Research Article

p -Uniform Convexity and q -Uniform Smoothness of Absolute Normalized Norms on $\mathbb{C}^2$

Tomonari Suzuki

Department of Basic Sciences, Kyushu Institute of Technology, Tobata, Kitakyushu 804-8550, Japan

Correspondence should be addressed to Tomonari Suzuki; suzuki-t@mns.kyutech.ac.jp

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We first prove characterizations of p -uniform convexity and q -uniform smoothness. We next give a formulation on absolute normalized norms on $\mathbb{C}^2$. Using these, we present some examples of Banach spaces. One of them is a uniformly convex Banach space which is not p -uniformly convex.

1. Introduction

Throughout this paper, we denote by $\mathbb{N}$, $\mathbb{R}$, and $\mathbb{C}$ the sets of positive integers, real numbers, and complex numbers, respectively.

Let X be a *nontrivial* Banach space, which means a real Banach space with $\dim X \geq 2$ or a complex Banach space with $\dim X \geq 1$. The *modulus of convexity* of X is defined as

$$\delta(\varepsilon) = \inf \left(1 - \frac{\|x + y\|}{2} \right) \quad (1)$$

for $\varepsilon \in [0, 2]$, where the infimum can be taken over all $x, y \in X$ with $\|x\| \leq 1$, $\|y\| \leq 1$, and $\|x - y\| \geq \varepsilon$. The *modulus of smoothness* of X is defined as

$$\rho(\tau) = \sup \left(\frac{\|x + \tau y\| + \|x - \tau y\|}{2} - 1 \right) \quad (2)$$

for $\tau \in (0, \infty)$, where the supremum can be taken over all $x, y \in X$ with $\|x\| \leq 1$ and $\|y\| \leq 1$. It is obvious that $\rho(\tau) \leq \tau$. We know that if X is a Hilbert space, then $\delta(\varepsilon) = 1 - \sqrt{1 - \varepsilon^2/4}$ and $\rho(\tau) = \sqrt{1 + \tau^2} - 1$.

We recall that X is said to be *uniformly convex* if $\delta(\varepsilon) > 0$ for all $\varepsilon > 0$. Also, X is said to be *uniformly smooth* if $\lim_{\tau \rightarrow +0} \rho(\tau)/\tau = 0$.

For $p \in [2, \infty)$, X is called *p -uniformly convex* if there exists $C > 0$ satisfying

$$\delta(\varepsilon) \geq C\varepsilon^p \quad (3)$$

for all $\varepsilon \in [0, 2]$. On the other hand, for $q \in (1, 2]$, X is called *q -uniformly smooth* if there exists $K > 0$ satisfying

$$\rho(\tau) \leq K\tau^q \quad (4)$$

for all $\tau \in (0, \infty)$. It is obvious that p -uniformly convex Banach spaces are uniformly convex, and q -uniformly smooth Banach spaces are uniformly smooth. We also know that, for $p \in (1, \infty)$, L^p spaces are $\max\{2, p\}$ -uniformly convex and $\min\{2, p\}$ -uniformly smooth. See [1–6] and others.

A norm $\|\cdot\|$ on $\mathbb{C}^2$ is said to be *absolute* if

$$\|(x_1, x_2)\| = \|(|x_1|, |x_2|)\| \quad (5)$$

for all $(x_1, x_2) \in \mathbb{C}^2$ and *normalized* if $\|(1, 0)\| = \|(0, 1)\| = 1$. The ℓ_p -norms $\|\cdot\|_p$ are such examples:

$$\|(x_1, x_2)\|_p = \begin{cases} (|x_1|^p + |x_2|^p)^{1/p}, & \text{if } 1 \leq p < \infty, \\ \max\{|x_1|, |x_2|\}, & \text{if } p = \infty. \end{cases} \quad (6)$$

Let AN_2 be the family of all absolute normalized norms on $\mathbb{C}^2$. We let Ψ_2 be the set of all convex functions ψ on $[0, 1]$ satisfying

$$\max\{1 - t, t\} \leq \psi(t) \leq 1 \quad (7)$$

for $t \in [0, 1]$. Bonsall and Duncan in [7] showed the following characterization of absolute normalized norms on $\mathbb{C}^2$. Namely, the set AN_2 of all absolute normalized norms on

$\mathbb{C}^2$ is in one-to-one correspondence with Ψ_2 . The correspondence is given by

$$\psi(t) = \|(1-t, t)\| \quad \text{for } t \in [0, 1]. \quad (8)$$

Indeed, for any $\psi \in \Psi_2$, the norm $\|\cdot\|_\psi$ on $\mathbb{C}^2$ defined as

$$\|(x_1, x_2)\|_\psi = \begin{cases} (|x_1| + |x_2|) \\ \times \psi\left(\frac{|x_2|}{|x_1| + |x_2|}\right), & \text{if } (x_1, x_2) \neq (0, 0), \\ 0, & \text{if } (x_1, x_2) = (0, 0) \end{cases} \quad (9)$$

belongs to AN_2 and satisfies (8). Saito et al. in [8] extended this result to $\mathbb{C}^n$.

In this paper, we first prove characterizations of p -uniform convexity and q -uniform smoothness. We next give another formulation on absolute normalized norms on $\mathbb{C}^2$. Using these, we present some examples, one of which is a uniformly convex Banach space which is not p -uniformly convex.

2. Characterizations

In this section, we prove characterizations of p -uniform convexity and q -uniform smoothness.

Proposition 1. *Let X be a Banach space and let $p \in [2, \infty)$. Then the following are equivalent:*

- (i) X is p -uniformly convex,
- (ii) $\liminf_{\varepsilon \rightarrow +0} \delta(\varepsilon)/\varepsilon^p > 0$.

Proof. We first assume that $\liminf_{\varepsilon \rightarrow +0} \delta(\varepsilon)/\varepsilon^p = 0$. Then for every $C > 0$, there exists a small $\varepsilon > 0$ such that $\delta(\varepsilon)/\varepsilon^p < C$. That is, X is not p -uniformly convex. Conversely, we next assume that X is not p -uniformly convex. That is, for every $C > 0$, there exists $\varepsilon \in (0, 2]$ such that $\delta(\varepsilon) < C\varepsilon^p$. Putting $C = 1/n$, we can define a sequence $\{\varepsilon_n\}$ in $(0, 2]$ such that $\delta(\varepsilon_n)/\varepsilon_n^p < 1/n$. In the case of $\liminf_{n \rightarrow \infty} \varepsilon_n = 0$, without loss of generality, we may assume $\lim_{n \rightarrow \infty} \varepsilon_n = 0$. We have

$$0 \leq \liminf_{\varepsilon \rightarrow +0} \frac{\delta(\varepsilon)}{\varepsilon^p} \leq \liminf_{n \rightarrow \infty} \frac{\delta(\varepsilon_n)}{\varepsilon_n^p} \leq \lim_{n \rightarrow \infty} \frac{1}{n} = 0 \quad (10)$$

and hence $\liminf_{\varepsilon \rightarrow +0} \delta(\varepsilon)/\varepsilon^p = 0$. In the other case, there exists $\varepsilon_0 > 0$ such that $\varepsilon_0 < \varepsilon_n$ for all $n \in \mathbb{N}$. Then since δ is nondecreasing, we have

$$0 \leq \frac{\delta(\varepsilon_0)}{2^p} \leq \frac{\delta(\varepsilon_n)}{\varepsilon_n^p} < \frac{1}{n} \quad (11)$$

for $n \in \mathbb{N}$ and hence $\delta(\varepsilon_0) = 0$. Therefore, $\delta(\varepsilon) = 0$ for $\varepsilon \in [0, \varepsilon_0]$. This implies $\liminf_{\varepsilon \rightarrow +0} \delta(\varepsilon)/\varepsilon^p = 0$. $\square$

Proposition 2. *Let X be a Banach space and let $q \in (1, 2]$. Then the following are equivalent:*

- (i) X is q -uniformly smooth,
- (ii) $\limsup_{\tau \rightarrow +0} \rho(\tau)/\tau^q < \infty$.

Proof. We first assume that $\limsup_{\tau \rightarrow +0} \rho(\tau)/\tau^q = \infty$. Then for every $K > 0$, there exists a small $\tau > 0$ such that $\rho(\tau)/\tau^q > K$. That is, X is not q -uniformly smooth. Conversely, we next assume that X is not q -uniformly smooth. That is, for every $K > 0$, there exists $\tau > 0$ such that $\rho(\tau) > K\tau^q$. Putting $K = n$, we can define a sequence $\{\tau_n\}$ in $(0, \infty)$ such that $\rho(\tau_n)/\tau_n^q > n$. Then we have

$$n < \frac{\rho(\tau_n)}{\tau_n^q} \leq \frac{\tau_n}{\tau_n^q} = \frac{1}{\tau_n^{q-1}}. \quad (12)$$

Hence, $\lim_{n \rightarrow \infty} \tau_n = 0$ because $q - 1 > 0$. Therefore, we obtain

$$\limsup_{\tau \rightarrow +0} \frac{\rho(\tau)}{\tau^q} \geq \limsup_{n \rightarrow \infty} \frac{\rho(\tau_n)}{\tau_n^q} \geq \lim_{n \rightarrow \infty} n = \infty. \quad (13)$$

This completes the proof. $\square$

We know that Hilbert spaces are 2-uniformly convex and 2-uniformly smooth Banach spaces. We can easily check this thing by Propositions 1 and 2.

3. Convex Functions

In this section, we discuss properties of convex functions belonging to Ψ_2 . We first note that functions ψ belonging to Ψ_2 are continuous and satisfy $\psi(0) = \psi(1) = 1$ and $\psi(t) \geq 1/2$ for all $t \in [0, 1]$.

Let $\psi \in \Psi_2$. Then we define ψ'_-, ψ'_+ , and $\partial\psi$ as follows:

$$\psi'_-(s) = \lim_{t \rightarrow s-0} \frac{\psi(t) - \psi(s)}{t - s} \quad (14)$$

for $s \in (0, 1]$,

$$\psi'_+(s) = \lim_{t \rightarrow s+0} \frac{\psi(t) - \psi(s)}{t - s} \quad (15)$$

for $s \in [0, 1)$, and

$$\partial\psi(s) = \{a \in \mathbb{R} : \psi(t) \geq \psi(s) + a(t - s) \quad \forall t \in [0, 1]\} \quad (16)$$

for $s \in [0, 1]$. See [9] and others.

We know the following.

Lemma 3 (see [9, 10]). *Let $\psi \in \Psi_2$. Then the following hold:*

- (i) For $s, t, u \in [0, 1]$ with $0 \leq s < t < u \leq 1$,

$$\frac{\psi(t) - \psi(s)}{t - s} \leq \frac{\psi(u) - \psi(s)}{u - s} \leq \frac{\psi(u) - \psi(t)}{u - t} \quad (17)$$

holds.

(ii) For $s, t, u \in [0, 1]$ with $0 \leq s < t < u \leq 1$,

$$\begin{aligned} \psi'_+(s) &\leq \frac{\psi(t) - \psi(s)}{t - s} \leq \psi'_-(t) \leq \psi'_+(t) \leq \frac{\psi(u) - \psi(t)}{u - t} \\ &\leq \psi'_-(u) \end{aligned} \quad (18)$$

holds.

(iii) For $t \in [0, 1]$,

$$\partial\psi(t) = \begin{cases} (-\infty, \psi'_+(0)], & \text{if } t = 0, \\ [\psi'_-(t), \psi'_+(t)], & \text{if } 0 < t < 1, \\ [\psi'_-(1), +\infty), & \text{if } t = 1 \end{cases} \quad (19)$$

holds.

(iv) $\bigcup\{\partial\psi(t) : t \in [0, 1]\} = \mathbb{R}$ holds.

(v) $-1 \leq \psi'_+(0)$ and $\psi'_-(1) \leq 1$ hold.

Remark 4. (i)–(iii) are stated in [9]. (iv) follows from Theorem 24.1 in [9]. (v) is proved in [10].

Using Lemma 3, we can easily prove the following.

Lemma 5. Let $\psi \in \Psi_2$. Then the following hold:

- (i) $\psi'_+(t) \leq (1 - \psi(t))/(1 - t)$ for every $t \in [0, 1]$,
- (ii) $\psi'_-(t) \geq (\psi(t) - 1)/t$ for every $t \in (0, 1]$.

Lemma 6. Let $\psi \in \Psi_2$ and $s, t, u \in [0, 1]$ with $s < t < u$. Then

$$\begin{aligned} -1 &\leq \frac{\psi(t) - \psi(s)}{t - s} \leq \frac{1 - \psi(t)}{1 - t}, \\ \frac{\psi(t) - 1}{t} &\leq \frac{\psi(u) - \psi(t)}{u - t} \leq 1 \end{aligned} \quad (20)$$

hold.

The following lemma is used in Section 5.

Lemma 7. Let $\psi \in \Psi_2$ and $s, u \in [0, 1]$ with $s < u$. Then

$$u - s \leq \psi(u)(1 - 2s) + \psi(s)(2u - 1) \leq 2(u - s) \quad (21)$$

holds.

Proof. In the case of $s \leq 1/2 \leq u$, we have

$$\begin{aligned} u - s &= \frac{1}{2}(1 - 2s) + \frac{1}{2}(2u - 1) \\ &\leq \psi(u)(1 - 2s) + \psi(s)(2u - 1) \\ &\leq (1 - 2s) + (2u - 1) \\ &= 2(u - s). \end{aligned} \quad (22)$$

Using Lemma 6, we will prove this lemma in the other cases. In the case of $s > 1/2$, since $2\psi(s) - ((\psi(s) - 1)/s)(2s - 1) \leq 2$, we have

$$\begin{aligned} u - s &\leq (2\psi(u) + 1 - 2u)(u - s) \\ &= 2\psi(u)(u - s) - (u - s)(2u - 1) \\ &\leq 2\psi(u)(u - s) - (\psi(u) - \psi(s))(2u - 1) \\ &= \psi(u)(1 - 2s) + \psi(s)(2u - 1) \\ &= 2\psi(s)(u - s) - (\psi(u) - \psi(s))(2s - 1) \\ &\leq 2\psi(s)(u - s) - \frac{\psi(s) - 1}{s}(u - s)(2s - 1) \\ &= \left(2\psi(s) - \frac{\psi(s) - 1}{s}(2s - 1)\right)(u - s) \\ &\leq 2(u - s). \end{aligned} \quad (23)$$

In the case of $u < 1/2$, since $2\psi(u) - ((1 - \psi(u))/(1 - u))(2u - 1) \leq 2$, we have

$$\begin{aligned} u - s &\leq (2\psi(s) + 2s - 1)(u - s) \\ &= 2\psi(s)(u - s) + (u - s)(2s - 1) \\ &\leq 2\psi(s)(u - s) - (\psi(u) - \psi(s))(2s - 1) \\ &= \psi(u)(1 - 2s) + \psi(s)(2u - 1) \\ &= 2\psi(u)(u - s) - (\psi(u) - \psi(s))(2u - 1) \\ &\leq 2\psi(u)(u - s) - \frac{1 - \psi(u)}{1 - u}(u - s)(2u - 1) \\ &= \left(2\psi(u) - \frac{1 - \psi(u)}{1 - u}(2u - 1)\right)(u - s) \\ &\leq 2(u - s). \end{aligned} \quad (24)$$

This completes the proof. $\square$

We also know the following.

Lemma 8 (Bonsall and Duncan [7] page 37). Let $\psi \in \Psi_2$. Then the following hold:

- (i) the function $t \mapsto \psi(t)/t$ is nonincreasing;
- (ii) the function $t \mapsto \psi(t)/(1 - t)$ is nondecreasing.

The following lemma follows from Lemma 8.

Lemma 9. Let $\psi \in \Psi_2$ and $s, u \in [0, 1]$ with $s < u$. Then

$$\frac{s}{\psi(s)} \leq \frac{u}{\psi(u)}, \quad \frac{1 - s}{\psi(s)} \geq \frac{1 - u}{\psi(u)} \quad (25)$$

hold.

4. Absolute Normalized Norms on $\mathbb{C}^2$

We denote by Γ_2 the set of nondecreasing functions γ from $[0, 1]$ into $[-1, 1]$ satisfying $\int_0^1 \gamma(s) ds = 0$. The following

proposition says there are many absolute normalized norms on $\mathbb{C}^2$, and we can make many such norms easily.

Proposition 10. Define a mapping D from Ψ_2 into Γ_2 by

$$(D\psi)(t) = \begin{cases} \psi'_+(t), & \text{if } t \in [0, 1), \\ \psi'_-(t), & \text{if } t = 1 \end{cases} \quad (26)$$

for $\psi \in \Psi_2$ and $t \in [0, 1]$, and define a mapping S from Γ_2 into Ψ_2 by

$$(S\gamma)(t) = 1 + \int_0^t \gamma(s) ds \quad (27)$$

for $\gamma \in \Gamma_2$ and $t \in [0, 1]$. Then $D \circ S\gamma = \gamma$ a.e. and $S \circ D\psi = \psi$ for all $\gamma \in \Gamma_2$ and $\psi \in \Psi_2$.

Proof. Fix $\psi \in \Psi_2$ and put $\gamma = D\psi$. We will show $\gamma \in \Gamma_2$. By Lemma 3, γ is nondecreasing, $-1 \leq \psi'_+(0) = \gamma(0)$ and $\gamma(1) = \psi'_-(1) \leq 1$. Hence $\gamma(t) \in [-1, 1]$ for all $t \in [0, 1]$. By the definition of D , we have

$$1 = \psi(1) = \psi(0) + \int_0^1 \gamma(s) ds = 1 + \int_0^1 \gamma(s) ds. \quad (28)$$

This implies $\int_0^1 \gamma(s) ds = 0$. Therefore, we have shown $\gamma \in \Gamma_2$. Next, we fix $\gamma \in \Gamma_2$ and put $\psi = S\gamma$. We will show $\psi \in \Psi_2$. Since γ is nondecreasing, we have that ψ is convex. It is obvious that $\psi(0) = \psi(1) = 1$. From the convexity of ψ , $\psi(t) \leq 1$ for all $t \in [0, 1]$. Since $-1 \leq \gamma(t)$ for $t \in [0, 1]$, we have

$$\psi(t) = 1 + \int_0^t \gamma(s) ds \geq 1 + \int_0^t (-1) ds = 1 - t \quad (29)$$

for $t \in [0, 1]$. Since $\gamma(t) \leq 1$ for $t \in [0, 1]$, we also have

$$\begin{aligned} \psi(t) &= 1 + \int_0^t \gamma(s) ds \\ &= 1 + \int_0^1 \gamma(s) ds - \int_t^1 \gamma(s) ds \\ &= 1 - \int_t^1 \gamma(s) ds \geq 1 - \int_t^1 1 ds = t \end{aligned} \quad (30)$$

for $t \in [0, 1]$. Therefore $\psi \in \Psi_2$. The remains are obvious. $\square$

We next discuss the convexity and smoothness. In [11], Takahashi et al. proved that $(\mathbb{C}^2, \|\cdot\|_\psi)$ is strictly convex if and only if ψ is strictly convex. See also [8]. Using this fact, we can obtain the following.

Proposition 11. Let $\psi \in \Psi_2$. Then $(\mathbb{C}^2, \|\cdot\|_\psi)$ is strictly convex if and only if $D\psi$ is injective.

Proof. We assume that $(\mathbb{C}^2, \|\cdot\|_\psi)$ is strictly convex. Then ψ is strictly convex. That is, for $s, t, u \in [0, 1]$ with $0 \leq s < t < u \leq 1$, we have

$$\psi'_+(s) < \psi'_-(t) \leq \psi'_+(t) < \psi'_-(u). \quad (31)$$

Hence $D\psi$ is injective. We can easily prove the converse implication. $\square$

In [10], Mitani et al. proved that $(\mathbb{C}^2, \|\cdot\|_\psi)$ is smooth if and only if ψ is differentiable at any $t \in (0, 1)$ and $\psi'_+(0) = -1$ and $\psi'_-(1) = 1$. Using this fact, we can prove the following.

Proposition 12. Let $\psi \in \Psi_2$. Then $(\mathbb{C}^2, \|\cdot\|_\psi)$ is smooth if and only if $D\psi$ is surjective.

Proof. We assume that $(\mathbb{C}^2, \|\cdot\|_\psi)$ is smooth. Then ψ is differentiable at any $t \in (0, 1)$ and $\psi'_+(0) = -1$ and $\psi'_-(1) = 1$. So $(D\psi)(0) = -1$ and $(D\psi)(1) = 1$ are obvious. We note that $\partial\psi(0) = (-\infty, -1]$ and $\partial\psi(1) = [1, +\infty)$. For $a \in (-1, 1)$, there exists $t \in [0, 1]$ with $a \in \partial\psi(t)$. From the above note, we have $t \in (0, 1)$. From the differentiability, we obtain

$$\begin{aligned} a \in \partial\psi(t) &= [\psi'_-(t), \psi'_+(t)] \\ &= \{\psi'_-(t)\} = \{\psi'_+(t)\} = \{(D\psi)(t)\}. \end{aligned} \quad (32)$$

That is, $(D\psi)(t) = a$. Therefore we have shown $D\psi$ is surjective. Conversely, we next assume that $D\psi$ is surjective. We suppose that ψ is not differentiable at some $t \in (0, 1)$. Then we have $\psi'_-(t) < \psi'_+(t)$. By Lemma 3, we have

$$(D\psi)([0, 1]) \subset [-1, 1] \setminus (\psi'_-(t), \psi'_+(t)) \subsetneq [-1, 1]. \quad (33)$$

This contradicts the surjectivity of $D\psi$. Hence, ψ is differentiable at any $t \in (0, 1)$. We next suppose that $-1 < \psi'_+(0)$. Then by Lemma 3 again, we have

$$(D\psi)([0, 1]) \subset [-1, 1] \setminus [-1, \psi'_+(0)) \subsetneq [-1, 1]. \quad (34)$$

This is a contradiction. Hence, $\psi'_+(0) = -1$. We can similarly prove $\psi'_-(1) = 1$. Therefore, $(\mathbb{C}^2, \|\cdot\|_\psi)$ is smooth. $\square$

5. Examples

In this section, we present examples of absolute normalized norms on $\mathbb{C}^2$ satisfying that $(\mathbb{C}^2, \|\cdot\|_\psi)$ is uniformly convex and is not p -uniformly convex. We also present examples of such norms satisfying that $(\mathbb{C}^2, \|\cdot\|_\psi)$ is uniformly smooth and is not q -uniformly smooth. We note that, in finite dimensional Banach spaces, strict convexity and uniform convexity are equivalent. Smoothness and uniform smoothness are also equivalent.

Theorem 13. Let $\gamma \in \Gamma_2$ and $p \in [2, \infty)$. Assume that there exist sequences $\{s_n\}$ and $\{u_n\}$ in $[0, 1]$ such that $s_n < u_n$ for $n \in \mathbb{N}$,

$$\lim_{n \rightarrow \infty} (u_n - s_n) = 0, \quad \lim_{n \rightarrow \infty} \frac{\gamma(u_n) - \gamma(s_n)}{(u_n - s_n)^{p-1}} = 0. \quad (35)$$

Then $(\mathbb{C}^2, \|\cdot\|_{S\gamma})$ is not p -uniformly convex.

Proof. Put $\psi = S\gamma$. Without loss of generality, we may assume

$$\frac{\gamma(u_n) - \gamma(s_n)}{(u_n - s_n)^{p-1}} \leq \frac{1}{n} \quad (36)$$

for $n \in \mathbb{N}$, and $\{s_n\}$ and $\{u_n\}$ converge to some number $t_0 \in [0, 1]$. We put

$$t_n = \frac{(s_n/\psi(s_n)) + (u_n/\psi(u_n))}{(1/\psi(s_n)) + (1/\psi(u_n))} \quad (37)$$

for $n \in \mathbb{N}$. It is clear that $s_n < t_n < u_n$ for $n \in \mathbb{N}$. Define sequences $\{x_n\}$ and $\{y_n\}$ in $\mathbb{C}^2$ by

$$x_n = \frac{1}{\psi(s_n)} (1 - s_n, s_n), \quad y_n = \frac{1}{\psi(u_n)} (1 - u_n, u_n) \quad (38)$$

for $n \in \mathbb{N}$. It is obvious $\|x_n\| = \|y_n\| = 1$. Then we have

$$\begin{aligned} x_n + y_n &= \left(\frac{1 - s_n}{\psi(s_n)} + \frac{1 - u_n}{\psi(u_n)}, \frac{s_n}{\psi(s_n)} + \frac{u_n}{\psi(u_n)} \right) \\ &= \left(\frac{1}{\psi(s_n)} + \frac{1}{\psi(u_n)} \right) (1 - t_n, t_n). \end{aligned} \quad (39)$$

Thus,

$$\|x_n + y_n\| = \left(\frac{1}{\psi(s_n)} + \frac{1}{\psi(u_n)} \right) \psi(t_n). \quad (40)$$

We put

$$v_n = \frac{(u_n/\psi(u_n)) - (s_n/\psi(s_n))}{((1 - 2s_n)/\psi(s_n)) + ((2u_n - 1)/\psi(u_n))}. \quad (41)$$

By Lemma 9,

$$0 \leq \frac{u_n}{\psi(u_n)} - \frac{s_n}{\psi(s_n)} \leq \frac{1 - 2s_n}{\psi(s_n)} + \frac{2u_n - 1}{\psi(u_n)}. \quad (42)$$

From this inequality and (46), $v_n \in [0, 1]$ holds. Using v_n , we also have

$$\begin{aligned} \|x_n - y_n\| &= \left\| \left(\frac{1 - s_n}{\psi(s_n)} - \frac{1 - u_n}{\psi(u_n)}, \frac{s_n}{\psi(s_n)} - \frac{u_n}{\psi(u_n)} \right) \right\| \\ &= \left\| \left(\frac{1 - s_n}{\psi(s_n)} - \frac{1 - u_n}{\psi(u_n)}, \frac{u_n}{\psi(u_n)} - \frac{s_n}{\psi(s_n)} \right) \right\| \\ &= \left(\frac{1 - 2s_n}{\psi(s_n)} + \frac{2u_n - 1}{\psi(u_n)} \right) \|(1 - v_n, v_n)\| \\ &= \left(\frac{1 - 2s_n}{\psi(s_n)} + \frac{2u_n - 1}{\psi(u_n)} \right) \psi(v_n). \end{aligned} \quad (43)$$

Therefore, we obtain

$$\begin{aligned} \delta \left(\left(\frac{1 - 2s_n}{\psi(s_n)} + \frac{2u_n - 1}{\psi(u_n)} \right) \psi(v_n) \right) \\ \leq 1 - \frac{1}{2} \left(\frac{1}{\psi(s_n)} + \frac{1}{\psi(u_n)} \right) \psi(t_n). \end{aligned} \quad (44)$$

We will show $\liminf_{\varepsilon \rightarrow +0} \delta(\varepsilon)/\varepsilon^p = 0$. Before showing it, we need some inequalities:

$$\begin{aligned} 2\psi(s_n)\psi(u_n) - (\psi(u_n) + \psi(s_n))\psi(t_n) \\ &= \psi(s_n)(\psi(u_n) - \psi(t_n)) - \psi(u_n)(\psi(t_n) - \psi(s_n)) \\ &= \psi(s_n) \int_{t_n}^{u_n} \gamma(s) ds - \psi(u_n) \int_{s_n}^{t_n} \gamma(s) ds \\ &\leq \psi(s_n) \gamma(u_n)(u_n - t_n) - \psi(u_n) \gamma(s_n)(t_n - s_n) \\ &= \psi(s_n) \gamma(u_n) \left(u_n - \frac{(s_n/\psi(s_n)) + (u_n/\psi(u_n))}{(1/\psi(s_n)) + (1/\psi(u_n))} \right) \\ &\quad - \psi(u_n) \gamma(s_n) \left(\frac{(s_n/\psi(s_n)) + (u_n/\psi(u_n))}{(1/\psi(s_n)) + (1/\psi(u_n))} - s_n \right) \\ &= \frac{1}{(1/\psi(s_n)) + (1/\psi(u_n))} (\gamma(u_n) - \gamma(s_n))(u_n - s_n) \\ &\leq (\gamma(u_n) - \gamma(s_n))(u_n - s_n) \\ &\leq \frac{1}{n} (u_n - s_n)^p, \end{aligned} \quad (45)$$

$$\begin{aligned} \left(\frac{1 - 2s_n}{\psi(s_n)} + \frac{2u_n - 1}{\psi(u_n)} \right) \psi(v_n) \\ &= (\psi(u_n)(1 - 2s_n) + \psi(s_n)(2u_n - 1)) \\ &\quad \times \frac{\psi(v_n)}{\psi(s_n)\psi(u_n)} \\ &\geq (u_n - s_n) \frac{\psi(v_n)}{\psi(s_n)\psi(u_n)} \\ &\geq \frac{1}{2} (u_n - s_n) > 0 \end{aligned} \quad (46)$$

by Lemma 7. From (45) and (46), we have

$$\begin{aligned} \frac{\delta(((1 - 2s_n)/\psi(s_n)) + ((2u_n - 1)/\psi(u_n)))\psi(v_n)}{(((1 - 2s_n)/\psi(s_n)) + ((2u_n - 1)/\psi(u_n)))\psi(v_n)^p} \\ \leq \frac{1 - (1/2)((1/\psi(s_n)) + (1/\psi(u_n)))\psi(t_n)}{(((1 - 2s_n)/\psi(s_n)) + ((2u_n - 1)/\psi(u_n)))\psi(v_n)^p} \\ = \frac{1}{2\psi(s_n)\psi(u_n)} \\ \times \frac{2\psi(s_n)\psi(u_n) - (\psi(u_n) + \psi(s_n))\psi(t_n)}{(((1 - 2s_n)/\psi(s_n)) + ((2u_n - 1)/\psi(u_n)))\psi(v_n)^p} \\ \leq \frac{1}{2\psi(s_n)\psi(u_n)} \end{aligned}$$

$$\begin{aligned}
& \times \frac{(u_n - s_n)^p}{n(((1 - 2s_n)/\psi(s_n)) + ((2u_n - 1)/\psi(u_n))) \psi(v_n))^p} \\
& \leq \frac{1}{2n\psi(s_n)\psi(u_n)} 2^p \\
& \leq \frac{2}{n} 2^p, \\
& \limsup_{n \rightarrow \infty} \left(\frac{1 - 2s_n}{\psi(s_n)} + \frac{2u_n - 1}{\psi(u_n)} \right) \psi(v_n) \\
& \leq \lim_{n \rightarrow \infty} \left(\frac{1 - 2s_n}{\psi(s_n)} + \frac{2u_n - 1}{\psi(u_n)} \right) \\
& = \frac{1 - 2t_0}{\psi(t_0)} + \frac{2t_0 - 1}{\psi(t_0)} \\
& = 0.
\end{aligned} \tag{47}$$

These imply $\liminf_{\varepsilon \rightarrow +0} \delta(\varepsilon)/\varepsilon^p = 0$. So by Proposition 1, we obtain the desired result. $\square$

Corollary 14. Let $\gamma \in \Gamma_2$. Assume that γ is injective, γ is infinitely differentiable on the neighborhood of some $t_0 \in (0, 1)$, and

$$\gamma'(t_0) = \gamma''(t_0) = \gamma'''(t_0) = \cdots = 0. \tag{48}$$

Then $(\mathbb{C}^2, \|\cdot\|_{S_\gamma})$ is uniformly convex and is not p -uniformly convex for all $p \in [2, \infty)$.

Proof. Put $\psi = S_\gamma$. By Proposition 11, since γ is injective, $(\mathbb{C}^2, \|\cdot\|_\psi)$ is strictly convex and hence it is uniformly convex. By the L'Hospital theorem, for $n \in \mathbb{N}$ with $n \geq 2$, we have

$$\begin{aligned}
0 &= \lim_{u \rightarrow t_0+0} \frac{\gamma^{(n-1)}(u)}{(n-1)!} \\
&= \lim_{u \rightarrow t_0+0} \frac{\gamma^{(n-2)}(u)}{(n-1)!1!(u-t_0)} \\
&= \lim_{u \rightarrow t_0+0} \frac{\gamma^{(n-3)}(u)}{(n-1)!2!(u-t_0)^2} \\
&\vdots \\
&= \lim_{u \rightarrow t_0+0} \frac{\gamma'(u)}{(n-1)(u-t_0)^{n-2}} \\
&= \lim_{u \rightarrow t_0+0} \frac{\gamma(u) - \gamma(t_0)}{(u-t_0)^{n-1}}.
\end{aligned} \tag{49}$$

So, by Theorem 13, we have that $(\mathbb{C}^2, \|\cdot\|_\psi)$ is not n -uniformly convex for every $n \in \mathbb{N}$ with $n \geq 2$. Therefore, we obtain the desired result. $\square$

FIGURE 1

It is well known that a function f from $\mathbb{R}$ into $\mathbb{R}$ defined by

$$f(t) = \begin{cases} 0, & \text{if } t \leq 0, \\ \exp(-t^{-2}), & \text{if } t > 0 \end{cases} \tag{50}$$

for $t \in \mathbb{R}$ is strictly increasing on $[0, \infty)$, infinitely differentiable and $f^{(n)}(0) = 0$ for all $n \in \mathbb{N}$.

Example 15. Define $\gamma \in \Gamma_2$ by

$$\gamma(t) = \begin{cases} -\exp\left(4 - \left(t - \frac{1}{2}\right)^{-2}\right), & \text{if } t < \frac{1}{2}, \\ 0, & \text{if } t = \frac{1}{2}, \\ +\exp\left(4 - \left(t - \frac{1}{2}\right)^{-2}\right), & \text{if } t > \frac{1}{2} \end{cases} \tag{51}$$

for $t \in [0, 1]$. Then $(\mathbb{C}^2, \|\cdot\|_{S_\gamma})$ is uniformly convex and not p -uniformly convex for all $p \in [2, \infty)$. See Figure 1.

Theorem 16. Let $\gamma \in \Gamma_2$ and $q \in (1, 2]$. Assume that there exist a constant $\lambda \in (0, 1/2)$ and sequences $\{s_n\}$ and $\{u_n\}$ in $[0, 1]$ such that $s_n < u_n$ for $n \in \mathbb{N}$,

$$\lim_{n \rightarrow \infty} (u_n - s_n) = 0, \quad \lim_{n \rightarrow \infty} \frac{\gamma(\lambda s_n + (1 - \lambda) u_n) - \gamma((1 - \lambda) s_n + \lambda u_n)}{(u_n - s_n)^{q-1}} = \infty. \quad (52)$$

Then $(\mathbb{C}^2, \|\cdot\|_{S\gamma})$ is not q -uniformly smooth.

Proof. Put $\psi = S\gamma$. Without loss of generality, we may assume

$$\frac{\gamma(\lambda s_n + (1 - \lambda) u_n) - \gamma((1 - \lambda) s_n + \lambda u_n)}{(u_n - s_n)^{q-1}} \geq n \quad (53)$$

for $n \in \mathbb{N}$, and $\{s_n\}$ and $\{u_n\}$ converge to some number $t_0 \in [0, 1]$. We define a sequence $\{t_n\}$ by (37). Since

$$\lim_{n \rightarrow \infty} \frac{\psi(s_n)}{\psi(s_n) + \psi(u_n)} = \frac{1}{2}, \quad \lim_{n \rightarrow \infty} \frac{\psi(u_n)}{\psi(s_n) + \psi(u_n)} = \frac{1}{2}, \quad (54)$$

we may also assume that

$$\begin{aligned} \frac{\psi(s_n)}{\psi(s_n) + \psi(u_n)} &\in [\lambda, 1 - \lambda], \\ \frac{\psi(u_n)}{\psi(s_n) + \psi(u_n)} &\in [\lambda, 1 - \lambda] \end{aligned} \quad (55)$$

for $n \in \mathbb{N}$. We note that

$$(1 - \lambda) s_n + \lambda u_n \leq t_n \leq \lambda s_n + (1 - \lambda) u_n \quad (56)$$

because

$$t_n = \frac{\psi(u_n)}{\psi(s_n) + \psi(u_n)} s_n + \frac{\psi(s_n)}{\psi(s_n) + \psi(u_n)} u_n \quad (57)$$

for $n \in \mathbb{N}$. Define sequences $\{x_n\}$ and $\{y_n\}$ in $\mathbb{C}^2$ by

$$\begin{aligned} x_n &= \frac{1}{\psi(t_n)} (1 - t_n, t_n), \\ y_n &= \frac{(\psi(s_n)(1 - u_n) - \psi(u_n)(1 - s_n), \psi(s_n)u_n - \psi(u_n)s_n)}{(\psi(s_n) + \psi(u_n))\psi(t_n)} \end{aligned} \quad (58)$$

for $n \in \mathbb{N}$. It is obvious that $\|x_n\| = 1$. We put $v_n \in [0, 1]$ by (41). We have

$$\begin{aligned} \|y_n\| &= \frac{\|(\psi(s_n)(1 - u_n) - \psi(u_n)(1 - s_n), \psi(s_n)u_n - \psi(u_n)s_n)\|}{(\psi(s_n) + \psi(u_n))\psi(t_n)} \\ &= \frac{\|(\psi(s_n)(u_n - 1) + \psi(u_n)(1 - s_n), \psi(s_n)u_n - \psi(u_n)s_n)\|}{(\psi(s_n) + \psi(u_n))\psi(t_n)} \\ &= \frac{\psi(s_n)(2u_n - 1) + \psi(u_n)(1 - 2s_n)}{(\psi(s_n) + \psi(u_n))\psi(t_n)} \|(1 - v_n, v_n)\| \\ &= \frac{\psi(s_n)(2u_n - 1) + \psi(u_n)(1 - 2s_n)}{(\psi(s_n) + \psi(u_n))\psi(t_n)} \psi(v_n) \\ &\leq 2(\psi(s_n)(2u_n - 1) + \psi(u_n)(1 - 2s_n)) \\ &\leq 4(u_n - s_n) \end{aligned} \quad (59)$$

by Lemma 7. We note that $\lim_n \|y_n\| = 0$. We will calculate $\|x_n + y_n\|$ and $\|x_n - y_n\|$. We have

$$\begin{aligned} x_n + y_n &= \frac{1}{\psi(t_n)} \left(1 - t_n + \frac{\psi(s_n)(1 - u_n) - \psi(u_n)(1 - s_n)}{\psi(s_n) + \psi(u_n)}, \right. \\ &\quad \left. t_n + \frac{\psi(s_n)u_n - \psi(u_n)s_n}{\psi(s_n) + \psi(u_n)} \right) \\ &= \frac{1}{\psi(t_n)} \left(1 + \frac{\psi(s_n) - \psi(u_n)}{\psi(s_n) + \psi(u_n)} \right) (1 - u_n, u_n) \\ &= \frac{1}{\psi(t_n)} \frac{2\psi(s_n)}{\psi(s_n) + \psi(u_n)} (1 - u_n, u_n) \end{aligned} \quad (60)$$

because

$$\frac{t_n + ((\psi(s_n)u_n - \psi(u_n)s_n) / (\psi(s_n) + \psi(u_n)))}{1 + ((\psi(s_n) - \psi(u_n)) / (\psi(s_n) + \psi(u_n)))} = u_n. \quad (61)$$

Hence,

$$\|x_n + y_n\| = \frac{1}{\psi(t_n)} \frac{2\psi(s_n)\psi(u_n)}{\psi(s_n) + \psi(u_n)} \quad (62)$$

for $n \in \mathbb{N}$. Similarly, we obtain

$$x_n - y_n = \frac{1}{\psi(t_n)} \frac{2\psi(u_n)}{\psi(s_n) + \psi(u_n)} (1 - s_n, s_n) \quad (63)$$

and hence

$$\|x_n - y_n\| = \frac{1}{\psi(t_n)} \frac{2\psi(s_n)\psi(u_n)}{\psi(s_n) + \psi(u_n)} \quad (64)$$

for $n \in \mathbb{N}$. Therefore, we obtain

$$\begin{aligned} \rho(\|y_n\|) &\geq \frac{\|x_n + y_n\| + \|x_n - y_n\|}{2} - 1 \\ &= \frac{1}{\psi(t_n)} \frac{2\psi(s_n)\psi(u_n)}{\psi(s_n) + \psi(u_n)} - 1. \end{aligned} \quad (65)$$

From

$$\begin{aligned} &2\psi(s_n)\psi(u_n) - (\psi(u_n) + \psi(s_n))\psi(t_n) \\ &= \psi(s_n)(\psi(u_n) - \psi(t_n)) - \psi(u_n)(\psi(t_n) - \psi(s_n)) \\ &= \psi(s_n) \left(\int_{t_n}^{\lambda s_n + (1-\lambda)u_n} \gamma(s) ds + \int_{\lambda s_n + (1-\lambda)u_n}^{u_n} \gamma(s) ds \right) \\ &\quad - \psi(u_n) \left(\int_{s_n}^{(1-\lambda)s_n + \lambda u_n} \gamma(s) ds + \int_{(1-\lambda)s_n + \lambda u_n}^{t_n} \gamma(s) ds \right) \\ &\geq \psi(s_n)\gamma(t_n)(\lambda s_n + (1-\lambda)u_n - t_n) \\ &\quad + \psi(s_n)\gamma(\lambda s_n + (1-\lambda)u_n)\lambda(u_n - s_n) \\ &\quad - \psi(u_n)\gamma((1-\lambda)s_n + \lambda u_n)\lambda(u_n - s_n) \\ &\quad - \psi(u_n)\gamma(t_n)(t_n - (1-\lambda)s_n - \lambda u_n) \\ &= \psi(s_n)\gamma(t_n) \\ &\quad \times \left(\lambda s_n + (1-\lambda)u_n - \frac{(s_n/\psi(s_n)) + (u_n/\psi(u_n))}{(1/\psi(s_n)) + (1/\psi(u_n))} \right) \\ &\quad + \psi(s_n)\gamma(\lambda s_n + (1-\lambda)u_n)\lambda(u_n - s_n) \\ &\quad - \psi(u_n)\gamma((1-\lambda)s_n + \lambda u_n)\lambda(u_n - s_n) \\ &\quad - \psi(u_n)\gamma(t_n) \\ &\quad \times \left(\frac{(s_n/\psi(s_n)) + (u_n/\psi(u_n))}{(1/\psi(s_n)) + (1/\psi(u_n))} - (1-\lambda)s_n - \lambda u_n \right) \\ &= -\psi(s_n)\gamma(t_n)\lambda(u_n - s_n) \\ &\quad + \psi(s_n)\gamma(\lambda s_n + (1-\lambda)u_n)\lambda(u_n - s_n) \\ &\quad - \psi(u_n)\gamma((1-\lambda)s_n + \lambda u_n)\lambda(u_n - s_n) \\ &\quad + \psi(u_n)\gamma(t_n)\lambda(u_n - s_n) \\ &= \psi(s_n)\lambda(u_n - s_n)(\gamma(\lambda s_n + (1-\lambda)u_n) - \gamma(t_n)) \\ &\quad + \psi(u_n)\lambda(u_n - s_n)(\gamma(t_n) - \gamma((1-\lambda)s_n + \lambda u_n)) \\ &\geq \frac{1}{2}\lambda(u_n - s_n)(\gamma(\lambda s_n + (1-\lambda)u_n) - \gamma(t_n)) \\ &\quad + \frac{1}{2}\lambda(u_n - s_n)(\gamma(t_n) - \gamma((1-\lambda)s_n + \lambda u_n)) \end{aligned}$$

$$\begin{aligned} &= \frac{1}{2}\lambda(u_n - s_n) \\ &\quad \times (\gamma(\lambda s_n + (1-\lambda)u_n) - \gamma((1-\lambda)s_n + \lambda u_n)) \\ &\geq \frac{1}{2}\lambda n(u_n - s_n)^q \end{aligned} \quad (66)$$

and (59), we have

$$\begin{aligned} \frac{\rho(\|y_n\|)}{\|y_n\|^q} &\geq \frac{1}{(4(u_n - s_n))^q} \frac{(1/2)\lambda n(u_n - s_n)^q}{\psi(t_n)(\psi(s_n) + \psi(u_n))} \\ &\geq \frac{1}{4^q} \frac{(1/2)\lambda n}{\psi(t_n)(\psi(s_n) + \psi(u_n))} \geq \frac{1}{4^{q+1}}\lambda n. \end{aligned} \quad (67)$$

Hence we obtain $\limsup_{\tau \rightarrow +0} \rho(\tau)/\tau^q = \infty$. So by Proposition 2, we obtain the desired result. $\square$

Corollary 17. Let f be a bijective and strictly increasing function from $[-1, 1]$ into $[0, 1]$ with $\int_{-1}^1 f(a)da = 1$. Assume that f is infinitely differentiable on the neighborhood of some $a_0 \in (-1, 1)$, and

$$f'(a_0) = f''(a_0) = f'''(a_0) = \dots = 0. \quad (68)$$

Then $f^{-1} \in \Gamma_2$ and $(\mathbb{C}^2, \|\cdot\|_{Sf^{-1}})$ is uniformly smooth and is not q -uniformly smooth for all $q \in (1, 2]$.

Proof. It is not difficult to check $f^{-1} \in \Gamma_2$. Put $\gamma = f^{-1}$ and $\psi = S\gamma$. By Proposition 12, since γ is surjective, $(\mathbb{C}^2, \|\cdot\|_\psi)$ is smooth and hence it is uniformly smooth. Fix $v \in \mathbb{N}$. As in the proof of Corollary 14, we can prove $\lim_{b \rightarrow a_0+0} (f(b) - f(a_0))/(b - a_0)^v = 0$. Since f is strictly increasing, we have

$$\lim_{b \rightarrow a_0+0} \frac{(b - a_0)^v}{f(b) - f(a_0)} = \infty. \quad (69)$$

Putting $u = f(b)$ and $t_0 = f(a_0)$, we have

$$\begin{aligned} \lim_{u \rightarrow t_0+0} \frac{\gamma(u) - \gamma(t_0)}{(u - t_0)^{1/v}} &= \lim_{b \rightarrow a_0+0} \frac{b - a_0}{(f(b) - f(a_0))^{1/v}} \\ &= \lim_{b \rightarrow a_0+0} \left(\frac{(b - a_0)^v}{f(b) - f(a_0)} \right)^{1/v} \\ &= \infty. \end{aligned} \quad (70)$$

We choose a strictly increasing sequence $\{s_n\}$ and a strictly decreasing sequence $\{u_n\}$ in $[0, 1]$ satisfying $t_0 = (2/3)s_n + (1/3)u_n$ for $n \in \mathbb{N}$ and $\lim_n s_n = \lim_n u_n = t_0$. Then it is obvious that $t_0 < (1/3)s_n + (2/3)u_n$ for $n \in \mathbb{N}$ and $\lim_n ((1/3)s_n + (2/3)u_n) = t_0$. We have

$$\begin{aligned} &\lim_{n \rightarrow \infty} \frac{\gamma((1/3)s_n + (2/3)u_n) - \gamma((2/3)s_n + (1/3)u_n)}{(u_n - s_n)^{1/v}} \\ &= \frac{1}{3^{1/v}} \lim_{n \rightarrow \infty} \frac{\gamma((1/3)s_n + (2/3)u_n) - \gamma(t_0)}{((1/3)s_n + (2/3)u_n - t_0)^{1/v}} = \infty. \end{aligned} \quad (71)$$

Thus, by Theorem 16, we have that $(\mathbb{C}^2, \|\cdot\|_\psi)$ is not $(1+1/\nu)$ -uniformly smooth. Since ν is arbitrary, $(\mathbb{C}^2, \|\cdot\|_\psi)$ is not q -uniformly smooth for every $q \in (1, 2]$. $\square$

Example 18. Define a function f from $[-1, 1]$ onto $[0, 1]$ by

$$f(a) = \begin{cases} \frac{-\exp(1-a^{-2})}{2} + \frac{1}{2}, & \text{if } a < 0, \\ \frac{1}{2}, & \text{if } a = 0, \\ \frac{+\exp(1-a^{-2})}{2} + \frac{1}{2} & \text{if } a > 0 \end{cases} \quad (72)$$

for $a \in [-1, 1]$. Then $(\mathbb{C}^2, \|\cdot\|_{Sf^{-1}})$ is uniformly smooth and not q -uniformly smooth for all $q \in (1, 2]$. See Figure 1.

Example 19. Let γ be as in Example 15 and let f be as in Example 18. Define a function η from $[0, 1]$ into $[-1, 1]$ by

$$\eta(t) = \begin{cases} \frac{f^{-1}(4t)}{4} - \frac{3}{4}, & \text{if } t \leq \frac{1}{4}, \\ \frac{\gamma(2t-1/2)}{2}, & \text{if } \frac{1}{4} \leq t \leq \frac{3}{4}, \\ \frac{f^{-1}(4t-3)}{4} + \frac{3}{4}, & \text{if } t \geq \frac{3}{4} \end{cases} \quad (73)$$

for $t \in [0, 1]$. Then $(\mathbb{C}^2, \|\cdot\|_{S\eta})$ is uniformly convex, uniformly smooth, not p -uniformly convex for all $p \in [2, \infty)$, and not q -uniformly smooth for all $q \in (1, 2]$. See Figure 1.

Conflict of Interests

The author declares that there is no conflict of interests regarding the publication of this paper.

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