,c) + d(c,b) \geq r \),

W₃: for each positive number \( k \) there is a positive number \( r = r(k) \) such that \( d(a,c) + d(c,b) \geq r \) for all \( c \in X \) and all \( a, b \in X \) with \( d(a,b) \geq k \).

2. Implications among the Axioms

**Proposition 1.** In a \( d \)-symmetric space \((X,d)\), \( C₃ \Rightarrow C₁ \Rightarrow C₅ \), \( C₃ \Rightarrow C₂ \), and \( C₄ \Rightarrow C₅ \).

**Proof.** Assume that \( C₃ \) holds in \((X,d)\) and let \( \lim d(xₙ,yₙ) = 0 \) and \( \lim d(xₙ,y) = 0 \). Put \( zₙ = x \) \( \forall n \) so that

\[
\lim d(xₙ,zₙ) = \lim d(xₙ,x) = 0 = \lim d(xₙ,yₙ) = \lim d(yₙ,xₙ).
\]

By \( C₃ \), \( \lim d(yₙ,zₙ) = 0 \); that is, \( \lim d(yₙ,x) = 0 \).

Hence

\[
C₃ \Rightarrow C₁.
\]

Assume that \( C₁ \) holds in \((X,d)\) and let \( \lim d(xₙ,yₙ) = 0 \) and \( \lim d(xₙ,y) = 0 \). Put \( yₙ = y \) \( \forall n \); then

\[
\lim d(xₙ,yₙ) = \lim d(xₙ,x) = 0.
\]

By \( C₃ \), \( \lim d(yₙ,xₙ) = 0 \); that is, \( \lim d(y,x) = 0 \).

Consider \( \lim d(x,y) = 0 \); this implies that \( x = y \). Hence \( C₃ \) holds. Thus

\[
C₁ \Rightarrow C₃.
\]

Assume that \( C₃ \) holds and let \( \lim d(xₙ,yₙ) = 0 \) and \( \lim d(xₙ,y) = 0 \). Put \( zₙ = x \) \( \forall n \); then \( \lim d(xₙ,zₙ) = \lim d(zₙ,yₙ) = 0 \).

By \( C₃ \), \( \lim d(xₙ,yₙ) = 0 \). Hence

\[
C₃ \Rightarrow C₂.
\]

Assume that \( C₄ \) holds and let \( \lim d(xₙ,yₙ) = 0 \) and \( \lim d(xₙ,y) = 0 \).

By \( C₄ \), \( \lim d(xₙ,y) = d(x,y) \). Hence \( d(x,y) = 0 \). Hence \( x = y \). □

The following proposition explains the relationship between Wilson’s axioms [4] \( W₁, W₂ \), and \( W₃ \) and the \( Cᵢ \)’s.

**Proposition 2.** Let \((X,d)\) be a \( d \)-symmetric space; then

(i) \( W₁ \Leftrightarrow C₅ \), (ii) \( W₂ \Leftrightarrow C₁ \), and (iii) \( W₃ \Leftrightarrow C₃ \).

**Proof.** (i) Assume \( W₁ \). Suppose \( \lim d(a,x_n) = \lim d(b,x_n) = 0 \) but \( a \neq b \).

Then

\[
\lim \{d(a,x_n) + d(b,x_n)\} = 0 \quad \text{but} \quad a \neq b.
\]

By

\[
W₁ \exists r > 0 \quad \forall x, \quad d(a,x) + d(b,x) \geq r,
\]

equations (6) and (7) are contradictory. Hence \( a = b \). Thus \( W₁ \Rightarrow C₅ \).

(ii) Assume \( W₂ \). Then for each \( a \in X \) and each \( k > 0 \) there corresponds \( r > 0 \) such that, for all \( b \in X \) with \( d(a,b) \geq k \) and \( \forall x \in X \), \( d(a,x) + d(b,x) \geq r \).

Suppose that \( W₂ \) fails. Then there exist \( a \neq b \) in \( X \) such that for every \( n \) there corresponds \( x_n \) in \( X \) such that \( d(a,x_n) + d(b,x_n) < 1/n \):

\[
\implies \lim d(a,x_n) = \lim d(b,x_n) = 0 \quad \text{but} \quad a \neq b.
\]

Thus if \( W₂ \) fails then \( C₃ \) fails. That is, \( C₃ \Rightarrow W₂ \). Hence \( W₁ \Rightarrow C₅ \).

(iii) Assume \( W₃ \). Suppose that \( C₃ \) fails. Then there exist \( a \in X \), \( \{b_n\} \), and \( \{c_n\} \) in \( X \) such that \( \lim d(a,b_n) = \lim d(b_n,c_n) = 0 \) but \( \lim d(a,c_n) \neq 0 \).

Since \( \lim d(a,c_n) \neq 0 \) there exists \( k > 0 \) and a subsequence \( \{c_{n_k}\} \) such that

\[
d(a,c_{n_k}) > k \quad \forall n_k.
\]

This implies that \( \lim d(a,c_{n_k}) = \lim d(b_n,c_{n_k}) = 0 \) but \( \lim d(a,b_{n_k}) \neq 0 \).

Hence \( C₄ \) fails.

(iv) Assume \( W₃ \). Suppose that \( C₄ \) fails. Then there exist sequences \( \{a_n\}, \{b_n\}, \) and \( \{c_n\} \) in \( X \) such that \( \lim d(a_n,b_n) = \lim d(b_n,c_n) = 0 \) but \( \lim d(a_n,c_n) \neq 0 \).

Since \( W₃ \) holds, \( \forall k > 0 \) there corresponds \( r > 0 \) such that for all \( a, b \) with

\[
d(a,b) \geq k, \quad d(a,c) + d(b,c) \geq r \quad \forall c.
\]

Since \( \lim d(a_n,c_n) \neq 0 \) there exists a positive number \( \epsilon \) and a subsequence of positive integers \( \{n_k\} \) such that \( d(a_{n_k},c_{n_k}) > \epsilon \). Choose \( r_1 \) corresponding to \( \epsilon \) so that

\[
d(a_{n_k},b_{n_k}) + d(b_{n_k},c_{n_k}) \geq r_1.
\]

Thus

\[
\lim \{d(a_{n_k},b_{n_k}) + d(b_{n_k},c_{n_k})\} \neq 0.
\]

This contradicts the assumption that \( \lim d(a_n,b_n) = \lim d(b_n,c_n) = 0 \).

Hence

\[
W₃ \Rightarrow C₃.
\]
Then there exists $k > 0$ such that, $\forall$ positive integer $n$, there exist $a_n$, $b_n$, and $c_n$ with
\[
d(a_n, b_n) \geq k \quad \text{but} \quad d(a_n, c_n) + d(b_n, c_n) < \frac{1}{n}.
\] (16)

Hence
\[
\lim d(a_n, b_n) \neq 0 \quad \text{but} \quad \lim d(a_n, c_n) = \lim d(b_n, c_n) = 0.
\] (17)

Hence $C_3$ fails.

\[C_3 \implies W_5. \quad \text{(18)}\]

This completes the proof of the proposition.

We introduce the following.

**Axiom C.** Every convergent sequence satisfies Cauchy criterion. That is, if $(x_n)$ is a sequence in $X$, $x \in X$ and $\lim d(x_n, x) = 0$; then given $\epsilon > 0 \exists N(\epsilon) \in \mathbb{N}$ such that $d(x_m, x_n) < \epsilon$ whenever $m, n \geq N(\epsilon)$ we have the following.

**Proposition 3.** In a $d$-symmetric space $(X, d)$, $C_1 \implies C \implies C_2$.

**Proof.** For $C_1 \implies C$, suppose that a sequence $(x_n)$ in $(X, d)$ is convergent to $x$ but does not satisfy Cauchy criterion. Then $\exists r > 0$ such that for every positive integer $k$ there correspond integers $m_k, n_k$ such that
\[
m_{k+1} > n_{k+1} > m_k > n_k, \quad d(x_{m_k}, x_{n_k}) > \epsilon \quad \forall k. \quad \text{(19)}
\]

Let
\[
y_k = x_{m_k}, \quad z_k = x_{n_k} \quad \forall k. \quad \text{(20)}
\]

Then
\[
\lim d(y_k, x) = 0, \quad \lim d(z_k, x) = 0. \quad \text{(21)}
\]

But $\lim d(y_k, z_k) \neq 0$; this contradicts $C_1$.

**Proof.** For $C \implies C_2$, suppose that $\lim d(x_n, x) = \lim d(y_n, x) = 0$.

Let $(z_n)$ be the sequence defined by $z_{2n-1} = x_n$ and $z_{2n} = y_n$. Then $\lim d(z_n, x) = 0$. Hence $(z_n)$ satisfies Cauchy criterion.

Given $\epsilon > 0 \exists N(\epsilon) \in \mathbb{N}$ such that $d(z_n, z_m) < \epsilon$ for $m, n \geq N(\epsilon)$:
\[
\implies d(z_{2n-1}, z_{2n}) < \epsilon \quad \text{for} \quad n \geq N(\epsilon),
\]
\[
\implies \lim d(x_n, y_n) < \epsilon \quad \text{for} \quad n \geq N(\epsilon),
\]
\[
\implies \lim d(x_n, y_n) = 0.
\]

\[\Box\]

**3. Examples for Nonimplications**

*Example 4.* A $d$-symmetric space in which the triangular inequality fails and $C_1$ holds.

Let $X = [0, 1]$. Define $d$ on $X \times X$ as follows:
\[
d(x, y) = \begin{cases} x+y & \text{if } x \neq y, \\ 1 & \text{if } x = y \neq 0, \\ 0 & \text{if } x = y = 0.
\end{cases}
\] (22)

Clearly $d$ is a $d$-symmetric space. $d$ does not satisfy the triangular inequality since $d(0.1, 0.2) + d(0.2, 0.1) = 0.6 < 1 = d(0.1, 0.1)$.

We show that $C_1$ through $C_5$ holds. We first show that $\lim d(x_n, x) = 0$ if $x = 0$ and $\lim x_n = 0$ in $R$.

If $x \neq 0$ then $\lim d(x_n, x) = x_n + x \geq x > 0$. Hence $\lim d(x_n, x) = x$.

Moreover if $x = 0$ then $\lim d(x_n, 0) = 0$ or $x_n$. Hence $\lim d(x_n, x) = 0$.

Now we show that $\lim d(x_n, y_n) = 0$ if and only if $\lim x_n = \lim y_n = 0$ in $R$.

Consider $\lim d(x_n, y_n) = 0 \implies d(x_n, y_n) < 1/2$ for large $n$:
\[
\implies d(x_n, y_n) = x_n + y_n \quad \text{or} \quad 0 \quad \text{for large } n,
\]

\[
\implies \text{either } x_n = y_n = 0 \quad \text{or} \quad d(x_n, y_n) = x_n + y_n \quad \text{for large } n
\]
\[
\implies \lim x_n = \lim y_n = 0 \quad \text{in } R.
\]

Conversely if $\lim x_n = \lim y_n = 0$ in $R$ then $\lim d(x_n, y_n) = 0$ or $x_n + y_n$ for large $n$.

Hence $\lim d(x_n, y_n) = 0$.

Verifications of validity of $C_1$ through $C_5$ is done as follows.

$C_1$: let $\lim d(x_n, y_n) = 0$ and $\lim d(x_n, x) = 0$; then $\lim x_n = \lim y_n = 0$ in $R$ and $x = 0$.

Hence $d(y_n, x) = d(y_n, 0) = y_n$ or $0$. This implies that $\lim d(y_n, x) = 0$.

$C_2$: let $d(x_n, x) = d(y_n, x) = 0$. Then $x = 0$ and $\lim x_n = y_n = 0$ in $R$.

Hence $\lim d(y_n, x_n) = 0$.

$C_3$: let $d(x_n, y_n) = d(y_n, z_n) = 0$; then $\lim x_n = \lim y_n = \lim z_n = 0$ in $R$.

Hence $\lim d(x_n, z_n) = 0$.

$C_4$: let $\lim d(x_n, x) = 0$. Then $x = 0$ and $\lim x_n = 0$.

If $y = 0, 0 \leq d(x_n, y) \leq x_n$. Hence $\lim d(x_n, y) = 0 = d(x, y)$.

If $y \neq 0$, $d(x_n, y) = x_n + y$. Hence $\lim d(x_n, y) = y + 0 = d(x, y)$.

$C_5$: let $\lim d(x_n, x) = 0$ and $\lim d(x_n, y) = 0$.

Then $x = 0$, $y = 0$ and $\lim x_n = 0$. Hence $x = y$.

*Example 5.* A $d$-symmetric space $(X, d)$ in which $C_1$ [hence $C_3$] holds while $C_j$ does not hold for $j = 2, 3, 4$.
Let \( X = [0, \infty) \). Define \( d \) on \( X \times X \) as follows:
\[
d(x, y) = \begin{cases}
x + y & \text{if } x \neq 0 \neq y, \\
1 & \text{if } x \neq 0 = y, \\
1/y & \text{if } x = 0 \neq y, \\
0 & \text{if } x = 0 = y.
\end{cases}
\] (23)

Clearly \((X, d)\) is a \( d \)-symmetric space. We show that \( C_1, C_5 \) hold.

Let \( \lim d(x_n, x) = 0 = \lim d(x_n, y_n) \).

If \( x \neq 0 \), \( d(x_n, x) > x \) if \( x_n \neq 0 \).

Thus \( \lim d(x_n, x) \geq \min\{x, 1/x\} > 0 \). (25)

This implies that
\[
\lim d(x_n, x) = 0 \implies \lim d(x_n^{(1)}, y_n^{(1)}) = \lim d(x_n^{(2)}, y_n^{(2)}) = 0.
\] (26)

If we show that \( y_n^{(2)} \) cannot be positive for infinitely many \( n \), it will follow that \( \lim d(x_n^{(2)}, y_n^{(2)}) = \lim d(x_n^{(2)}, 0) = 0 \) so that \( \lim d(0, y_n) = 0 \). Hence \( C_1 \) holds.

If \( y_n^{(2)} \neq 0 \) for infinitely many \( n \), say \( \{y_{n_k}^{(2)}\} \) is the infinite subsequence of \( \{y_n^{(2)}\} \) with \( y_{n_k}^{(2)} \neq 0 \) \( \forall n_k \), then \( d(x_n^{(2)}, y_{n_k}^{(2)}) = x_{n_k}^{(2)} + y_{n_k}^{(2)} > x_{n_k}^{(2)} \) so that \( \lim d(x_n^{(2)}, y_{n_k}^{(2)}) \geq \lim x_{n_k}^{(2)} \geq \infty \) contradicting the assumption that \( \lim d(x_n, y_n) = 0 \). Thus \( C_1 \) holds.

Since \( C_1 \Rightarrow C_5, C_2 \) holds.

\( C_2 \) does not hold since \( d(n, 0) = 1/n \) while \( d(n, n) = 2n \) \( \forall n \) so that \( \lim d(n, n) \neq 0 \).

\( C_3 \) does not hold since \( \lim d(n, 0) = \lim d(0, n) = \lim d(n, n) = \infty \).

\( C_4 \) does not hold since \( \lim d(n, 0) = 0 \) but \( \lim d(n, 2) = \infty \) while \( d(0, 2) = 1/2 \).

Example 6. A \( d \)-symmetric space \((X, d)\) in which \( C_2 \) holds but \( C_1, C_3, C_4, C_5 \) fail.
Example 7. A $d$-symmetric space $(X, d)$ in which $C_4$ holds but $C_1$ fails.

Let $X = N \cup \{0\}$. Define $d$ on $X \times X$ as follows:

\[
d(m, n) = d(n, m) \quad \forall m, n \in X,
\]

\[
d(0, n) = \begin{cases} 
\frac{1}{n} & \text{if } n \text{ is odd}, \\
1 & \text{if } n \text{ is even},
\end{cases}
\]

\[
d(0, 0) = 0,
\]

\[
d(m, n) = \begin{cases} 
\frac{1}{|m - n|} & \text{if } m + n \text{ is even and } |m - n| \leq 1, \\
1 & \text{if } m + n \text{ is even and } |m - n| > 1.
\end{cases}
\]

(32)

If $\{x_n\}$ in $X$ and $\lim d(x_n, 0) = 0$ then $x_n$ is eventually odd.

If $x \neq 0$, $d(x_n, x)$ cannot be 1 so $x_n + x$ is even or odd and $|x_n - x| = 1$.

But in this case $d(x_n, x) = |1/x_n - 1/x|$ so that $d(x_n, x) \neq 0$.

Thus $d(x_n, x) = 0 \Leftrightarrow x = 0$ and $x_n$ is eventually odd.

If $m$ is a fixed even integer and $x_n$ is odd, $x_n + m$ is odd and eventually $2 > 1$.

So

\[
\lim d(x_n, m) = 1 = d(0, m).
\]

(33)

If $m$ is a fixed odd integer and $x_n$ is odd, $x_n + m$ is even.

So $d(x_n, m) = |1/m - 1/x_n|$ so that $\lim d(x_n, 0) = 0 \Rightarrow \lim d(x_n, m) = d(0, m)$.

If $m = 0$ and $x_n$ is odd eventually

\[
d(x_n, 0) = \frac{1}{n} \quad \text{so } \lim d(x_n, m) = \lim \frac{1}{n} = 0 = d(0, m).
\]

(34)

If $m = 0$ and $x_n = 0$ eventually

\[
d(x_n, 0) = \frac{1}{n} \quad \text{so } \lim d(x_n, m) = \lim \frac{1}{n} = 0 = d(0, m).
\]

(35)

Hence $C_4$ holds in $(X, d)$.

$C_1$ does not hold: let $x_n = 2n - 1$ and $y_n = 2n$:

\[
d(x_n, 0) = \frac{1}{2n - 1}, \quad d(x_n, y_n) = \frac{1}{2n - 1} - \frac{1}{2n},
\]

(36)

\[
d(y_n, 0) = 1.
\]

Hence $d(x_n, 0) = d(x_n, y_n) = 0$ and $d(y_n, 0) \neq 0$.

Example 8. A $d$-symmetric space $(X, d)$ in which $C_3$ holds but $C_4$ does not hold.

Let $X = [0, 1] \cup \{2\}$. Define $d$ on $X \times X$ as follows:

\[
d(x, y) = \begin{cases} 
x + y & \text{if } 0 \leq x \neq y \leq 1, \\
1 & \text{if } x = y \neq 0 \text{ or } x = y = 2, \\
2 & \text{if } x = 0 \text{ or } y = 2 \text{ or } x = 2 \text{ and } y = 0, \\
0 & \text{if } x = y = 0.
\end{cases}
\]

(37)

Clearly $(X, d)$ is a $d$-symmetric space which is not a symmetric space.

We first show that if $\{x_n\}$ converges to $x$ in $(X, d)$ then $x \in [0, 2]$.

Suppose that $0 \neq x \neq 2$; then $x \in (0, 1)$. Consider

\[
\lim d(x_n, y_n) = 0 \quad \text{if and only if } \lim x_n = \lim y_n = 0.
\]

(38)

\[
\Leftrightarrow \lim d(x_n, y_n) = 0 \quad \text{if and only if } \lim x_n = \lim y_n = 0.
\]

(39)

Conversely if $\lim x_n = \lim y_n = 0$ then $\forall N \in N$, $d(x_n, y_n) < 1$ for $n \geq N$.

\[
\Rightarrow \lim d(x_n, y_n) = 0 \quad \text{for } n \geq N.
\]

(40)

Hence $C_4$ holds in $(X, d)$.

$C_3$ fails: $x_n = 1/(n + 1)$ for $n \geq 1$:

\[
d(x_n, 0) = \frac{1}{n + 1} \Rightarrow \lim d(x_n, 0) = 0,
\]

(41)

\[
d(x_n, 2) = 1 \quad \forall n \Rightarrow \lim d(x_n, 2) = 1 \quad \text{but } d(0, 2) = 2.
\]

Example 9. A $d$-symmetric space $(X, d)$ in which $C_4$ holds but $C_2, C_3$ fail to hold.
Let $X = N \cup \{0, \infty\}$. Define $d$ on $X \times X$ as follows:

$$d(m, \infty) = d(\infty, m) = 1 \text{ if } m \in X,$$

$$d(m, 0) = d(0, m) \quad \text{if } m \in N,$$  \hspace{1cm} \text{(42)}

$$d(0, 0) = 0.$$  \hspace{1cm} \text{(43)}

If $m, n \in N$, then

$$d(m, n) = \begin{cases} \frac{1}{m} - \frac{1}{n} & \text{if } |m - n| \geq 2, \\ 1 & \text{if } |m - n| \leq 1. \end{cases}$$  \hspace{1cm} \text{(44)}

Clearly $(X, d)$ is a $d$-symmetric space which is not a symmetric space.

We show that if $\lim d(x_n, x) = 0$ then $x = 0$ and one of the subsequences $(x_{2m})$ and $(x_{2m - 1})$ of $x_n$ may possibly be finite, where $x_n = 0$ for all $n$ and $x_n \neq x_{n+1}$ for all $n$ and $x_n \neq 0$ for any $n$. This holds since $(X, d)$ is a $d$-symmetric space.

We show that $C_3$ holds. Assume that $\lim d(x_n, x) = 0$. Then $x = 0$.

Let $m \in N$ and $y_n = 0$ for all $n$. Then $d(y_n, m) = d(o, m) = 1/m$.

So $\lim d(y_n, m) = d(o, m)$.

If $m \neq 0$ for all $n$ and $m = 1/n$ for all $n > m$ then $\lim d(y_n, m) = 1/m = d(0, m)$.

Thus if $m \in N$ and $|x_n - x| = 0$ then $\lim d(x_n, m) = d(x, m) = 0$.

Clearly this holds when $m = \infty$ or $m = 0$ as well. Hence $C_3$ holds.

Remark: From this example we can conclude that

(1) $C_3$ does not imply $C_4$ as otherwise, since $C_3 \Rightarrow C_4$ it would follow that $C_4 \Rightarrow C_3$, which does not hold as is evident from the above example,

(2) in a $d$-symmetric space, convergent sequences are necessarily Cauchy sequences.

**Example 10.** A $d$-symmetric space $(X, d)$ in which $C_4$ holds but $C_2, C_3$ fail to hold.

Let $X = N \cup \{0\}$. Define $d$ on $X \times X$ as follows:

$$d(x, y) = d(y, x) = 1 \text{ for every } x, y \in X,$$

$$d(2m, 0) = 1, \quad d(2m - 1, 0) = \frac{1}{2m - 1}, \quad \forall m,$$

$$d(0, 0) = 0.$$  \hspace{1cm} \text{(45)}

Clearly $(X, d)$ is a $d$-symmetric space.

We first characterize all convergent sequences in $(X, d)$.

Suppose that $\lim d(x_n, x) = 0$. We show that $x = 0$.

If $x$ is odd and $x_n$ is even then $d(x_n, x) = 1$ if $x_n > x + 2$.

So $\lim d(x_n, x) \neq 0$. Thus $x_n$ is even for at most finitely many $n$.

We may thus assume that $x_n$ is odd for all $n$.

The $d(x_n, x) = 1/x_n + 1/x$ so that $d(x_n, x) \geq 1/x > 0$.

Hence $x$ cannot be odd. Now suppose that $x > 0$ and $x$ is even.

Then $d(x_n, x) = 1$ if $x_n = 0$ if $x_n$ is even and $|x_n - x| > 2$ while $d(x_n, x) = 1/x_n + 1/x$ if $x_n > x + 2$.

In all cases $\lim d(x_n, x) \neq 0$.

Hence the only possibility is $x = 0$.

We now show that the following are equivalent.

(a) $\lim d(x_n, x) = 0$ in $R$,

(b) there exists a positive integer $N$ such that $x_n$ is positive and even, only if $n < N$.

Assumption (b): $x_n$ is odd or zero if $n \geq N$ so that $\lim d(x_n, 0) = 0$.

Hence (b) implies (a).

Assumption (a): since $d(2m, 0) = 1$ for $m \in N$, it follows that at most finitely many terms of $\{x_n\}$ can be even. This proves (b). Thus $\lim d(x_n, x) = 0$ if $x = 0$ and $\exists N \in N \ni \forall n \geq N$ is "0" or odd for $n \geq N$.

Consequently $C_3$ holds.

**Example 11.** $C_1$ does not hold: let $x_n = 2n + 1, y_n = 2n$ and $x = 0$:

$$\lim d(x_n, x) = \lim \frac{1}{2n + 1} = 0,$$

$$\lim d(x_n, y_n) = \lim \frac{1}{2n + 1} + \frac{1}{2n} = 0.$$  \hspace{1cm} \text{(46)}

But $\lim d(y_n, x) = 1$ since $\lim d(2n, 0) = 1 \forall n$. 

$$\lim d(x_n, y_n) = \lim \frac{1}{2n + 1} + \frac{1}{2n} = 0.$$  \hspace{1cm} \text{(47)}

But $\lim d(y_n, x) = 1$ since $\lim d(2n, 0) = 1 \forall n$. 

$$\lim d(x_n, y_n) = \lim \frac{1}{2n + 1} = 0.$$  \hspace{1cm} \text{(48)}
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\[ C_2 \] holds: assume that \( \lim d(x_n, x) = 0 = \lim d(y_n, x) \). Then \( x = 0 \) and then there exists \( N \) such that \( x_n \) is “0” or odd and \( y_n = 0 \) or odd for \( n \geq N \) and \( \lim(1/x_n) = 0 \).

If \( x_n = y_n = 0 \), \( d(x_n, y_n) = 0 \).

If \( x_n = 0 \), \( y_n \) is odd, \( d(x_n, y_n) = 1/y_n \).

If \( y_n = 0 \), \( x_n \) is odd, \( d(x_n, y_n) = 1/x_n \).

If \( x_n \) is odd and \( y_n \) is odd, \( d(x_n, y_n) = 1/x_n + 1/y_n \).

Consequently \( \lim d(x_n, y_n) = 0 \).

\[ C_3 \] does not hold: let \( x_n = 0 \), \( y_n = 2n + 1 \), and \( z_n = 2n \):

\[
\begin{align*}
d(x_n, y_n) &= \frac{1}{2n + 1}, \\
d(y_n, z_n) &= \frac{1}{2n + 1} + \frac{1}{2n},
\end{align*}
\]

so that \( \lim d(x_n, y_n) = \lim d(y_n, z_n) = 0 \) but \( \lim d(x_n, z_n) = 1 \).

\[ C_4 \] does not hold: let \( x_n = 2n + 1 \), \( x = 0 \), and \( y = 3 \):

\[
\lim d(x_n, 0) = \lim \frac{1}{2n + 1} = 0,
\]

\[
\lim d(x_n, 3) = 1, \quad \lim d(0, 3) = \frac{1}{3}.
\]

Example 11. The following example shows that there exist symmetric spaces in which \( C \) does not hold.

Let \( X = \{0, 1/2, 1/3, 1/4, \ldots\} \).

Define \( d(x, y) = 0, d(x, y) = d(y, x) \)

\[
\begin{align*}
d\left(\frac{1}{n}, 0\right) &= d\left(0, \frac{1}{n}\right) = \frac{1}{n} \quad \forall n \text{ in } N, \\
d\left(\frac{1}{n}, \frac{1}{m}\right) &= 1 \quad \forall n, m \text{ in } N.
\end{align*}
\]

Then \( (X, d) \) is a symmetric space; \( \{1/n\} \) converges to 0 but is not a Cauchy sequence.

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Professor I. Ramabhadra Sarma is a retired professor from Acharya Nagarjuna University.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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