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Research Article

Invariant Distributionally Scrambled Manifolds for an Annihilation Operator

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This note proves that the annihilation operator of a quantum harmonic oscillator admits an invariant distributionally ε-scrambled linear manifold for any $0 < \varepsilon < 2$. This is a positive answer to Question 1 by Wu and Chen (2013).

A *dynamical system* is a pair (X, f), where X is a complete metric space without isolated points and the map $f : X \to X$ is continuous. Throughout this paper, let $\mathbb{N} = \{1, 2, 3, \ldots\}$ and $\mathbb{Z}^+ = \{0, 1, 2, \ldots\}$.

Sharkovskii's amazing discovery [1], as well as Li and Yorke's famous work which introduced the concept of "chaos" known as the Li-Yorke chaos today in a mathematically rigorous way [2], has activated sustained interest and provoked the rapid advancement of discrete chaos theory in the last decades. Since then, several other rigorous definitions of chaos have been proposed. Each of their definitions tries to describe one kind of unpredictability in the evolution of the system dynamics. This was also the original idea of Li and Yorke. In Li and Yorke's study [2], they suggested considering "divergent pairs" (x, y), which are *proximal* but not *asymptotic*, in the sense that

$$\begin{aligned} &\liminf_{n\to\infty} d\left(f^n(x), f^n(y)\right) = 0, \\ &\limsup_{n\to\infty} d\left(f^n(x), f^n(y)\right) > 0, \end{aligned} \tag{1}$$

where f^n denotes the nth iteration of f.

A generalization of the concept of Li-Yorke chaos is *distributional chaos*, introduced by Schweizer and Smítal [3] in 1994.

Let (X, f) be a dynamical system. For any pair of points $x, y \in X$ and any $n \in \mathbb{N}$, let

$$F(x, y, t, n) = \left|\left\{j \in \mathbb{N} : d\left(f^j(x), f^j(y)\right) < t, 1 \le j \le n\right\}\right|, \tag{2}$$

where $|A|$ denotes the cardinality of set A. Define *lower* and *upper distributional functions* $\mathbb{R} \to [0, 1]$ generated by f, x, and y, as

$$F_{x,y}(t, f) = \liminf_{n\to\infty} \frac{1}{n} F(x, y, t, n), \tag{3}$$

$$F^*_{x,y}(t, f) = \limsup_{n\to\infty} \frac{1}{n} F(x, y, t, n), \tag{4}$$

respectively. A dynamical system (X, f) is said to be *distributionally ε-chaotic for a given $\varepsilon > 0$* if there exists an uncountable subset $D \subset X$ such that for any pair of distinct points $x, y \in D$, one has $F^*_{x,y}(t, f) = 1$ for all $t > 0$ and $F_{x,y}(\varepsilon, f) = 0$. The set D is a *distributionally ε-scrambled set* and the pair (x, y) a *distributionally ε-chaotic pair*. If (X, f) is distributionally ε-chaotic for any given $0 < \varepsilon < \operatorname{diam} X$, then (X, f) is said to exhibit *maximal distributional chaos*.

The quantum harmonic oscillator is the quantum-mechanical analog of the classical harmonic oscillator. It is one of the most important models in quantum mechanics [4, 5], because an arbitrary potential can be approximated by

a harmonic potential at the vicinity of a stable equilibrium point. Transmutation of the quantum harmonic oscillator may be described by the (time-dependent) Schrödinger equation as

$$-\frac{\hbar^2}{2m}\frac{\partial^2\psi}{\partial x^2}+\frac{m\omega^2}{2}x^2\psi=i\hbar\frac{\partial\psi}{\partial t}, \tag{5}$$

with a wave function $\psi(x,t)$, displacement x, mass m, frequency ω, and Planck constant $\hbar$. The nondimensionalized steady states in terms of eigenfunctions in the separable Hilbert space $\mathscr{H}=L^2(-\infty,+\infty)$ form an orthonormal basis:

$$\psi_n(x)=\frac{e^{-x^2/2}H_n(x)}{\sqrt{\sqrt{\pi}2^n n!}},\quad n=0,1,\ldots, \tag{6}$$

where $H_n(x)=(-1)^n e^{x^2}(\mathrm{d}^n/\mathrm{d}x^n)e^{-x^2}$ is the nth *Hermite polynomial*. The natural phase space for the quantum harmonic oscillator is the Schwartz class, also called Schwartz space Φ of rapidly decreasing functions in $\mathscr{H}$, defined as

$$\Phi=\left\{\phi\in\mathscr{H}:\phi=\sum_{n=0}^{\infty}c_n\psi_n,\sum_{n=0}^{\infty}|c_n|^2(n+1)^r<+\infty,\forall r>0\right\}. \tag{7}$$

Here, Φ is an infinite-dimensional Fréchet space with topology defined by the system of seminorms $p_r(\cdot)$ of the form

$$p_r(\phi)=p_r\left(\sum_{n=0}^{\infty}c_n\psi_n\right)=\left(\sum_{n=0}^{\infty}|c_n|^2(n+1)^r\right)^{1/2}. \tag{8}$$

This topology on Φ can be equivalently introduced by the metric

$$\rho(\phi,\psi)=\sum_{m=0}^{\infty}\frac{1}{2^m}\frac{p_m(\phi-\psi)}{1+p_m(\phi-\psi)}. \tag{9}$$

It follows directly from (9) that $\operatorname{diam}\Phi=2$. For any $\phi=\sum_{n=0}^{\infty}c_n\psi_n\in\Phi$ and any $i\in\mathbb{Z}^+$, denote $\phi_i=c_i$. The quantum harmonic oscillator may be equivalently described in terms of the annihilation operator $\widehat{a}=(1/\sqrt{2})(x+(\mathrm{d}/\mathrm{d}x))$ and its adjoint $a^\dagger=(1/\sqrt{2})(x-(\mathrm{d}/\mathrm{d}x))$. According to the basic properties of Hermite polynomials, one has $\widehat{a}:\Phi\to\Phi$ given by

$$\widehat{a}(\psi_n)=\frac{1}{\sqrt{2}}\left(x+\frac{\mathrm{d}}{\mathrm{d}x}\right)\psi_n=\sqrt{n}\psi_{n-1}. \tag{10}$$

Meanwhile, it is not difficult to check that for any $i\in\mathbb{N}$ and any $\phi=\sum_{n=0}^{\infty}c_n\psi_n\in\Phi$,

$$\widehat{a}^i(\phi)=\sum_{n=i}^{\infty}\sqrt{A_n^i}\cdot c_n\cdot\psi_{n-i}=\sum_{n=0}^{\infty}\sqrt{A_{n+i}^i}\cdot c_{n+i}\cdot\psi_n, \tag{11}$$

where $A_n^i=n\cdot(n-1)\cdots(n-i+1)$. The $\widehat{a}$ acts as a kind of backward shift on the space Φ. In fact, it is a special weighted backward shift on the Fréchet space (see [6–8] for the recent results on this topic).

Applying a result of Godefroy and Shapiro [9], Gulisashvili and MacCluer [10] proved that the annihilation operator $\widehat{a}$ is Devaney chaotic. Then, Duan et al. [11] obtained that $\widehat{a}$ is also Li-Yorke chaotic. However, it follows directly from [12, Theorem 4.1] that this holds trivially. Oprocha [13] showed that $\widehat{a}$ is distributionally ε-chaotic with $\varepsilon=1/16$. Recently, in [14] it was further shown that $\widehat{a}$ exhibits distributional ε-chaos for any $0<\varepsilon<2$ and that the principal measure of $\widehat{a}$ is 1. Moreover, Wu and Chen [15] proved that $\widehat{a}$ admits an invariant distributionally ε-scrambled set for any $0<\varepsilon<2$ and posed the following question.

Question. Is there an invariant manifold D of Φ such that D is a distributionally ε-scrambled set for any $0<\varepsilon<2$?

This paper gives a positive answer to the question above; see the following theorem.

Theorem 1. *There exists an invariant manifold $E\subset\Phi$ such that E is a distributionally ε-scrambled set under $\widehat{a}$ for any $0<\varepsilon<2=\operatorname{diam}\Phi$.*

Proof. Let $L_1=\mathscr{L}_1=2$, $L_n=2^{L_1+\cdots+L_{n-1}}$, and $\mathscr{L}_n=\sum_{j=1}^{n}L_j$ for $n>1$. Arrange all odd prime numbers by the natural order "<" and denote them by $\mathrm{P}_1,\mathrm{P}_2,\ldots$. For any $n,m\in\mathbb{N}$, set

$$\mathscr{A}_{n,m}=\left\{j\in\mathbb{Z}^+:\mathscr{L}_{\mathrm{P}_n^m}\le j<\mathscr{L}_{\mathrm{P}_n^m+1},j-\mathscr{L}_{\mathrm{P}_n^m}\equiv 0\pmod{}\right\}. \tag{12}$$

Take a point $\xi=\sum_{j=0}^{\infty}\xi_j\psi_j$ such that

$$\xi_j=\begin{cases}\sqrt{\dfrac{\mathrm{P}_n^m}{j!}}, & j\in\mathscr{A}_{n,m},\ n,m\in\mathbb{N},\\ 0, & \text{otherwise.}\end{cases} \tag{13}$$

Since $j\ge\mathscr{L}_{\mathrm{P}_n^m}\ge\mathrm{P}_n^m$ if $j\in\mathscr{A}_{n,m}$, we have that, for any $j\in\mathbb{N}$,

$$|\xi_j|\le\frac{1}{\sqrt{(j-1)!}}. \tag{14}$$

This implies that, for any $r\ge 0$,

$$\sum_{j=0}^{\infty}|\xi_j|^2(j+1)^r\le\sum_{j=1}^{\infty}\frac{1}{(j-1)!}(j+1)^r<+\infty. \tag{15}$$

Hence $\xi\in\Phi$.

Take $E:=\operatorname{span}\{\xi^{(n)}:n\in\mathbb{Z}^+\}$, where $\xi^{(n)}=\widehat{a}^n(\xi)$. Clearly, E is an invariant linear manifold under $\widehat{a}$. Given two fixed points $\phi,\psi\in E$ with $\phi\ne\psi$, according to the construction of E, there exist $\alpha_0,\beta_0,\ldots,\alpha_N,\beta_N\in\mathbb{K}$, such that $\phi=\alpha_0\xi^{(0)}+\cdots+\alpha_N\xi^{(N)}$ and $\psi=\beta_0\xi^{(0)}+\cdots+\beta_N\xi^{(N)}$.

Now, we assert that (ϕ,ψ) is a distributionally ε-chaotic pair for any $0<\varepsilon<2$.

First, observe that for any $\mathscr{L}_{2n}\le j<\mathscr{L}_{2n+1}$, $\xi_j=0$ and $\widehat{a}^j(\xi)=\sum_{n=0}^{\infty}\sqrt{A_{n+j}^j}\cdot\xi_{n+j}\cdot\psi_n$. Combining this with (14), it

follows that for any fixed $m \in \mathbb{Z}^+$ and any $\mathscr{L}_{2n} \le j \le \mathscr{L}_{2n} + (L_{2n+1}/2)$,

$$\begin{aligned} p_m\left(\widehat{a}^j(\xi)\right) &= \left(\sum_{k=0}^{\infty}\left|\sqrt{A_{k+j}^j}\cdot\xi_{k+j}\right|^2(k+1)^m\right)^{1/2} \\ &\le \left(\sum_{k=\mathscr{L}_{2n+1}-(j+1)}^{\infty}\left|\sqrt{A_{k+j}^j}\cdot\xi_{k+j}\right|^2(k+1)^m\right)^{1/2} \\ &\le \left(\sum_{k=(L_{2n+1}/2)-1}^{\infty}\frac{k+j}{k!}(k+1)^m\right)^{1/2}, \end{aligned} \tag{16}$$

as $\mathscr{L}_{2n+1}-(j+1) \ge \mathscr{L}_{2n+1}-(\mathscr{L}_{2n}+(L_{2n+1}/2)+1) \ge (L_{2n+1}/2)-1$. Meanwhile, it is easy to see that, for any $\mathscr{L}_{2n} \le j \le \mathscr{L}_{2n} + (L_{2n+1}/2)$,

$$\begin{aligned} &\sum_{k=(L_{2n+1}/2)-1}^{\infty}\frac{k+j}{k!}(k+1)^m \\ &\quad\le \sum_{k=(L_{2n+1}/2)-1}^{\infty}\frac{k+\mathscr{L}_{2n}+(L_{2n+1}/2)}{k!}(k+1)^m \\ &\quad= \sum_{k=(L_{2n+1}/2)-1}^{\infty}\frac{k+\log_2 L_{2n+1}+(L_{2n+1}/2)}{k!}(k+1)^m \longrightarrow 0 \\ &\qquad (n\longrightarrow\infty). \end{aligned} \tag{17}$$

For any fixed $t > 0$, one can choose a $K_1 \in \mathbb{N}$ such that $\sum_{n=K_1+1}^{\infty}(1/2^n) < (t/2(N+1))$. It is clear that, for any $j, m \in \mathbb{Z}^+$,

$$p_m\left(\widehat{a}^j(\phi)-\widehat{a}^j(\psi)\right) \le \sum_{k=0}^{N}|\alpha_k-\beta_k|\cdot p_m\left(\widehat{a}^{j+k}(\xi)\right). \tag{18}$$

This, together with (16) and (17), leads to that there exists a $K_2 \in \mathbb{N}$ such that for any $n \ge K_2$ and any $\mathscr{L}_{2n} \le j \le \mathscr{L}_{2n} + (L_{2n+1}/2) - N$,

$$\begin{aligned} &\max\left\{|\alpha_k-\beta_k|\cdot p_m\left(\widehat{a}^{j+k}(\xi)\right) : 0\le k\le N, 0\le m\le K_1\right\} \\ &\quad< \frac{t}{4(N+1)}. \end{aligned} \tag{19}$$

This implies that, for any $\mathscr{L}_{2n} \le j \le \mathscr{L}_{2n} + (L_{2n+1}/2) - N$ $(n \ge K_2)$,

$$\begin{aligned} &\rho\left(\widehat{a}^j(\phi),\widehat{a}^j(\psi)\right) \\ &\quad\le \sum_{k=0}^{N}\rho\left(\widehat{a}^{j+k}(\alpha_k\xi),\widehat{a}^{j+k}(\beta_k\xi)\right) \\ &\quad\le \sum_{k=0}^{N}\left(\sum_{m=0}^{K_1}\frac{1}{2^m}\frac{|\alpha_k-\beta_k|\cdot p_m\left(\widehat{a}^{j+k}(\xi)\right)}{1+|\alpha_k-\beta_k|\cdot p_m\left(\widehat{a}^{j+k}(\xi)\right)}+\sum_{m=K_1+1}^{\infty}\frac{1}{2^m}\right) \\ &\quad< \sum_{k=0}^{N}\left(\sum_{m=0}^{K_1}\frac{1}{2^m}\frac{(t/4(N+1))}{1+(t/4(N+1))}+\frac{t}{2(N+1)}\right) \\ &\quad< \sum_{k=0}^{N}\frac{t}{N+1} = t. \end{aligned} \tag{20}$$

Consequently,

$$\begin{aligned} &F_{\phi,\psi}^*(t,\widehat{a}) \\ &\quad= \limsup_{n\to\infty}\frac{1}{n}\left|\left\{j : \rho\left(\widehat{a}^j(\phi),\widehat{a}^j(\psi)\right) < t, 1\le j\le n\right\}\right| \\ &\quad\ge \limsup_{n\to\infty}\frac{1}{\mathscr{L}_{2n}+(L_{2n+1}/2)-N} \\ &\qquad\times\left|\left\{j : \rho\left(\widehat{a}^j(\psi),\widehat{a}^j(\phi)\right) < t, 1\le j\le \mathscr{L}_{2n}+\frac{L_{2n+1}}{2}-N\right\}\right| \\ &\quad\ge \limsup_{n\to\infty}\frac{(L_{2n+1}/2)-N}{\mathscr{L}_{2n}+(L_{2n+1}/2)-N} \\ &\quad= \limsup_{n\to\infty}\frac{2^{\mathscr{L}_{2n}-1}-N}{\mathscr{L}_{2n}+2^{\mathscr{L}_{2n}-1}-N} = 1. \end{aligned} \tag{21}$$

Second, since $\phi \ne \psi$, there exists $0 \le \ell \le N$ such that $\alpha_\ell \ne \beta_\ell$. It follows from (11) that, for any $j \in \mathbb{Z}^+$,

$$\begin{aligned} &\widehat{a}^j(\phi)-\widehat{a}^j(\psi) \\ &\quad= \sum_{k=0}^{N}(\alpha_k-\beta_k)\cdot\widehat{a}^{j+k}(\xi) \\ &\quad= \sum_{n=0}^{\infty}\left(\sum_{k=0}^{N}(\alpha_k-\beta_k)\cdot\sqrt{A_{n+j+k}^{j+k}}\cdot\xi_{n+j+k}\cdot\psi_n\right). \end{aligned} \tag{22}$$

For any fixed $\mathscr{L}_{\mathrm{P}_n^{N+1}} \le j \le \mathscr{L}_{\mathrm{P}_n^{N+1}} + (L_{\mathrm{P}_n^{N+1}+1}/2)(n \ge N)$, there exists $0 \le i_0 \le N$ such that $j+\ell+i_0 \in \mathscr{A}_{n,N+1}$. According to the choice of ξ, it follows that for all $i \in [0,N]\setminus\{i_0\}$, $\xi_{j+\ell+i} = 0$. This implies that

$$\begin{aligned} &\left(\widehat{a}^j(\phi)-\widehat{a}^j(\psi)\right)_{i_0} \\ &\quad= (\alpha_\ell-\beta_\ell)\cdot\sqrt{A_{i_0+j+\ell}^{j+\ell}}\cdot\sqrt{\frac{\mathrm{P}_n^{N+1}}{(i_0+j+\ell)!}} \\ &\quad= (\alpha_\ell-\beta_\ell)\cdot\sqrt{\frac{\mathrm{P}_n^{N+1}}{i_0!}}. \end{aligned} \tag{23}$$

Then, for any $m \in \mathbb{Z}^+$,

$$\begin{aligned} & p_m\left(\widehat{a}^j(\phi)-\widehat{a}^j(\psi)\right) \\ & \quad \geq p_0\left(\widehat{a}^j(\phi)-\widehat{a}^j(\psi)\right) \\ & \quad \geq \left|\alpha_\ell-\beta_\ell\right| \cdot \sqrt{\frac{\mathrm{P}_n^{N+1}}{i_0!}} \\ & \quad \geq \left|\alpha_\ell-\beta_\ell\right| \cdot \sqrt{\frac{\mathrm{P}_n^{N+1}}{N!}}. \end{aligned} \tag{24}$$

Combining this with the fact that the function $t \in [0,+\infty) \mapsto (t/(1+t)) \in \mathbb{R}$ is increasing, it follows that, for any $\mathscr{L}_{\mathrm{P}_n^{N+1}} \leq j \leq \mathscr{L}_{\mathrm{P}_n^{N+1}} + (L_{\mathrm{P}_n^{N+1}+1}/2)$,

$$\begin{aligned} & \rho\left(\widehat{a}^j(\phi), \widehat{a}^j(\psi)\right) \\ & \quad = \sum_{m=0}^{\infty} \frac{1}{2^m} \frac{p_m\left(\widehat{a}^j(\phi)-\widehat{a}^j(\psi)\right)}{1+p_m\left(\widehat{a}^j(\phi)-\widehat{a}^j(\psi)\right)} \\ & \quad \geq \sum_{m=0}^{\infty} \frac{1}{2^m} \cdot \frac{\left|\alpha_\ell-\beta_\ell\right| \cdot \sqrt{\left(\mathrm{P}_n^{N+1}/N!\right)}}{1+\left|\alpha_\ell-\beta_\ell\right| \cdot \sqrt{\left(\mathrm{P}_n^{N+1}/N!\right)}} \\ & \quad = \frac{2\left|\alpha_\ell-\beta_\ell\right| \cdot \sqrt{\left(\mathrm{P}_n^{N+1}/N!\right)}}{1+\left|\alpha_\ell-\beta_\ell\right| \cdot \sqrt{\left(\mathrm{P}_n^{N+1}/N!\right)}} \longrightarrow 2 \quad (n \longrightarrow +\infty). \end{aligned} \tag{25}$$

Hence, for any $0 < \varepsilon < 2$,

$$\begin{aligned} & F_{\phi,\psi}(\varepsilon, \widehat{a}) \\ & \quad = \liminf_{n\to\infty} \frac{1}{n}\left|\left\{j : \rho\left(\widehat{a}^j(\phi), \widehat{a}^j(\psi)\right) < \varepsilon, 1 \leq j \leq n\right\}\right| \\ & \quad \leq \liminf_{n\to\infty} \frac{1}{\mathscr{L}_{\mathrm{P}_n^{N+1}} + \left(L_{\mathrm{P}_n^{N+1}+1}/2\right)} \\ & \qquad \times \left|\left\{j : \rho\left(\widehat{a}^j(\phi), \widehat{a}^j(\psi)\right) < \varepsilon, 1 \leq j \leq \mathscr{L}_{\mathrm{P}_n^{N+1}} + \frac{L_{\mathrm{P}_n^{N+1}+1}}{2}\right\}\right| \\ & \quad \leq \liminf_{n\to\infty} \frac{\mathscr{L}_{\mathrm{P}_n^{N+1}}}{\mathscr{L}_{\mathrm{P}_n^{N+1}} + \left(L_{\mathrm{P}_n^{N+1}+1}/2\right)} \\ & \quad = \liminf_{n\to\infty} \frac{\mathscr{L}_{\mathrm{P}_n^{N+1}}}{\mathscr{L}_{\mathrm{P}_n^{N+1}} + 2^{\mathscr{L}_{\mathrm{P}_n^{N+1}}-1}} = 0. \end{aligned} \tag{26}$$

Summing up the above discussions, since both ϕ and ψ are arbitrary, it follows that E is an invariant distributionally ε-scrambled linear manifold for any $0 < \varepsilon < 2$. □

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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