Research Article

Least Squares Estimation for α -Fractional Bridge with Discrete Observations

Guangjun Shen and Xiuwei Yin

Department of Mathematics, Anhui Normal University, Wuhu 241000, China

Correspondence should be addressed to Guangjun Shen; gjsen@163.com

Received 15 November 2013; Accepted 7 December 2013; Published 23 January 2014

Academic Editor: Litan Yan

Copyright © 2014 G. Shen and X. Yin. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

We consider a fractional bridge defined as $dX_t = -\alpha(X_t/(T-t))dt + dB_t^H$, $0 \leq t < T$, where B^H is a fractional Brownian motion of Hurst parameter $H > 1/2$ and parameter $\alpha > 0$ is unknown. We are interested in the problem of estimating the unknown parameter $\alpha > 0$. Assume that the process is observed at discrete time $t_i = i\Delta_n$, $i = 0, \dots, n$, and $T_n = n\Delta_n$ denotes the length of the “observation window.” We construct a least squares estimator $\hat{\alpha}_n$ of α which is consistent; namely, $\hat{\alpha}_n$ converges to α in probability as $n \rightarrow \infty$.

1. Introduction

Self-similar stochastic processes with long range dependence are of practical interest in various applications, including econometrics, internet traffic, and hydrology. These are processes $X = \{X_t : t \geq 0\}$ whose dependence on the time parameter t is self-similar, in the sense that there exists a (self-similarity) parameter $H \in (0, 1)$ such that, for any constant $c \geq 0$, $\{X_{ct} : t \geq 0\}$ and $\{c^H X_t : t \geq 0\}$ have the same distribution. These processes are often endowed with other distinctive properties.

The fractional Brownian motion (fBm) is the usual candidate to model phenomena in which the self-similarity property can be observed from the empirical data. The fBm is a suitable generalization of the standard Brownian motion, which exhibits long-range dependence and self-similarity and has stationary increments. Some surveys and complete literatures could be found in Biagini et al. [1], Hu [2], Mishura [3], and Nualart [4].

Recently, Es-Sebaï and Nourdin [5] study the asymptotic properties of a least squares estimator for the parameter α of a fractional bridge defined as

$$X_0 = 0, \quad dX_t = -\alpha \frac{X_t}{T-t} dt + dB_t^H, \quad 0 \leq t < T, \quad (1)$$

where B^H is a fBm with Hurst parameter $H > 1/2$ and the process X was observed continuously. In particular,

when $H = 1/2$, Barczy and Pap [6, 7] study the various problems related to the α -Wiener bridge. The parametric estimation problems for fractional diffusion processes based on continuous-time observations have been studied, for example, in Tudor and Viens [8], Hu and Nualart [9], and Belfadli et al. [10].

In applications usually the process cannot be observed continuously. Only discrete-time observations are available. There exists a rich literature on the parameter estimation problem for diffusion processes driven by fBm based on discrete observations (see, e.g., Hu and Song [11], Es-Sebaï [12]).

Motivated by all these results, in this paper, we will consider the α fractional bridge (1). Assume that the process X is observed equidistantly in time with the step size $t_i = i\Delta_n$, $i = 0, \dots, n$, and $T_n = n\Delta_n$ denotes the length of the “observation window.” We also assume that $T_n + \Delta_n = T$ and $\Delta_n \rightarrow 0$ when $n \rightarrow \infty$. Our goal is to study the asymptotic behavior of the least squares estimator (LSE for short) $\hat{\alpha}_n$ of α based on the sampling data X_{t_i} , $i = 0, \dots, n$. Our techniques used in this work are inspired from Es-Sebaï [12].

The least squares estimator $\hat{\alpha}_n$ aims to minimize

$$\alpha \mapsto \sum_{i=1}^n \int_{t_{i-1}}^{t_i} \left| \dot{X}_t + \alpha \frac{X_{t_{i-1}}}{T-t_{i-1}} \right|^2 dt. \quad (2)$$

This is a quadratic function of α . The minimum is achieved when

$$\hat{\alpha}_n = -\frac{\sum_{i=1}^n \int_{t_{i-1}}^{t_i} (X_{t_{i-1}} / (T - t_{i-1})) \delta^H X_t}{\Delta_n \sum_{i=1}^n (X_{t_{i-1}}^2 / (T - t_{i-1})^2)}. \quad (3)$$

By (1), we can get the following result

$$\hat{\alpha}_n - \alpha = -\frac{\sum_{i=1}^n M_i}{\Delta_n \sum_{i=1}^n (X_{t_{i-1}}^2 / (T - t_{i-1})^2)}, \quad (4)$$

where $M_i = \alpha(X_{t_{i-1}} / (T - t_{i-1})) \int_{t_{i-1}}^{t_i} ((X_{t_{i-1}} / (T - t_{i-1})) - (X_s / (T - s))) ds + \int_{t_{i-1}}^{t_i} (X_{t_{i-1}} / (T - t_{i-1})) \delta^H B_t^H, i = 1, \dots, n$.

The paper is organized as follows. In Section 2 some known results that we will use are recalled. The consistency of estimator is proved in Section 3.

2. Preliminaries

Recall that fBm B^H with index $H \in (0, 1)$ is a mean zero Gaussian process $B^H = \{B_t^H, t \geq 0\}$ with $B_0^H = 0$ and the covariance

$$R^H(t, s) := E(B_t B_s) = \frac{1}{2} (t^{2H} + s^{2H} - |t - s|^{2H}) \quad (5)$$

for all $s, t \geq 0$. For $H = 1/2$, B^H coincides with the standard Brownian motion B . B^H is neither a semimartingale nor a Markov process unless $H = 1/2$, so many of the powerful techniques from stochastic analysis are not available when dealing with B^H . It is possible to construct a stochastic calculus of variations with respect to the Gaussian process B^H , which will be related to the Malliavin calculus. Some surveys and complete literatures could be found in Alòs et al. [13], Nualart [4] and the reference. We recall here the basic definitions and results of this calculus. The crucial ingredient is the canonical Hilbert space $\mathcal{H}$ (it is also said to be reproducing kernel Hilbert space) associated with the fBm which is defined as the closure of the linear space $\mathcal{E}$ generated by the indicator functions $\{1_{[0,t]}, t \in [0, T]\}$ with respect to the scalar product

$$\langle 1_{[0,t]}, 1_{[0,s]} \rangle_{\mathcal{H}} = R_H(t, s) = \frac{1}{2} (t^{2H} + s^{2H} - |t - s|^{2H}). \quad (6)$$

The mapping $1_{[0,s]} \rightarrow B_s^H$ can be extended to a linear isometry between $\mathcal{H}$ and the Gaussian space associated with B^H . We will denote the isometry by $\varphi \rightarrow B^H(\varphi)$. For $1/2 < H < 1$ we denote by $\mathcal{S}$ the set of smooth functionals of the form

$$F = f(B^H(\varphi_1), \dots, B^H(\varphi_n)), \quad (7)$$

where $f \in C_b^\infty(\mathbb{R}^n)$ and $\varphi_i \in \mathcal{H}$. The Malliavin derivative of a functional F as above is given by

$$D^H F = \sum_{i=1}^n \frac{\partial f}{\partial x_i} (B^H(\varphi_1), \dots, B^H(\varphi_n)) \varphi_i, \quad (8)$$

and this operator can be extended to the closure $\mathbb{D}^{m,2}$ ($m \geq 1$) of $\mathcal{S}$ with respect to the norm

$$\|F\|_{m,2}^2 \equiv E|F|^2 + E\|D^H F\|_{\mathcal{H}}^2 + \dots + E\|D^{H,m} F\|_{\mathcal{H}^{\otimes m}}^2, \quad (9)$$

where $\mathcal{H}^{\otimes m}$ denotes the m fold symmetric tensor product and the m th derivative $D^{H,m}$ is defined by iteration. The divergence integral δ^H is the adjoint operator of D^H . Concretely, a random variable $u \in L^2(\Omega, \mathcal{H})$ belongs to the domain of the divergence operator δ^H (in symbol $\text{Dom}(\delta^H)$) if

$$E|\langle D^H F, u \rangle_{\mathcal{H}}| \leq c\|F\|_{L^2} \quad (10)$$

for every $F \in \mathcal{S}$. In this case $\delta^H(u)$ is given by the duality relationship

$$E(F \delta^H(u)) = E\langle D^H F, u \rangle_{\mathcal{H}} \quad (11)$$

for any $F \in \mathbb{D}^{1,2}$, and we have the following integration by parts:

$$F \delta^H(u) = \delta^H(Fu) + \langle D^H F, u \rangle_{\mathcal{H}} \quad (12)$$

for any $u \in \text{Dom}(\delta^H)$, $F \in \mathbb{D}^{1,2}$ such that $Fu \in L^2(\Omega, \mathcal{H})$. It follows that

$$E[\delta^H(u)^2] = E\|u\|_{\mathcal{H}}^2 + E\langle D^H u, (D^H u)^* \rangle_{\mathcal{H} \otimes \mathcal{H}}, \quad (13)$$

where $(D^H u)^*$ is the adjoint of $D^H u$ in the Hilbert space $\mathcal{H} \otimes \mathcal{H}$, and

$$\|u\|_{\mathcal{H}}^2 = \iint_0^T u_s u_r \phi_H(s, r) ds dr, \quad (14)$$

where

$$\phi_H(s, r) = \frac{\partial^2 R_H}{\partial s \partial r}(s, r) = H(2H - 1)|s - r|^{2H-2} \geq 0, \quad (15)$$

and, for $\varphi : [0, T]^2 \rightarrow \mathbb{R}$, we have

$$\begin{aligned} \|\varphi\|_{\mathcal{H} \otimes \mathcal{H}}^2 &= \int_{[0,T]^4} \varphi(t, s) \varphi(t', s') \phi_H(t, t') \phi_H(s, s') dt ds dt' ds'. \end{aligned} \quad (16)$$

We denote by $|\mathcal{H}|$ the subspace of $\mathcal{H}$, which is defined as the set of measurable functions f on $[0, T]$ with

$$\|f\|_{|\mathcal{H}|}^2 := \iint_0^T |f(s)| |f(r)| \phi_H(s, r) ds dr < \infty. \quad (17)$$

Note that, if $\varphi, \psi \in |\mathcal{H}|$, then

$$EB^H(\varphi) B^H(\psi) = \iint_0^T \varphi(s) \psi(r) \phi_H(s, r) ds dr. \quad (18)$$

It follows actually from Pipiras and Taqqu [14] that the space $|\mathcal{H}| \subset \mathcal{H}$ is a Banach space for the norm $\|\cdot\|_{|\mathcal{H}|}$. Moreover,

$$L^2([0, T]) \subset L^{1/H}([0, T]) \subset |\mathcal{H}| \subset \mathcal{H}. \quad (19)$$

If $u \in \mathbb{D}^{1,2}(|\mathcal{H}|)$, $u \in \text{Dom}(\delta^H)$, then we have (Nualart [4])

$$E(\delta^H(u))^2 \leq C_H \left(E\|u\|_{|\mathcal{H}|}^2 + E\|D^H(u)\|_{|\mathcal{H}| \otimes |\mathcal{H}|}^2 \right) \quad (20)$$

and if $\varphi : [0, T]^2 \rightarrow \mathbb{R}$, then

$$\begin{aligned} & \|\varphi\|_{|\mathcal{H}| \otimes |\mathcal{H}|}^2 \\ &= \int_{[0, T]^4} |\varphi(t, s)| |\varphi(t', s')| \phi_H(t, t') \phi_H(s, s') dt ds dt' ds'. \end{aligned} \quad (21)$$

As a consequence, we have

$$E(\delta^H(u))^2 \leq C_H \left(E\|u\|_{L^{1/H}([0, T])}^2 + E\|D^H(u)\|_{L^{1/H}([0, T]^2)}^2 \right). \quad (22)$$

For every $n \geq 1$, let $\mathcal{H}_n$ be the n th Wiener chaos of B^H , that is, the closed linear subspace of $L^2(\Omega)$ generated by the random variables $\{H_n(B^H(h)), h \in \mathcal{H}, \|h\|_{\mathcal{H}} = 1\}$, where H_n is the n th Hermite polynomial. The mapping $I_n(h^{\otimes n}) = n!H_n(B^H(h))$ provides a linear isometry between the symmetric tensor product $\mathcal{H}^{\otimes n}$ (equipped with the modified norm $\|\cdot\|_{\mathcal{H}^{\otimes n}} = (1/\sqrt{n!})\|\cdot\|_{\mathcal{H}^{\otimes n}}$) and $\mathcal{H}_n$. For every $f, g \in \mathcal{H}^{\otimes n}$ the following multiplication formula holds

$$E(I_n(f)I_n(g)) = n! \langle f, g \rangle_{\mathcal{H}^{\otimes n}}. \quad (23)$$

Let $f, g : [0, T] \rightarrow \mathbb{R}$ be Hölder continuous functions of orders $\alpha \in (0, 1)$ and $\beta \in (0, 1)$ with $\alpha + \beta > 1$. Young proved that the Riemann-Stieltjes integral (so-called Young integral) $\int_0^T f_s dg_s$ exists. Moreover, if $\alpha = \beta \in (1/2, 1)$ and $F : \mathbb{R}^2 \rightarrow \mathbb{R}$ is a function of class $\mathcal{C}^1$, the integrals $\int_0^T (\partial F / \partial f)(f_u, g_u) df_u$ and $\int_0^T (\partial F / \partial g)(f_u, g_u) dg_u$ exist in the Young sense and the following change of variables formula holds:

$$\begin{aligned} F(f_t, g_t) &= F(f_0, g_0) + \int_0^t \frac{\partial F}{\partial f}(f_u, g_u) df_u \\ &\quad + \int_0^t \frac{\partial F}{\partial g}(f_u, g_u) dg_u, \quad t \in [0, T]. \end{aligned} \quad (24)$$

As a consequence, if $H \in (1/2, 1)$ and $(u_t, t \in [0, T])$ is a process with Hölder paths of order $\alpha < (1-H, 1)$, the integral $\int_0^T u_s dB_s^H$ is well defined as Young integral. Suppose that, for any $t \in [0, T]$, $u_t \in \mathbb{D}^{1,2}(|\mathcal{H}|)$, and

$$\int_0^T |D_s u_t| |t-s|^{2H-2} ds dt < \infty \quad \text{a.s.} \quad (25)$$

Then, following from Alòs and Nualart [15], we have

$$\begin{aligned} \int_0^t u_s dB_s^H &= \int_0^t u_s \delta^H B_s^H + H(2H-1) \\ &\quad \times \int_0^t D_s u_r |r-s|^{2H-2} dr ds. \end{aligned} \quad (26)$$

In particular, when φ is a nonrandom Hölder continuous function of order $\alpha \in (1-H, 1)$, we have

$$\int_0^t \varphi_s dB_s^H = \int_0^t \varphi_s \delta^H B_s^H = B^H(\varphi). \quad (27)$$

In addition, for all $\varphi, \psi \in |\mathcal{H}|$,

$$\begin{aligned} E\left(\int_0^T \varphi_s dB_s^H \int_0^T \psi_s dB_s^H\right) \\ = H(2H-1) \iint_0^T \varphi_u \psi_v |u-v|^{2H-2} du dv. \end{aligned} \quad (28)$$

3. Asymptotic Behavior of the Least Squares Estimator

Throughout this paper we assume $H \in (1/2, 1)$. We will study (1) driven by a fractional Brownian motion B^H with Hurst parameter H and $\alpha > 0$ being the unknown parameter to be estimated for discretely observed X . It is readily checked that we have the following explicit expression for X_t :

$$X_t = (T-t)^\alpha \int_0^t (T-s)^{-\alpha} dB_s^H, \quad 0 \leq t < T, \quad (29)$$

where the integral can be understood as Young integral. In order to study the asymptotic behavior of the least squares estimator, let us introduce the following processes:

$$A_t := \int_0^t (T-s)^{-\alpha} dB_s^H, \quad 0 \leq t < T. \quad (30)$$

Hence, we have

$$X_t = (T-t)^\alpha A_t, \quad 0 \leq t < T. \quad (31)$$

For simplicity, we assume that the notation $a_n \geq b_n$ means that there exists positive constants $C = C_{H,\alpha} > 0$ (depending only on H, α and its value may differ from line to line) so that

$$\sup_{n \geq 1} \frac{|a_n|}{|b_n|} < C < \infty. \quad (32)$$

We firstly give the following lemmas.

Lemma 1. *Let $\alpha > 0$, $1/2 < H < 1$. Then*

$$\int_0^{T_n} \frac{X_s}{T-s} dB_s^H = \int_0^{T_n} \frac{X_s}{T-s} \delta^H B_s^H + \beta_n, \quad (33)$$

where

$$\beta_n = H(2H-1) \int_0^{T_n} \int_0^r (T-r)^{\alpha-1} (T-s)^{-\alpha} (r-s)^{2H-2} ds dr,$$

$$\lim_{n \rightarrow \infty} \beta_n = HB(\alpha, 2H-1) T^{2H-1}. \quad (34)$$

Proof. By (26), we have

$$\begin{aligned}
 \int_0^{T_n} \frac{X_s}{T-s} dB_s^H &= \int_0^{T_n} \frac{X_s}{T-s} \delta^H B_s^H + H(2H-1) \\
 &\quad \times \int_0^{T_n} D_s^H \frac{X_r}{T-r} |s-r|^{2H-2} dr ds \\
 &= \int_0^{T_n} \frac{X_s}{T-s} \delta^H B_s^H + H(2H-1) \\
 &\quad \times \int_0^{T_n} \int_0^r (T-r)^{\alpha-1} (T-s)^{-\alpha} (r-s)^{2H-2} ds dr \\
 &= \int_0^{T_n} \frac{X_s}{T-s} \delta^H B_s^H + \beta_n.
 \end{aligned} \tag{35}$$

On the other hand,

$$\begin{aligned}
 \lim_{n \rightarrow \infty} \beta_n &= H(2H-1) \\
 &\quad \times \lim_{n \rightarrow \infty} \int_0^{T_n} \int_0^r (T-r)^{\alpha-1} (T-s)^{-\alpha} (r-s)^{2H-2} ds dr \\
 &= H(2H-1) \lim_{n \rightarrow \infty} \int_{T-T_n}^T \int_r^T r^{\alpha-1} s^{-\alpha} (s-r)^{2H-2} ds dr \\
 &= H(2H-1) \lim_{n \rightarrow \infty} \int_0^{T_n} \int_r^{T_n} (T-T_n+r)^{\alpha-1} \\
 &\quad \times (T-T_n+s)^{-\alpha} (s-r)^{2H-2} ds dr \\
 &= H(2H-1) \int_0^T \int_r^T r^{\alpha-1} s^{-\alpha} (s-r)^{2H-2} ds dr \\
 &= H(2H-1) \int_0^T s^{-\alpha} \int_0^s r^{\alpha-1} (s-r)^{2H-2} dr ds \\
 &= HB(\alpha, 2H-1) T^{2H-1}.
 \end{aligned} \tag{36}$$

This completes the proof. $\square$

The following Lemma 2 comes from Lemma 3.2 of Es-Sebaï and Nourdin [5].

Lemma 2. Letting $0 < \alpha < H$, $1/2 < H < 1$, one has

$$\begin{aligned}
 E\left(\frac{X_t}{T-t}\right)^2 &\leq \frac{H(2H-1)}{H-\alpha} B(1-\alpha, 2H-1) (T-t)^{2\alpha-2} T^{2H-2\alpha}, \\
 0 \leq t &< T.
 \end{aligned} \tag{37}$$

Lemma 3. Assume $1-H < \alpha < H$, $1/2 < H < 1$, and let $F_{T_n} = \int_0^{T_n} (X_t/(T-t)) \delta^H B_t^H$. Then

$$\begin{aligned}
 \lim_{n \rightarrow \infty} E(F_{T_n}^2) &= \frac{H^2(2H-1)^2 B(\alpha, 2H-1) B(1-\alpha, 2H-1)}{2(H+\alpha-1)(H-\alpha)} T^{4H-2}.
 \end{aligned} \tag{38}$$

Proof. By the isometry property of the double stochastic integral I_2 , the variance of F_{T_n} is given by

$$E(F_{T_n}^2) = \frac{H^2(2H-1)^2}{2} I_{T_n}, \tag{39}$$

where

$$\begin{aligned}
 I_{T_n} &= \int_{[0, T_n]^4} (T-t_1)^{\alpha-1} (T-s_1)^{-\alpha} (T-t_2)^{\alpha-1} (T-s_2)^{-\alpha} \\
 &\quad \times |s_1-s_2|^{2H-2} |t_1-t_2|^{2H-2} ds_1 ds_2 dt_1 dt_2.
 \end{aligned} \tag{40}$$

Now, we study I_{T_n} , by setting

$$\begin{aligned}
 I_1 &= \int_{[0, T_n]^2} (T-s_1)^{-\alpha} (T-s_2)^{-\alpha} |s_1-s_2|^{2H-2} ds_1 ds_2, \\
 I_2 &= \int_{[0, T_n]^2} (T-t_1)^{\alpha-1} (T-t_2)^{\alpha-1} |t_1-t_2|^{2H-2} dt_1 dt_2.
 \end{aligned} \tag{41}$$

We have $I_{T_n} = I_1 I_2$. By (17.40) of Es-Sebaï and Nourdin [5], we have

$$\begin{aligned}
 I_1 &= \int_{[0, T_n]^2} (T-s_1)^{-\alpha} (T-s_2)^{-\alpha} |s_1-s_2|^{2H-2} ds_1 ds_2 \\
 &= \frac{B(1-\alpha, 2H-1)}{H-\alpha} T_n^{2H-2\alpha} \\
 &\longrightarrow \frac{B(1-\alpha, 2H-1)}{H-\alpha} T^{2H-2\alpha}, \quad n \longrightarrow \infty.
 \end{aligned} \tag{42}$$

Similarly

$$I_2 \longrightarrow \frac{B(\alpha, 2H-1)}{H+\alpha-1} T^{2H+2\alpha-2}. \tag{43}$$

Thus, the proof is finished. $\square$

The following theorem gives the consistency of the least squares estimator $\hat{\alpha}_n$ of α .

Theorem 4. Let $1/2 < \alpha < H < 1$. If $\Delta_n \rightarrow 0$, $T_n = n\Delta_n \rightarrow T$ as $n \rightarrow \infty$, and $T_n + \Delta_n = T$, then, one has

$$\hat{\alpha}_n \xrightarrow{P} \alpha, \quad n \longrightarrow \infty, \tag{44}$$

where $\xrightarrow{P}$ means convergence in probability.

Proof. By (4), we have

$$\hat{\alpha}_n - \alpha = -\frac{(\alpha/n) \sum_{i=1}^n M_i}{(\alpha \Delta_n/n) \sum_{i=1}^n (X_{t_{i-1}}^2/(T-t_{i-1})^2)}. \quad (45)$$

Letting $0 < \varepsilon < 1$, we obtain

$$\begin{aligned} P(|\hat{\alpha}_n - \alpha| > \varepsilon) &= P\left(\left|\frac{(\alpha/n) \sum_{i=1}^n M_i}{(\alpha \Delta_n/n) \sum_{i=1}^n (X_{t_{i-1}}^2/(T-t_{i-1})^2)}\right| > \varepsilon\right) \\ &\leq P\left(\left|\frac{\alpha}{n} \sum_{i=1}^n M_i\right| > \varepsilon(1-\varepsilon)\right) \\ &\quad + P\left(\left|\frac{\alpha \Delta_n}{n} \sum_{i=1}^n \frac{X_{t_{i-1}}^2}{(T-t_{i-1})^2} - 1\right| > \varepsilon\right) \\ &:= B_1(n) + B_2(n). \end{aligned} \quad (46)$$

First, we considering the term $B_1(n)$, we have

$$\begin{aligned} B_1(n) &= P\left(\left|\frac{\alpha}{n} \sum_{i=1}^n M_i\right| > \varepsilon(1-\varepsilon)\right) \\ &\leq P\left(\left|\frac{\alpha}{n} \sum_{i=1}^n \left[M_i - \int_{t_{i-1}}^{t_i} \frac{X_{t_{i-1}}}{T-t_{i-1}} \delta^H B_t^H\right]\right| > \frac{1}{3}\varepsilon(1-\varepsilon)\right) \\ &\quad + P\left(\left|\frac{\alpha}{n} \sum_{i=1}^n \int_{t_{i-1}}^{t_i} \left(\frac{X_{t_{i-1}}}{T-t_{i-1}} - \frac{X_t}{T-t}\right) \delta^H B_t^H\right| > \frac{1}{3}\varepsilon(1-\varepsilon)\right) \\ &\quad + P\left(\left|\frac{\alpha}{n} \int_0^{T_n} \frac{X_t}{T-t} \delta^H B_t^H\right| > \frac{1}{3}\varepsilon(1-\varepsilon)\right) \\ &:= B_{1,1}(n) + B_{1,2}(n) + B_{1,3}(n). \end{aligned} \quad (47)$$

For the term $B_{1,1}(n)$, using Lemma 2, we obtain

$$\begin{aligned} \sum_{i=1}^n E\left[\left|M_i - \int_{t_{i-1}}^{t_i} \frac{X_{t_{i-1}}}{T-t_{i-1}} \delta^H B_t^H\right|\right] &\leq \alpha \sum_{i=1}^n \left(E\left(\frac{X_{t_{i-1}}}{T-t_{i-1}}\right)^2\right)^{1/2} \\ &\quad \times \int_{t_{i-1}}^{t_i} \left(E\left(\frac{X_{t_{i-1}}}{T-t_{i-1}} - \frac{X_t}{T-t}\right)^2\right)^{1/2} dt \end{aligned}$$

$$\begin{aligned} &\leq \sum_{i=1}^n (T-t_{i-1})^{\alpha-1} \int_{t_{i-1}}^{t_i} \left(E\left(\frac{X_{t_{i-1}}}{T-t_{i-1}} - \frac{X_t}{T-t}\right)^2\right)^{1/2} dt \\ &\leq \Delta_n^{\alpha-1} \sum_{i=1}^n \int_{t_{i-1}}^{t_i} \left(E\left(\frac{X_{t_{i-1}}}{T-t_{i-1}} - \frac{X_t}{T-t}\right)^2\right)^{1/2} dt \\ &\leq \Delta_n^{\alpha-1} \left[\sum_{i=1}^n \int_{t_{i-1}}^{t_i} \left(E\left(\frac{X_{t_{i-1}}}{T-t_{i-1}}\right)^2\right)^{1/2} dt \right. \\ &\quad \left. + \sum_{i=1}^n \int_{t_{i-1}}^{t_i} \left(E\left(\frac{X_t}{T-t}\right)^2\right)^{1/2} dt \right] \\ &\leq \Delta_n^{\alpha-1} \left[\sum_{i=1}^n \int_{t_{i-1}}^{t_i} \left(E\left(\frac{X_{t_{i-1}}}{T-t_{i-1}}\right)^2\right)^{1/2} dt \right. \\ &\quad \left. + \sum_{i=1}^n \int_{t_{i-1}}^{t_i} \left(E\left(\frac{X_{t_i}}{T-t_i}\right)^2\right)^{1/2} dt \right] \\ &\leq n \Delta_n^{2\alpha-1}. \end{aligned} \quad (48)$$

So, we get

$$\frac{\alpha}{n} \sum_{i=1}^n E\left[\left|M_i - \int_{t_{i-1}}^{t_i} \frac{X_{t_{i-1}}}{T-t_{i-1}} \delta^H B_t^H\right|\right] \leq \Delta_n^{2\alpha-1}. \quad (49)$$

Hence,

$$B_{1,1}(n) \leq \frac{\Delta_n^{2\alpha-1}}{\varepsilon(1-\varepsilon)}. \quad (50)$$

For the term $B_{1,2}(n)$, it follows the fact that, for $0 \leq t < T$,

$$\begin{aligned} \frac{X_{t_{i-1}}}{T-t_{i-1}} - \frac{X_t}{T-t} &= -\left[\left((T-t)^{\alpha-1} - (T-t_{i-1})^{\alpha-1}\right) A_{t_{i-1}} \right. \\ &\quad \left. + (T-t)^{\alpha-1} (A_t - A_{t_{i-1}})\right]. \end{aligned} \quad (51)$$

We have

$$\begin{aligned} E\left[\sum_{i=1}^n \int_{t_{i-1}}^{t_i} \left(\frac{X_{t_{i-1}}}{T-t_{i-1}} - \frac{X_t}{T-t}\right) \delta^H B_t^H\right] &\leq E\left[\sum_{i=1}^n \int_{t_{i-1}}^{t_i} \left((T-t)^{\alpha-1} - (T-t_{i-1})^{\alpha-1}\right) A_{t_{i-1}} \delta^H B_t^H\right] \\ &\quad + E\left[\sum_{i=1}^n \int_{t_{i-1}}^{t_i} (T-t)^{\alpha-1} (A_t - A_{t_{i-1}}) \delta^H B_t^H\right]. \end{aligned} \quad (52)$$

Using inequality (22) and $EA_t = 0$, $D_s^H A_t = (T-s)^{-\alpha} 1_{[0,t]}(s)$, we have

$$\begin{aligned}
& E \left| \sum_{i=1}^n \int_{t_{i-1}}^{t_i} ((T-t)^{\alpha-1} - (T-t_{i-1})^{\alpha-1}) A_{t_{i-1}} \delta^H B_t^H \right| \\
&= E \left| \int_0^{T_n} \sum_{i=1}^n ((T-t)^{\alpha-1} - (T-t_{i-1})^{\alpha-1}) A_{t_{i-1}} 1_{(t_{i-1}, t_i]}(t) \delta^H B_t^H \right| \\
&\leq \left(E \left| \int_0^{T_n} \sum_{i=1}^n ((T-t)^{\alpha-1} - (T-t_{i-1})^{\alpha-1}) \right. \right. \\
&\quad \left. \left. \times A_{t_{i-1}} 1_{(t_{i-1}, t_i]}(t) \delta^H B_t^H \right|^2 \right)^{1/2} \\
&\leq C_H \left(\int_0^{T_n} \left| \sum_{i=1}^n ((T-t)^{\alpha-1} - (T-t_{i-1})^{\alpha-1}) \right. \right. \\
&\quad \left. \left. \times D_s^H A_{t_{i-1}} 1_{(t_{i-1}, t_i]}(t) \right|^{1/H} ds dt \right)^H \\
&= C_H \left(\int_0^{T_n} \sum_{i=1}^n |((T-t)^{\alpha-1} - (T-t_{i-1})^{\alpha-1}) (T-s)^{-\alpha}|^{1/H} \right. \\
&\quad \left. \times 1_{(t_{i-1}, t_i]}(t) 1_{[0, t_{i-1}]}(s) ds dt \right)^H \\
&= C_H \left(\sum_{i=1}^n \int_{t_{i-1}}^{t_i} ((T-t)^{\alpha-1} - (T-t_{i-1})^{\alpha-1})^{1/H} dt \right. \\
&\quad \left. \times \int_0^{t_{i-1}} (T-s)^{-\alpha/H} ds \right)^H \\
&\leq C_H T^{H-\alpha} \left(\sum_{i=1}^n \int_{t_{i-1}}^{t_i} \Delta_n^{(\alpha-1)/H} dt \right)^H \leq C_H T^{H-\alpha} n \Delta_n^{H+\alpha-1}. \tag{53}
\end{aligned}$$

On the other hand,

$$\begin{aligned}
& E \left| \sum_{i=1}^n \int_{t_{i-1}}^{t_i} (T-t)^{\alpha-1} (A_t - A_{t_{i-1}}) \delta^H B_t^H \right| \\
&= E \left| \int_0^{T_n} \sum_{i=1}^n (T-t)^{\alpha-1} (A_t - A_{t_{i-1}}) 1_{(t_{i-1}, t_i]}(t) \delta^H B_t^H \right| \\
&\leq C_H \left(\int_0^{T_n} \left| \sum_{i=1}^n (T-t)^{\alpha-1} D_s^H (A_t - A_{t_{i-1}}) \right. \right. \\
&\quad \left. \left. \times 1_{(t_{i-1}, t_i]}(t) \right|^{1/H} ds dt \right)^H
\end{aligned}$$

$$\begin{aligned}
&\leq C_H \left(\int_0^{T_n} \sum_{i=1}^n |(T-t)^{\alpha-1} D_s^H (A_t - A_{t_{i-1}})|^{1/H} \right. \\
&\quad \left. \times 1_{(t_{i-1}, t_i]}(t) ds dt \right)^H \\
&= C_H \left(\int_0^{T_n} \sum_{i=1}^n ((T-t)^{\alpha-1} (T-s)^{-\alpha})^H 1_{[t_{i-1}, t]}(s) \right. \\
&\quad \left. \times 1_{(t_{i-1}, t_i]}(t) ds dt \right)^H \\
&= C_H \left(\sum_{i=1}^n \int_{t_{i-1}}^{t_i} (T-t)^{(\alpha-1)/H} dt \int_{t_{i-1}}^t (T-s)^{-\alpha/H} ds \right)^H \\
&\leq C_H \left(\sum_{i=1}^n \int_{t_{i-1}}^{t_i} (T-t)^{(\alpha-1)/H} dt \int_{t_{i-1}}^t (T-t_n)^{-\alpha/H} ds \right)^H \\
&\leq C_H (n \Delta_n^{(2H-1)/H})^H \leq C_H n \Delta_n^{2H-1}. \tag{54}
\end{aligned}$$

So, we get

$$E \left| \sum_{i=1}^n \int_{t_{i-1}}^{t_i} \left(\frac{X_{t_{i-1}}}{T-t_{i-1}} - \frac{X_t}{T-t} \right) \delta^H B_t^H \right| \geq n \Delta_n^{H+\alpha-1}. \tag{55}$$

Thus,

$$\frac{\alpha}{n} E \left| \sum_{i=1}^n \int_{t_{i-1}}^{t_i} \left(\frac{X_{t_{i-1}}}{T-t_{i-1}} - \frac{X_t}{T-t} \right) \delta^H B_t^H \right| \geq \Delta_n^{H+\alpha-1}. \tag{56}$$

Hence

$$B_{1,2}(n) \geq \frac{\Delta_n^{H+\alpha-1}}{\varepsilon(1-\varepsilon)}. \tag{57}$$

For the term $B_{1,3}(n)$, by setting $F_{T_n} = \int_0^{T_n} (X_t/(T-t)) \delta^H B_t^H$ and by using Lemma 3, we get

$$\begin{aligned}
B_{1,3}(n) &= P \left(\left| \frac{\alpha}{n} \int_0^{T_n} \frac{X_t}{T-t} \delta^H B_t^H \right| > \frac{1}{3} \varepsilon (1-\varepsilon) \right) \\
&\leq \left[\frac{3\alpha}{\varepsilon(1-\varepsilon)n} \right]^2 E(F_{T_n}^2) \geq \frac{1}{\varepsilon^2(1-\varepsilon)^2 n^2}. \tag{58}
\end{aligned}$$

As a consequence,

$$B_1(n) \geq \frac{\Delta_n^{2\alpha-1}}{\varepsilon(1-\varepsilon)} + \frac{\Delta_n^{H+\alpha-1}}{\varepsilon(1-\varepsilon)} + \frac{1}{\varepsilon^2(1-\varepsilon)^2 n^2}. \tag{59}$$

Second, we estimate the term $B_2(n)$:

$$\begin{aligned}
 B_2(n) &= P\left(\left|\frac{\alpha\Delta_n}{n}\sum_{i=1}^n\left(\frac{X_{t_{i-1}}}{T-t_{i-1}}\right)^2-1\right|>\varepsilon\right) \\
 &\leq P\left(\left|\frac{\alpha}{n}\sum_{i=1}^n\int_{t_{i-1}}^{t_i}\left[\left(\frac{X_{t_{i-1}}}{T-t_{i-1}}\right)^2-\left(\frac{X_t}{T-t}\right)^2\right]dt\right|>\varepsilon/2\right) \\
 &\quad + P\left(\left|\frac{\alpha}{n}\int_0^{T_n}\left(\frac{X_t}{T-t}\right)^2dt-1\right|>\varepsilon/2\right) \\
 &:= B_{2,1}(n) + B_{2,2}(n).
 \end{aligned} \tag{60}$$

We firstly consider $B_{2,1}(n)$, since

$$\begin{aligned}
 E\left|\frac{\alpha}{n}\sum_{i=1}^n\int_{t_{i-1}}^{t_i}\left[\left(\frac{X_{t_{i-1}}}{T-t_{i-1}}\right)^2-\left(\frac{X_t}{T-t}\right)^2\right]dt\right| \\
 \leq \frac{\alpha}{n}\sum_{i=1}^n\int_{t_{i-1}}^{t_i}E\left|\left(\frac{X_{t_{i-1}}}{T-t_{i-1}}\right)^2-\left(\frac{X_t}{T-t}\right)^2\right|dt \\
 \leq \frac{\alpha}{n}\sum_{i=1}^n\int_{t_{i-1}}^{t_i}\left(E\left(\frac{X_{t_{i-1}}}{T-t_{i-1}}\right)^2+E\left(\frac{X_t}{T-t}\right)^2\right)dt \tag{61} \\
 \leq \frac{\alpha}{n}\sum_{i=1}^n\int_{t_{i-1}}^{t_i}\left(E\left(\frac{X_{t_{i-1}}}{T-t_{i-1}}\right)^2+E\left(\frac{X_{t_i}}{T-t_i}\right)^2\right)dt \\
 \leq \frac{2\alpha}{n}\sum_{i=1}^n\Delta_n^{2\alpha-1}\triangleq\Delta_n^{2\alpha-1}.
 \end{aligned}$$

By Markov inequality, we obtain

$$B_{2,1}(n)\triangleq\frac{\Delta_n^{2\alpha-1}}{\varepsilon}. \tag{62}$$

Now, we estimate the term $B_{2,2}(n)$. Applying the change of variable formula (24), we get

$$\begin{aligned}
 \frac{\alpha}{n}\int_0^{T_n}\left(\frac{X_t}{T-t}\right)^2dt-1 &= \frac{1}{n(\alpha-(1/2))} \\
 &\quad \times\left(\frac{X_{T_n}}{2\Delta_n}-\int_0^{T_n}\frac{X_t}{T-t}\delta^HB_t^H-\beta_n\right).
 \end{aligned} \tag{63}$$

Hence,

$$\begin{aligned}
 B_{2,2}(n) &\leq P\left(\left|\frac{X_{T_n}}{T_n(2\alpha-1)}\right|>\frac{\varepsilon}{6}\right) \\
 &\quad + P\left(\left|\frac{1}{n(\alpha-(1/2))}\int_0^{T_n}\frac{X_t}{T-t}\delta B_t^H\right|>\frac{\varepsilon}{6}\right) \tag{64} \\
 &\quad + P\left(\left|\frac{\beta_n}{n(\alpha-(1/2))}\right|>\frac{\varepsilon}{6}\right).
 \end{aligned}$$

By Markov inequality and Lemma 2, we obtain

$$B_{2,2}(n)\triangleq\frac{\Delta_n^{2\alpha}}{\varepsilon^2T_n^2}+\frac{1}{\varepsilon n^2}+\frac{1}{\varepsilon n}. \tag{65}$$

Therefore

$$B_2(n)\triangleq\frac{\Delta_n^{2\alpha-1}}{\varepsilon}+\frac{\Delta_n^{2\alpha}}{\varepsilon^2T_n^2}+\frac{1}{\varepsilon n^2}+\frac{1}{\varepsilon n}\leq\frac{\Delta_n^{2\alpha-1}}{\varepsilon}+\frac{\Delta_n^{2\alpha}}{\varepsilon^2T_n^2}+\frac{1}{\varepsilon n}. \tag{66}$$

Combining (59) and (66), this completes the proof. $\square$

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

Acknowledgments

The authors would like to thank anonymous earnest referee whose remarks and suggestions greatly improved the presentation of the paper. Guangjun Shen is partially supported by the National Natural Science Foundation of China (11271020) and the Natural Science Foundation of Anhui Province (1208085MA11). Xiuwei Yin is partially supported by Mathematical Tianyuan Foundation of China (11326171) and the Natural Science Foundation of Anhui Province (1308085QA14).

References

- [1] F. Biagini, Y. Hu, B. Øksendal, and T. Zhang, *Stochastic Calculus for fBm and Applications. Probability and Its Application*, Springer, Berlin, Germany, 2008.
- [2] Y. Hu, "Integral transformations and anticipative calculus for fractional Brownian motions," *Memoirs of the American Mathematical Society*, vol. 175, no. 825, 2005.
- [3] Y. S. Mishura, *Stochastic Calculus for Fractional Brownian Motion and Related Processes*, vol. 1929 of *Lecture Notes in Mathematics*, Springer, Berlin, Germany, 2008.
- [4] D. Nualart, *The Malliavin Calculus and Related Topics*, Springer, Berlin, Germany, 2nd edition, 2006.
- [5] K. Es-Sebaïy and I. Nourdin, "Parameter estimation for α fractional bridges," in *Malliavin Calculus and Stochastic Analysis: A Festschrift in Honor of David Nualart*, F. Viens, J. Feng, Y. Hu, and E. Nualart, Eds., vol. 34 of *Springer Proceedings in Mathematics and Statistics*, pp. 385–412, 2013.
- [6] M. Barczy and G. Pap, "Asymptotic behavior of maximum likelihood estimator for time inhomogeneous diffusion processes," *Journal of Statistical Planning and Inference*, vol. 140, no. 6, pp. 1576–1593, 2010.
- [7] M. Barczy and G. Pap, " α -Wiener bridges: singularity of induced measures and sample path properties," *Stochastic Analysis and Applications*, vol. 28, no. 3, pp. 447–466, 2010.
- [8] C. A. Tudor and F. G. Viens, "Statistical aspects of the fractional stochastic calculus," *The Annals of Statistics*, vol. 35, no. 3, pp. 1183–1212, 2007.
- [9] Y. Hu and D. Nualart, "Parameter estimation for fractional Ornstein-Uhlenbeck processes," *Statistics & Probability Letters*, vol. 80, no. 11-12, pp. 1030–1038, 2010.

- [10] R. Belfadli, K. Es-Sebaiy, and Y. Ouknine, "Parameter estimation for fractional Ornstein-Uhlenbeck processes: non-ergodic case," *Frontiers in Science and Engineering*, vol. 1, pp. 1–16, 2011.
- [11] Y. Hu and J. Song, "Parameter estimation for fractional Ornstein-Uhlenbeck processes with discrete observations," in *Malliavin Calculus and Stochastic Analysis: A Festschrift in Honor of David Nualart*, F. Viens, J. Feng, Y. Hu, and E. Nualart, Eds., vol. 34 of *Springer Proceedings in Mathematics and Statistics*, pp. 427–442, 2013.
- [12] K. Es-Sebaiy, "Berry-Esséen bounds for the least squares estimator for discretely observed fractional Ornstein-Uhlenbeck processes," *Statistics & Probability Letters*, vol. 83, no. 10, pp. 2372–2385, 2013.
- [13] E. Alòs, O. Mazet, and D. Nualart, "Stochastic calculus with respect to Gaussian processes," *The Annals of Probability*, vol. 29, no. 2, pp. 766–801, 2001.
- [14] V. Pipiras and M. S. Taqqu, "Integration questions related to fractional Brownian motion," *Probability Theory and Related Fields*, vol. 118, no. 2, pp. 251–291, 2000.
- [15] E. Alòs and D. Nualart, "Stochastic integration with respect to the fractional Brownian motion," *Stochastics and Stochastics Reports*, vol. 75, no. 3, pp. 129–152, 2003.