Research Article

Ulam's Type Stability of Involutional-Exponential Functional Equations

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Let S be a commutative semigroup, $f, g : S \rightarrow \mathbb{C}$ and $\sigma : S \rightarrow S$ an involution. In this paper we consider the stability of involution-exponential functional equations $|f(x + \sigma y) - g(x)f(y)| \leq \phi(x)$ [resp., $\phi(y)$], $|f(x + \sigma y) - f(x)g(y)| \leq \phi(x)$ [resp., $\phi(y)$] for all $x, y \in S$, where $\phi : S \rightarrow \mathbb{R}^+$ satisfies the growth condition: there exists $C > 1$ such that $\lim_{k \rightarrow \infty} C^{-k}\phi(kx) = 0$ for each $x \in S$. We also consider the stability of L^∞ -version $|f(x + \sigma y) - f(x)f(y)|_{L^\infty(\mathbb{R}^{2n})} \leq \epsilon$, where $f : \mathbb{R}^n \rightarrow \mathbb{C}$ is a locally integrable function.

1. Introduction

Throughout this paper we denote by $S, \mathbb{R}, \mathbb{R}^+, \mathbb{C}, \mathbb{R}^n$, a commutative semigroup with an identity element, the set of real numbers, nonnegative real numbers, complex numbers, and the n -dimensional Euclidean space, respectively, and $\mathbb{R}_0^n = \mathbb{R}^n \setminus \{0\}$, $\mathbb{C}_0 = \mathbb{C} \setminus \{0\}$, $\phi : S \rightarrow \mathbb{R}^+$, $\epsilon \geq 0$. A function $m : S \rightarrow \mathbb{C}$ is called *exponential* provided that $m(x + y) = m(x)m(y)$ for all $x, y \in S$, and $\sigma : S \rightarrow S$ is called an *involution* provided that $\sigma(x + y) = \sigma(x) + \sigma(y)$ and $\sigma(\sigma(x)) = x$ for all $x, y \in S$. An exponential function $m : S \rightarrow \mathbb{C}$ is called σ -*exponential* if m satisfies $m(\sigma x) = m(x)$ for all $x \in S$ and denote by m_σ a σ -exponential function.

In [1], the following functional inequalities with involution are investigated:

$$|f(x + \sigma y) - g(x)f(y)| \leq \phi(x) \text{ [resp. } \phi(y)], \quad \forall x, y \in S, \quad (1)$$

$$|f(x + \sigma y) - f(x)g(y)| \leq \phi(x) \text{ [resp. } \phi(y)], \quad \forall x, y \in S. \quad (2)$$

As a result, all unbounded functions f, g satisfying the inequalities (1) and (2) are exactly described only when ϕ is a constant function while only one of unbounded functions f, g satisfying each of (1) and (2) is exactly described when ϕ is an arbitrary unbounded function.

In this paper we investigate the functional inequalities (1) and (2) by imposing some growth conditions on ϕ, f , or g . First, we introduce the condition on $h : S \rightarrow \mathbb{C}$:

$$\inf_{x \in S} \frac{1 + \phi(x)}{|h(x)|} = 0, \quad (3)$$

where h will stand for f and g .

Secondly, we introduce the condition on ϕ ; there exists $C > 1$ such that

$$\lim_{k \rightarrow \infty} C^{-k}\phi(kx) = 0 \quad (4)$$

for all $x \in S$.

As a result, we completely determine f and g satisfying each of the inequalities (1) and (2): if g satisfies (3) [resp., f satisfies (3)] or ϕ satisfies (4), then (f, g) satisfying (1) [resp., (2)] are of the form

$$f(x) = f(0)m_\sigma(x),$$

$$g(x) = m_\sigma(x) \quad (5)$$

$$\times [\text{resp. } f(x) = f(0)m(x), g(x) = m(\sigma x)]$$

for all $x \in S$, where m_σ is a σ -exponential function and m is an exponential function.

As an application of our result, we determine all unbounded functions $f, g : \mathbb{R}_0^2 \rightarrow \mathbb{R}$ satisfying the functional inequalities

$$|f(ux + vy, uy - vx) - g(x, y) f(u, v)| \leq \phi(x, y) [\phi(u, v)], \quad (6)$$

$$|f(ux + vy, uy - vx) - f(x, y) g(u, v)| \leq \phi(x, y) [\phi(u, v)] \quad (7)$$

for all $(x, y), (u, v) \in \mathbb{R}_0^2$, where f, g satisfy (3) or $\phi : \mathbb{R}_0^2 \rightarrow \mathbb{R}^+$ satisfies (4) (see [2–5] for related equations) and determine all unbounded functions $f, g : \mathbb{R}_0^4 \rightarrow \mathbb{R}$ satisfying the functional inequalities

$$\begin{aligned} & |f(x_1, y_1, u_1, v_1) g(x_2, y_2, u_2, v_2) \\ & - f(x_1 x_2 + y_1 y_2 + u_1 u_2 + v_1 v_2, x_1 y_2 - y_1 x_2 \\ & + u_1 v_2 - v_1 u_2, x_1 u_2 - y_1 v_2 - u_1 x_2 \\ & + v_1 y_2, x_1 v_2 + y_1 u_2 - u_1 y_2 - v_1 x_2)| \\ & \leq \psi(x_1, y_1, u_1, v_1) [\psi(x_2, y_2, u_2, v_2)], \end{aligned} \quad (8)$$

for all $x_1, y_1, u_1, v_1, x_2, y_2, u_2, v_2 \in \mathbb{R}$, where f, g satisfy (3) or $\psi : \mathbb{R}_0^4 \rightarrow \mathbb{R}^+$ satisfies (4) (see [2, 4] for related equations). Finally, we consider the stability of L^∞ -version

$$|f(x + \sigma y) - f(x) f(y)|_{L^\infty(\mathbb{R}^{2n})} \leq \epsilon, \quad (9)$$

where $f : \mathbb{R}^n \rightarrow \mathbb{C}$ is a locally integrable function. As a result, we prove that every unbounded solution f (i.e., $\|f\|_{L^\infty(\mathbb{R}^n)} = \infty$) of (9) satisfies

$$f(x) = m\left(\frac{x + \sigma x}{2}\right) \quad (10)$$

for almost every $x \in \mathbb{R}^n$, where $m : \mathbb{R}^n \rightarrow \mathbb{C}$ is an unbounded exponential function. Every bounded solution f (i.e., $\|f\|_{L^\infty(\mathbb{R}^n)} < \infty$) satisfies

$$\|f\|_{L^\infty(\mathbb{R}^n)} \leq \frac{1}{2} (1 + \sqrt{1 + 4\epsilon}); \quad (11)$$

If $\epsilon < 1/4$, then f satisfies either

$$\frac{1}{2} (1 + \sqrt{1 - 4\epsilon}) \leq \|f\|_{L^\infty(\mathbb{R}^n)} \leq \frac{1}{2} (1 + \sqrt{1 + 4\epsilon}) \quad (12)$$

or

$$\|f\|_{L^\infty(\mathbb{R}^n)} \leq \frac{1}{2} (1 - \sqrt{1 - 4\epsilon}). \quad (13)$$

We refer the reader to [1, 6–16] for related functional equations and their stabilities. We also refer the reader to [17–19] for some recent developments on the issues of stability and superstability for functional equations.

2. Stability of (1) and (2)

In this section we investigate unbounded functions f, g satisfying (1) and (2) when some of f and g satisfy (3) or ϕ satisfies (4). For bounded solutions of (1) and (2) we refer the reader to [1].

Lemma 1. Assume that $m : S \rightarrow \mathbb{C}$ is an unbounded exponential function and $\phi : S \rightarrow \mathbb{R}^+$ satisfies (4). Then m satisfies (3).

Proof. Since m is unbounded, we can choose a $x_0 \in S$ such that $|m(x_0)| \geq C$, where $C > 1$ is the constant in (4). Since ϕ satisfies (4) we have

$$\begin{aligned} \lim_{k \rightarrow \infty} \frac{1 + \phi(kx_0)}{|m(kx_0)|} &= \lim_{k \rightarrow \infty} \frac{1 + \phi(kx_0)}{|m(x_0)|^k} \\ &\leq \lim_{k \rightarrow \infty} (C^{-k} + C^{-k} \phi(kx_0)) = 0. \end{aligned} \quad (14)$$

This completes the proof. $\square$

Theorem 2. Let $f, g : S \rightarrow \mathbb{C}$ be unbounded functions satisfying

$$|f(x + \sigma y) - g(x) f(y)| \leq \phi(x) \quad (15)$$

for all $x, y \in S$. Then g is a σ -exponential function. In particular if g satisfies (3) or ϕ satisfies (4), then there exists a σ -exponential function $m_\sigma : S \rightarrow \mathbb{C}$ such that

$$f(x) = f(0) m_\sigma(x), \quad g(x) = m_\sigma(x) \quad (16)$$

for all $x \in S$.

Proof. Choosing a sequence $y_n \in S$, $n = 1, 2, 3, \dots$, such that $|f(y_n)| \rightarrow \infty$ as $n \rightarrow \infty$, putting $y = y_n$, $n = 1, 2, 3, \dots$, in (15), dividing the result by $|f(y_n)|$, and letting $n \rightarrow \infty$ we have

$$g(x) = \lim_{n \rightarrow \infty} \frac{f(x + \sigma y_n)}{f(y_n)} \quad (17)$$

for all $x \in S$. Putting $x = 0$ in (15) we have

$$|f(\sigma y) - g(0) f(y)| \leq \phi(0) \quad (18)$$

for all $y \in S$. Multiplying both sides of (17) by $g(y)$ and using (15), (17), and (18) we have

$$\begin{aligned} g(y) g(x) &= \lim_{n \rightarrow \infty} \frac{g(y) f(x + \sigma y_n)}{f(y_n)} = \lim_{n \rightarrow \infty} \frac{f(y + \sigma x + y_n)}{f(y_n)} \\ &= \lim_{n \rightarrow \infty} \frac{g(0) f(x + \sigma y + \sigma y_n)}{f(y_n)} = g(0) g(x + \sigma y) \end{aligned} \quad (19)$$

for all $x, y \in S$. Dividing (19) by $g(0)^2$ we have

$$g_0(x) g_0(y) = g_0(x + \sigma y) \quad (20)$$

for all $x, y \in S$, where $g_0(x) = g(x)/g(0)$. From (20) we have

$$g(x) = g(0) m_\sigma(x) \quad (21)$$

for some σ -exponential m_σ . If g satisfies (3) or ϕ satisfies (4), then, by Lemma 1, we can choose a sequence $x_n \in S$, $n = 1, 2, 3, \dots$, such that $(1 + \phi(x_n))/|g(x_n)| \rightarrow 0$ as $n \rightarrow \infty$. Putting $x = x_n$, $n = 1, 2, 3, \dots$, in (15), dividing the result by $|g(x_n)|$, and letting $n \rightarrow \infty$ we have

$$f(y) = \lim_{n \rightarrow \infty} \frac{f(x_n + \sigma y)}{g(x_n)} \quad (22)$$

for all $y \in S$. Multiplying both sides of (22) by $g(x)$ and using (15), (18), and (22) we have

$$\begin{aligned} g(x) f(y) &= \lim_{n \rightarrow \infty} \frac{g(x) f(x_n + \sigma y)}{g(x_n)} \\ &= \lim_{n \rightarrow \infty} \frac{f(x + \sigma x_n + y)}{g(x_n)} \\ &= \lim_{n \rightarrow \infty} \frac{g(0) f(\sigma x + \sigma y + x_n)}{g(x_n)} = g(0) f(x + y) \end{aligned} \quad (23)$$

for all $x, y \in S$. Putting $y = 0$ in (23) and dividing the result by $g(0)$ we have

$$f(x) = f(0) g_0(x) = f(0) m_\sigma(x) \quad (24)$$

for all $x \in S$. Putting $x = 0$ in (15) and using (24) we have

$$|f(0)(1 - g(0))| |m_\sigma(y)| \leq \phi(0) \quad (25)$$

for all $y \in S$. Since m_σ is unbounded, from (25) we have $g(0) = 1$. Now, from (21) and (24) we get (16). This completes the proof. $\square$

We denote by $c \cdot x$ the inner product of $c = (c_1, c_2, \dots, c_n) \in \mathbb{C}^n$ and $x = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n$ which is defined as $c \cdot x = \sum_{j=1}^n c_j x_j$, and $\Re c = (\Re c_1, \dots, \Re c_n)$, where $\Re c_j$ are the real parts of c_j , $j = 1, 2, \dots, n$. It is easy to see that if S is uniquely 2-divisible (i.e., for each $x \in S$ there exists a unique $y \in S$ such that $2y = x$), then m_σ is σ -exponential if and only if

$$m_\sigma(x) = m\left(\frac{x + \sigma x}{2}\right), \quad x \in S \quad (26)$$

for some exponential function $m : S \rightarrow \mathbb{C}$.

Corollary 3. Let $P(x), x \in \mathbb{R}^n$, be a polynomial. Suppose that $f, g : \mathbb{R}^n \rightarrow \mathbb{C}$ are unbounded function satisfying

$$|f(x + \sigma y) - g(x) f(y)| \leq |P(x)| \quad (27)$$

for all $x, y \in \mathbb{R}^n$. Then there exists a σ -exponential function $m_\sigma : \mathbb{R}^n \rightarrow \mathbb{C}$ such that

$$f(x) = f(0) m_\sigma(x), \quad g(x) = m_\sigma(x) \quad (28)$$

for all $x \in \mathbb{R}^n$. In particular if g is continuous, then there exists $c \in \mathbb{C}^n$, $\Re(c + c\sigma) \neq 0$ such that

$$f(x) = f(0) e^{(1/2)(c+c\sigma) \cdot x}, \quad g(x) = e^{(1/2)(c+c\sigma) \cdot x} \quad (29)$$

for all $x \in \mathbb{R}^n$.

Proof. It is easy to see that $|P(x)|$ satisfies (4). Thus, by Theorem 2 we get (28). Assume that g is continuous. It is well known that every continuous exponential functional $m : \mathbb{R}^n \rightarrow \mathbb{C}$ is given by $m(x) = e^{c \cdot x}$ for some $c \in \mathbb{C}^n$. Thus, from (26) we have $m_\sigma(x) = e^{(1/2)(c+c\sigma) \cdot x}$ for all $x \in \mathbb{R}^n$, where $c\sigma$ denotes matrix multiplication. Thus, we get (29). This completes the proof. $\square$

Remark 4. Let $a, b \in \mathbb{R}^n$ be two nonzero vectors that are not parallel; that is, $b \neq ra$ for all $r \in \mathbb{R}$. Then, the hyperplane $b \cdot x = 0$ is not parallel to $(b-a) \cdot x = 0$ and hence there exists $x_0 \in \mathbb{R}^n$ such that $b \cdot x_0 > 0$ and $(b-a) \cdot x_0 > 0$. If $b = ta$ for some $t \in \mathbb{R}$, then there exists $x_0 \in \mathbb{R}^n$ such that $b \cdot x_0 > 0$ and $(b-a) \cdot x_0 > 0$ if and only if $t > 1$. Thus, if $b \neq ta$ for all $t \leq 1$, then there exists $x_0 \in \mathbb{R}^n$ such that $b \cdot x_0 > 0$ and $(b-a) \cdot x_0 > 0$.

Corollary 5. Let $\gamma \in \mathbb{R}^n$ be fixed. Suppose that $f, g : \mathbb{R}^n \rightarrow \mathbb{C}$ are unbounded continuous function satisfying

$$|f(x + \sigma y) - g(x) f(y)| \leq e^{\gamma \cdot x} \quad (30)$$

for all $x, y \in \mathbb{R}^n$. Then there exists $c \in \mathbb{C}^n$, $\Re(c + c\sigma) \neq 0$ such that $g(x) = e^{(1/2)(c+c\sigma) \cdot x}$ for all $x \in \mathbb{R}^n$. If $(1/2)\Re(c + c\sigma) \neq t\gamma$ for all $t \leq 1$, then one has

$$f(x) = f(0) e^{(1/2)(c+c\sigma) \cdot x}, \quad g(x) = e^{(1/2)(c+c\sigma) \cdot x} \quad (31)$$

for all $x \in \mathbb{R}^n$.

Proof. Recall that every continuous σ -exponential functional $m_\sigma : \mathbb{R}^n \rightarrow \mathbb{C}$ is given by

$$m_\sigma(x) = e^{(1/2)(c+c\sigma) \cdot x} \quad (32)$$

for all $x \in \mathbb{R}^n$, where $c\sigma$ denotes matrix multiplication. If $(1/2)\Re(c + c\sigma) \neq t\gamma$ for all $t \leq 1$, then by Remark 4 there exists $x_0 \in \mathbb{R}^n$ such that

$$\frac{1}{2}(c + c\sigma) \cdot x_0 > 0, \quad \frac{1}{2}(c + c\sigma) \cdot x_0 > \gamma \cdot x_0. \quad (33)$$

From (32) and (33) we have

$$\lim_{k \rightarrow \infty} \frac{1 + e^{\gamma \cdot kx_0}}{g(kx_0)} = \frac{1 + e^{\gamma \cdot kx_0}}{e^{(1/2)(c+c\sigma) \cdot kx_0}} = 0, \quad (34)$$

which implies that g satisfies the condition (3). Thus, we get (31). This completes the proof. $\square$

Theorem 6. Let $f, g : S \rightarrow \mathbb{C}$ be unbounded functions satisfying

$$|f(x + \sigma y) - g(x) f(y)| \leq \phi(y) \quad (35)$$

for all $x, y \in S$. Then there exists an unbounded σ -exponential function $m_\sigma : S \rightarrow \mathbb{C}$ such that $f(x) = f(0)m_\sigma(x)$ for all $x \in S$. In particular if f satisfies (3) or ϕ satisfies (4), then one has

$$f(x) = f(0) m_\sigma(x), \quad g(x) = m_\sigma(x) \quad (36)$$

for all $x \in S$.

Proof. Putting $y = 0$ in (35) we have

$$|f(x) - f(0)g(x)| \leq \phi(0) \quad (37)$$

for all $x \in S$. Choose a sequence $x_n \in S$, $n = 1, 2, 3, \dots$, such that $|g(x_n)| \rightarrow \infty$ as $n \rightarrow \infty$. Putting $x = x_n$, $n = 1, 2, 3, \dots$, in (35), dividing the result by $|g(x_n)|$, letting $n \rightarrow \infty$, and using (37) we have

$$f(y) = \lim_{n \rightarrow \infty} \frac{f(x_n + \sigma y)}{g(x_n)} = \lim_{n \rightarrow \infty} \frac{f(0)g(x_n + \sigma y)}{g(x_n)} \quad (38)$$

for all $y \in S$. Multiplying both sides of (38) by $f(x)$ and using (35) and (38) we have

$$\begin{aligned} f(y)f(x) &= \lim_{n \rightarrow \infty} \frac{f(0)g(x_n + \sigma y)f(x)}{g(x_n)} \\ &= \lim_{n \rightarrow \infty} \frac{f(0)f(x_n + \sigma y + \sigma x)}{g(x_n)} = f(0)f(x + y) \end{aligned} \quad (39)$$

for all $x, y \in S$. From (39) we have $f_0(x) := f(x)/f(0)$ is an exponential function, say m . Now, from (37) we can write

$$f(x) = f(0)m(x), \quad g(x) = m(x) + r(x) \quad (40)$$

for all $x \in S$, where $|r(x)| \leq \phi(0)/|f(0)|$ for all $x \in S$. Putting (40) in (35) and using the triangle inequality we have

$$\begin{aligned} |f(0)m(x)| |m(\sigma y) - m(y)| &\leq \phi(y) + r(x) |f(y)| \\ &\leq \phi(y) + \frac{\phi(0)|f(y)|}{|f(0)|} \end{aligned} \quad (41)$$

for all $x, y \in S$. Since m is unbounded, from (41) we have $m(\sigma y) = m(y)$ for all $y \in S$. Assume that f satisfies (3) or ϕ satisfies (4). Choose a sequence $y_n \in S$, $n = 1, 2, 3, \dots$, such that $(1 + \phi(y_n))/|f(y_n)| \rightarrow 0$ as $n \rightarrow \infty$. Putting $y = y_n$, $n = 1, 2, 3, \dots$, in (35), dividing the result by $|f(y_n)|$, letting $n \rightarrow \infty$, and using (37) we have

$$g(x) = \lim_{n \rightarrow \infty} \frac{f(x + \sigma y_n)}{f(y_n)} = \lim_{n \rightarrow \infty} \frac{f(0)g(x + \sigma y_n)}{f(y_n)} \quad (42)$$

for all $x \in S$. Multiplying both sides of (42) by $f(y)$ and using (35) and (42) we have

$$\begin{aligned} g(x)f(y) &= \lim_{n \rightarrow \infty} \frac{f(0)g(x + \sigma y_n)f(y)}{f(y_n)} \\ &= \lim_{n \rightarrow \infty} \frac{f(0)f(x + \sigma y + \sigma y_n)}{f(y_n)} = f(0)g(x + \sigma y) \end{aligned} \quad (43)$$

for all $x, y \in S$. Putting $x = 0$ in (43), replacing σy by x , and dividing the result by $f(0)$ we have

$$g(x) = g(0)f_0(\sigma x) = g(0)m_\sigma(\sigma x) = g(0)m_\sigma(x) \quad (44)$$

for all $x \in S$. Putting $y = 0$ in (35) and using (40) and (44) we get $g(0) = 1$. This completes the proof. $\square$

Using Theorem 6 and applying the same method as in the proof of Corollary 3 we have the following.

Corollary 7. Let $P(x)$, $x \in \mathbb{R}^n$, be a polynomial. Suppose that $f, g : \mathbb{R}^n \rightarrow \mathbb{C}$ are unbounded function satisfying

$$|f(x + \sigma y) - g(x)f(y)| \leq |P(y)| \quad (45)$$

for all $x, y \in \mathbb{R}^n$. Then there exists a σ -exponential function $m_\sigma : \mathbb{R}^n \rightarrow \mathbb{C}$ such that

$$f(x) = f(0)m_\sigma(x), \quad g(x) = m_\sigma(x) \quad (46)$$

for all $x \in \mathbb{R}^n$.

Using Theorem 6 and applying the same method as in the proof of Corollary 5 we have the following.

Corollary 8. Let $\gamma \in \mathbb{R}^n$ be fixed. Suppose that $f, g : \mathbb{R}^n \rightarrow \mathbb{C}$ are unbounded continuous function satisfying

$$|f(x + \sigma y) - g(x)f(y)| \leq e^{\gamma \cdot y} \quad (47)$$

for all $x, y \in \mathbb{R}^n$. Then there exists $c \in \mathbb{C}^n$, $\Re(c + c\sigma) \neq 0$ such that $f(x) = f(0)e^{(1/2)(c+c\sigma) \cdot x}$ for all $x \in \mathbb{R}^n$. If $(1/2)\Re(c+c\sigma) \neq t\gamma$ for all $t \leq 1$, then one has

$$f(x) = f(0)e^{(1/2)(c+c\sigma) \cdot x}, \quad g(x) = e^{(1/2)(c+c\sigma) \cdot x} \quad (48)$$

for all $x \in \mathbb{R}^n$.

Theorem 9. Let $f, g : S \rightarrow \mathbb{C}$ be unbounded functions satisfying

$$|f(x + \sigma y) - f(x)g(y)| \leq \phi(x) \quad (49)$$

for all $x, y \in S$. Then there exists an unbounded exponential $m : S \rightarrow \mathbb{C}$ such that $f(x) = f(0)m(x)$ for all $x \in S$. In particular if f satisfies (3) or ϕ satisfies (4), then one has

$$f(x) = f(0)m(x), \quad g(x) = m(\sigma x) \quad (50)$$

for all $x \in S$.

Proof. Putting $x = 0$ in (49) we have

$$|f(\sigma y) - f(0)g(y)| \leq \phi(0) \quad (51)$$

for all $y \in S$. Choose a sequence $y_n \in S$, $n = 1, 2, 3, \dots$, such that $|g(y_n)| \rightarrow \infty$ as $n \rightarrow \infty$. Putting $y = y_n$, $n = 1, 2, 3, \dots$, in (49), dividing the result by $|g(y_n)|$, letting $n \rightarrow \infty$, and using (51) we have

$$f(x) = \lim_{n \rightarrow \infty} \frac{f(x + \sigma y_n)}{g(y_n)} = \lim_{n \rightarrow \infty} \frac{f(0)g(\sigma x + y_n)}{g(y_n)} \quad (52)$$

for all $x \in S$. Multiplying both sides of (52) by $f(y)$ and using (49) and (52) we have

$$\begin{aligned} f(y)f(x) &= \lim_{n \rightarrow \infty} \frac{f(0)f(y)g(\sigma x + y_n)}{g(y_n)} \\ &= \lim_{n \rightarrow \infty} \frac{f(0)f(y + x + \sigma y_n)}{g(y_n)} = f(0)f(x + y) \end{aligned} \quad (53)$$

for all $x, y \in S$. From (53) we have $f_0(x) := f(x)/f(0)$ is an exponential function, say m . Assume that g satisfies (3) or ϕ satisfies (4). Choose a sequence $x_n \in S, n = 1, 2, 3, \dots$, such that $(1 + \phi(x_n))/|f(x_n)| \rightarrow 0$ as $n \rightarrow \infty$. Putting $x = x_n, n = 1, 2, 3, \dots$, in (49), dividing the result by $|f(x_n)|$, letting $n \rightarrow \infty$, and using (51) we have

$$g(y) = \lim_{n \rightarrow \infty} \frac{f(x_n + \sigma y)}{f(x_n)} = \lim_{n \rightarrow \infty} \frac{f(0)g(y + \sigma x_n)}{f(x_n)} \quad (54)$$

for all $y \in S$. Multiplying both sides of (54) by $f(x)$ and using (49) and (54) we have

$$\begin{aligned} f(x)g(y) &= \lim_{n \rightarrow \infty} \frac{f(0)f(x)g(y + \sigma x_n)}{f(x_n)} \\ &= \lim_{n \rightarrow \infty} \frac{f(0)f(x + \sigma y + x_n)}{f(x_n)} = f(0)g(\sigma x + y) \end{aligned} \quad (55)$$

for all $x, y \in S$. Putting $y = 0$ in (55), replacing σx by x , and dividing the result by $f(0)$ we have

$$g(x) = g(0)f_0(\sigma x) = g(0)m(\sigma x) \quad (56)$$

for all $x \in S$. Putting $x = 0$ in (49) and using (56) we get $g(0) = 1$. This completes the proof. $\square$

Using Theorem 9 we have the following.

Corollary 10. Let $P(x), x \in \mathbb{R}^n$, be a polynomial. Suppose that $f, g : \mathbb{R}^n \rightarrow \mathbb{C}$ are unbounded function satisfying

$$|f(x + \sigma y) - f(x)g(y)| \leq |P(x)| \quad (57)$$

for all $x, y \in \mathbb{R}^n$. Then there exists an exponential function $m : \mathbb{R}^n \rightarrow \mathbb{C}$ such that

$$f(x) = f(0)m(x), \quad g(x) = m(\sigma x) \quad (58)$$

for all $x \in \mathbb{R}^n$.

Corollary 11. Let $\gamma \in \mathbb{R}^n$ be fixed. Suppose that $f, g : \mathbb{R}^n \rightarrow \mathbb{C}$ are unbounded continuous function satisfying

$$|f(x + \sigma y) - f(x)g(y)| \leq e^{\gamma \cdot x} \quad (59)$$

for all $x, y \in \mathbb{R}^n$. Then there exists $c \in \mathbb{C}^n, \Re c \neq 0$ such that $f(x) = f(0)e^{c \cdot x}$ for all $x \in \mathbb{R}^n$. If $\Re c \neq t\gamma$ for all $t \leq 1$, then we have

$$f(x) = f(0)e^{c \cdot x}, \quad g(x) = e^{c \sigma \cdot x} \quad (60)$$

for all $x \in \mathbb{R}^n$.

Theorem 12. Let $f, g : S \rightarrow \mathbb{C}$ be unbounded functions satisfying

$$|f(x + \sigma y) - f(x)g(y)| \leq \phi(y) \quad (61)$$

for all $x, y \in S$. Then g is an exponential function. In particular, if g satisfies the condition (3) or ϕ satisfies (4), then there exists an unbounded exponential $m : S \rightarrow \mathbb{C}$ such that

$$f(x) = f(0)m(x), \quad g(x) = m(\sigma x) \quad (62)$$

for all $x \in S$.

Proof. Choose a sequence $x_n \in S, n = 1, 2, 3, \dots$, such that $|f(x_n)| \rightarrow \infty$ as $n \rightarrow \infty$. Putting $x = x_n, n = 1, 2, 3, \dots$, in (61), dividing the result by $|f(x_n)|$, and letting $n \rightarrow \infty$ we have

$$g(y) = \lim_{n \rightarrow \infty} \frac{f(x_n + \sigma y)}{f(x_n)} \quad (63)$$

for all $y \in S$. Multiplying both sides of (63) by $g(x)$ and using (61) and (63) we have

$$\begin{aligned} g(y)g(x) &= \lim_{n \rightarrow \infty} \frac{f(x_n + \sigma y)g(x)}{f(x_n)} \\ &= \lim_{n \rightarrow \infty} \frac{f(x_n + \sigma y + \sigma x)}{f(x_n)} = g(x + y) \end{aligned} \quad (64)$$

for all $x, y \in S$. Therefore, g is an exponential function, say m . Assume that g satisfies (3) or ϕ satisfies (4). Choose a sequence $y_n \in S, n = 1, 2, 3, \dots$, such that $(1 + \phi(y_n))/|g(y_n)| \rightarrow 0$ as $n \rightarrow \infty$. Putting $y = y_n, n = 1, 2, 3, \dots$, in (61), dividing the result by $|g(y_n)|$, and letting $n \rightarrow \infty$ we have

$$f(x) = \lim_{n \rightarrow \infty} \frac{f(x + \sigma y_n)}{g(y_n)} \quad (65)$$

for all $x \in S$. Multiplying both sides of (65) by $g(y)$ and using (61) and (65) we have

$$\begin{aligned} f(x)g(y) &= \lim_{n \rightarrow \infty} \frac{f(x + \sigma y_n)g(y)}{g(y_n)} \\ &= \lim_{n \rightarrow \infty} \frac{f(x + \sigma y_n + \sigma y)}{g(y_n)} = f(x + \sigma y) \end{aligned} \quad (66)$$

for all $x, y \in S$. Putting $x = 0$ and replacing y by σx in (66) we have $f(x) = f(0)m(\sigma x)$ for all $x \in S$. Replacing $m(x)$ by $m(\sigma x)$, we get (62). This completes the proof. $\square$

Using Theorem 12 we have the following.

Corollary 13. Let $P(x), x \in \mathbb{R}^n$, be a polynomial. Suppose that $f, g : \mathbb{R}^n \rightarrow \mathbb{C}$ are unbounded function satisfying

$$|f(x + \sigma y) - f(x)g(y)| \leq |P(y)| \quad (67)$$

for all $x, y \in \mathbb{R}^n$. Then there exists an exponential function $m : \mathbb{R}^n \rightarrow \mathbb{C}$ such that

$$f(x) = f(0)m(x), \quad g(x) = m(\sigma x) \quad (68)$$

for all $x \in \mathbb{R}^n$.

Corollary 14. Let $\gamma \in \mathbb{R}^n$ be fixed. Suppose that $f, g : \mathbb{R}^n \rightarrow \mathbb{C}$ are unbounded continuous function satisfying

$$|f(x + \sigma y) - f(x)g(y)| \leq e^{\gamma \cdot y} \quad (69)$$

for all $x, y \in \mathbb{R}^n$. Then there exists $c \in \mathbb{C}^n, \Re c \neq 0$ such that $g(x) = e^{c \sigma \cdot x}$ for all $x \in S$. If $\Re c \neq t\gamma$ for all $t \leq 1$, then we have

$$f(x) = f(0)e^{c \cdot x}, \quad g(x) = e^{c \sigma \cdot x} \quad (70)$$

for all $x \in S$.

3. Applications

In this section we consider the stability of (6)~(8). A function $M : (0, \infty) \rightarrow \mathbb{R}$ is called *multiplicative function* provided that $M(xy) = M(x)M(y)$ for all $x, y > 0$. Let $F(x + yi) = f(x, y)$, $G(x + iy) = g(x, y)$, and $\Phi(x + yi) = \phi(x, y)$ for all $(x, y) \in \mathbb{R}_0^2$. Then the functional inequalities (6) and (7) are converted to

$$|F(z\bar{w}) - G(z)F(w)| \leq \Phi(z) [\text{resp.}, \Phi(w)], \quad (71)$$

$$|F(z\bar{w}) - F(z)G(w)| \leq \Phi(z) [\text{resp.}, \Phi(w)] \quad (72)$$

for all $z, w \in \mathbb{C}_0$.

Viewing $\mathbb{C}_0$ as a multiplicative group, letting $\sigma(z) = \bar{z}$, and applying Theorems 2 and 6 to the inequalities (71) we have the following.

Theorem 15. *Let $f, g : \mathbb{R}_0^2 \rightarrow \mathbb{R}$ be unbounded functions satisfying (6). Then f, g are of the form*

$$\begin{aligned} f(x, y) &= f(1, 0) M\left(\sqrt{x^2 + y^2}\right), \\ g(x, y) &= M\left(\sqrt{x^2 + y^2}\right) \end{aligned} \quad (73)$$

for all $x, y \in \mathbb{R}$, where $M : (0, \infty) \rightarrow \mathbb{R}$ is a multiplicative function.

Applying Theorems 9 and 12 to the inequalities (72) we have the following.

Theorem 16. *Let $f, g : \mathbb{R}_0^2 \rightarrow \mathbb{R}$ be unbounded functions satisfying (7). Then f, g are of the form*

$$\begin{aligned} f(x, y) &= f(1, 0) M\left(\sqrt{x^2 + y^2}\right) E\left(\tan^{-1}\left(\frac{y}{x}\right)\right), \\ g(x, y) &= M\left(\sqrt{x^2 + y^2}\right) E\left(-\tan^{-1}\left(\frac{y}{x}\right)\right) \end{aligned} \quad (74)$$

for all $(x, y) \in \mathbb{R}_0^2$, where $M : (0, \infty) \rightarrow \mathbb{R}$ is a multiplicative function and $E : \mathbb{R} \rightarrow \mathbb{R}$ is an exponential function satisfying $E(2\pi) = 1$.

Let $\mathbb{H} = \{a + bi + cj + dk \mid a, b, c, d \in \mathbb{R}\}$ be the set of quaternions. Recall that $i^2 = j^2 = k^2 = -1$, $ij = k$, $jk = i$, $ki = j$, $ji = -k$, $kj = -i$, and $ik = -j$ and the conjugate of $q = a + bi + cj + dk \in \mathbb{H}$ is given by $q^* = a - bi - cj - dk$. We denote $\|q\| = \sqrt{qq^*} = \sqrt{a^2 + b^2 + c^2 + d^2}$. Let $F(x + yi + uj + vk) = f(x, y, u, v)$, $G(x + iy + uj + vk) = g(x, y, u, v)$, and $\Phi(x + yi + uj + vk) = \phi(x, y, u, v)$ for all $(x, y, u, v) \in \mathbb{R}_0^4$. Then the functional inequalities (8) are converted to

$$|F(qp^*) - G(q)F(p)| \leq \Phi(q) [\text{resp.}, \Phi(p)] \quad (75)$$

for all $p, q \in \mathbb{H} \setminus \{0\}$.

Applying Theorems 2 and 6 to the inequalities (75) we have the following.

Theorem 17. *Let $f, g : \mathbb{R}_0^4 \rightarrow \mathbb{R}$ be unbounded functions satisfying (8). Then f, g are of the form*

$$\begin{aligned} f(x, y, u, v) &= f(1, 0, 0, 0) M\left(\sqrt{x^2 + y^2 + u^2 + v^2}\right), \\ g(x, y, u, v) &= M\left(\sqrt{x^2 + y^2 + u^2 + v^2}\right) \end{aligned} \quad (76)$$

for all $(x, y, u, v) \in \mathbb{R}_0^4$, where $M : (0, \infty) \rightarrow \mathbb{R}$ is a multiplicative function.

4. Stability in L^∞ -Version

Let $f : \mathbb{R}^n \rightarrow \mathbb{C}$ be a locally integrable function and $\sigma : \mathbb{R}^n \rightarrow \mathbb{R}^n$ an involution. In this section, we consider an L^∞ -version of the stability of functional equation

$$f(x + \sigma y) = f(x)f(y) \quad (77)$$

for almost every $(x, y) \in \mathbb{R}^{2n}$. More precisely, we study the functional inequality

$$\|f(x + \sigma y) - f(x)f(y)\|_{L^\infty(\mathbb{R}^{2n})} \leq \epsilon. \quad (78)$$

As is well known, inequality (78) implies

$$\left| \int (f(x + \sigma y) - f(x)f(y)) \varphi(x, y) dx dy \right| \leq \epsilon \|\varphi\|_{L^1(\mathbb{R}^{2n})} \quad (79)$$

for all $\varphi \in L^1(\mathbb{R}^4)$.

We first employ $\delta : \mathbb{R}^n \rightarrow \mathbb{R}$ defined by

$$\delta(x) = \begin{cases} qe^{-(1-|x|^2)^{-1}}, & \text{if } |x| < 1 \\ 0, & \text{if } |x| \geq 1, \end{cases} \quad (80)$$

where $q = \left(\int_{|x|<1} e^{-(1-|x|^2)^{-1}} dx\right)^{-1}$. It is easy to see that δ is an infinitely differentiable function with support $\{x : |x| \leq 1\}$.

Let f be a locally integrable function on $\mathbb{R}^n$ and $\delta_t(x) := t^{-n}\delta(x/t)$, $t > 0$. Then for each $t > 0$,

$$f * \delta_t(x) = \int_{\mathbb{R}^n} f(\xi) \delta_t(x - \xi) d\xi \quad (81)$$

is a smooth function and $f * \delta_t(x) \rightarrow f(x)$ for almost every $x \in \mathbb{R}^n$ as $t \rightarrow 0^+$.

In the following, we exclude the case when $f(x) = 0$ for almost every $x \in \mathbb{R}^n$.

Theorem 18. *Let $f : \mathbb{R}^n \rightarrow \mathbb{C}$ satisfy (78). Then either there exists an unbounded exponential function $m : \mathbb{R}^n \rightarrow \mathbb{C}$ such that*

$$f(x) = m\left(\frac{x + \sigma x}{2}\right) \quad (82)$$

for almost every $x \in \mathbb{R}^n$, or else

$$\|f\|_{L^\infty(\mathbb{R}^n)} \leq \frac{1}{2} (1 + \sqrt{1 + 4\epsilon}). \quad (83)$$

If $\epsilon < 1/4$, then either

$$\frac{1}{2} (1 + \sqrt{1 - 4\epsilon}) \leq \|f\|_{L^\infty(\mathbb{R}^n)} \leq \frac{1}{2} (1 + \sqrt{1 + 4\epsilon}) \quad (84)$$

or

$$\|f\|_{L^\infty(\mathbb{R}^n)} \leq \frac{1}{2} (1 - \sqrt{1 - 4\epsilon}). \quad (85)$$

Proof. Applying $\varphi(x', y') = \delta_t(x - x')\delta_s(y - y')$ in (79) we have

$$\begin{aligned} & \iint f(x' + \sigma y') \delta_t(x - x') \delta_s(y - y') dx' dy' \\ &= \int f(z') \left(\int \delta_t(x - z' + \sigma y') \delta_s(y - y') dy' \right) dz' \\ &= \int f(z') \left(\int \delta_t(x - z' + \sigma y') (\delta_s \circ \sigma) \right. \\ & \quad \left. \times (\sigma y - \sigma y') dy' \right) dz' \\ &= \int f(z') \left(\int \delta_t(x - z' + y') (\delta_s \circ \sigma) (\sigma y - y') dy' \right) dz' \\ &= \int f(z') \left(\int \delta_t(y') (\delta_s \circ \sigma) (x + \sigma y - z' - y') dy' \right) dz' \\ &= \int f(z') \delta_t * (\delta \circ \sigma)_s (x + \sigma y - z') dz' \\ &= (f * \delta_t * (\delta \circ \sigma)_s)(x + \sigma y). \end{aligned} \quad (86)$$

We also have

$$\begin{aligned} & \iint f(x') f(y') \delta_t(x - x') \delta_s(y - y') dx' dy' \\ &= \int f(y') \left(\int f(x') \delta_t(x - x') dx' \right) \delta_s(y - y') dy' \\ &= (f * \delta_t)(x) \int f(y') (f * \delta_s)(y - y') dy' \\ &= (f * \delta_t)(x) (f * \delta_s)(y). \end{aligned} \quad (87)$$

Thus, the inequality (78) is converted to the classical functional inequality

$$|(f * \delta_t * \delta_s^\sigma)(x + \sigma y) - (f * \delta_t)(x) (f * \delta_s)(y)| \leq \epsilon \quad (88)$$

for all $x, y \in \mathbb{R}^n$, where $\delta^\sigma = \delta \circ \sigma$.

Choosing $y_0 \in \mathbb{R}$, $s_0 > 0$ such that $(f * \delta_{s_0})(y_0) \neq 0$, putting $y = y_0$, $s = s_0$ in (88), using the triangle inequality, and dividing the result by $|(f * \delta_{s_0})(y_0)|$ we have

$$|(f * \delta_t)(x)| \leq \frac{|(f * \delta_t * \delta_{s_0}^\sigma)(x + \sigma y_0)| + \epsilon}{|(f * \delta_{s_0})(y_0)|} \quad (89)$$

for all $x \in \mathbb{R}^n$. Since $(f * \delta_t * \delta_{s_0}^\sigma)(x + \sigma y_0) \rightarrow (f * \delta_{s_0}^\sigma)(x + \sigma y_0)$ as $t \rightarrow 0^+$, it follows that

$$F(x) := \limsup_{t \rightarrow 0^+} (f * \delta_t)(x) \quad (90)$$

exists for all $x \in \mathbb{R}^n$. Since $(f * \delta_t)(x) \rightarrow f(x)$ for almost every $x \in \mathbb{R}^n$, it follows from (90) that

$$F(x) = f(x) \quad (91)$$

for almost every $x \in \mathbb{R}^n$.

Fixing $y \in \mathbb{R}^n$ and letting $s \rightarrow 0^+$ so that $(f * \delta_s)(y) \rightarrow F(y)$ in (88), we have

$$|(f * \delta_t)(x + \sigma y) - (f * \delta_t)(x) F(y)| \leq \epsilon \quad (92)$$

for all $x, y \in \mathbb{R}^n$. We first consider the case when F is unbounded. Let $y_n \in \mathbb{R}^n$, $n = 1, 2, 3, \dots$ be a sequence such that $|F(y_n)| \rightarrow \infty$. Putting $y = y_n$ in (92), dividing the result by $|F(y_n)|$, and letting $n \rightarrow \infty$ we have

$$(f * \delta_t)(x) = \lim_{n \rightarrow \infty} \frac{(f * \delta_t)(x + \sigma y_n)}{F(y_n)} \quad (93)$$

for all $(x, y) \in \mathbb{R}^n$. Multiplying $F(y)$ in (93) and using (92) and (93) we have

$$\begin{aligned} (f * \delta_t)(x) F(y) &= \lim_{n \rightarrow \infty} \frac{(f * \delta_t)(x + \sigma y_n) F(y)}{F(y_n)} \\ &= \lim_{n \rightarrow \infty} \frac{(f * \delta_t)(x + \sigma y + \sigma y_n)}{F(y_n)} \\ &= (f * \delta_t)(x + \sigma y) \end{aligned} \quad (94)$$

for all $x, y \in \mathbb{R}^n$, $t > 0$. Putting $x = 0$ in (94) we have

$$(f * \delta_t)(0) F(y) = (f * \delta_t)(\sigma y) \quad (95)$$

for all $y \in \mathbb{R}^n$, $t > 0$. From (95) we have $(f * \delta_t)(0) \neq 0$ for some $t > 0$. Putting (95) in (94) we have

$$F(\sigma x) F(y) = F(y + \sigma x) \quad (96)$$

for all $x, y \in \mathbb{R}^n$. From (96) F is an exponential function. Now, we prove that

$$F(x) = F(\sigma x) \quad (97)$$

for all $x \in \mathbb{R}^n$. In view of (94), replacing $(f * \delta_t)(x)$ by $(f * \delta_t)(0) F(\sigma x)$ and $(f * \delta_s)(y)$ by $(f * \delta_s)(0) F(\sigma y)$ in (88) and letting $s \rightarrow 0^+$ so that $(f * \delta_s)(0) \rightarrow F(0)$ we have

$$|(f * \delta_t)(x + \sigma y) - (f * \delta_t)(0) F(\sigma x) F(\sigma y)| \leq \epsilon \quad (98)$$

for all $x, y \in \mathbb{R}^n$, $t > 0$. Using (95) and (98) we have

$$|(f * \delta_t)(0) F(y + \sigma x) - (f * \delta_t)(0) F(\sigma x) F(\sigma y)| \leq \epsilon. \quad (99)$$

Letting $t \rightarrow 0^+$ in (99) so that $(f * \delta_t)(0) \rightarrow F(0)$ we have

$$|F(y + \sigma x) - F(\sigma x)F(\sigma y)| \leq \epsilon \quad (100)$$

for all $x, y \in \mathbb{R}$. Since F is an exponential function, it follows from (100) that

$$|F(\sigma x)| |F(y) - F(\sigma y)| \leq \epsilon \quad (101)$$

for all $x, y \in \mathbb{R}^n$. Since F is unbounded, from (101) we have $F(y) = F(\sigma y)$ for all $y \in \mathbb{R}^n$. Now, F is written in the form

$$\begin{aligned} F(x) &= F\left(\frac{x}{2} + \frac{x}{2}\right) = F\left(\frac{x}{2}\right)F\left(\frac{x}{2}\right) \\ &= F\left(\frac{x}{2}\right)F\left(\frac{\sigma x}{2}\right) = F\left(\frac{x + \sigma x}{2}\right) \end{aligned} \quad (102)$$

for all $x, y \in \mathbb{R}^n$. Conversely, let $F(x) = m((x + \sigma x)/2)$, where $m : \mathbb{R}^n \rightarrow \mathbb{C}$ is an arbitrary exponential function. Then F is an exponential function satisfying $F(x) = F(\sigma x)$ for all $x \in \mathbb{R}^n$. Thus, we get (82). From now on, we assume that F is bounded, say $|F(x)| \leq M$ for all $x \in \mathbb{R}^n$. Then, it follows from (91) that $\|f\|_{L^\infty(\mathbb{R}^n)} \leq M$. Thus, we have

$$\begin{aligned} |(f * \delta_t)(x)| &= \left| \int f(x') \delta(x - x') dx' \right| \\ &\leq M \int |\delta(x - x')| dx' = M \end{aligned} \quad (103)$$

for all $x \in \mathbb{R}^n$. From the inequality (92), using the method in Theorem 10 of [1] we have

$$|(f * \delta_t)(x)| (|F(y)| - 1) \leq \epsilon \quad (104)$$

for all $x, y \in \mathbb{R}^n, t > 0$. Fixing $x \in \mathbb{R}^n$ and letting $t \rightarrow 0^+$ in (104) we have

$$|F(x)| (|F(y)| - 1) \leq \epsilon \quad (105)$$

for all $x, y \in \mathbb{R}$. From (105), using the method in Theorem 10 of [1] we have

$$|F(x)| \leq \frac{1}{2} (1 + \sqrt{1 - 4\epsilon}) \quad (106)$$

for all $x \in \mathbb{R}$, and if $\epsilon < 1/4$, then we have either

$$\frac{1}{2} (1 - \sqrt{1 + 4\epsilon}) \leq |F(x)| \leq \frac{1}{2} (1 + \sqrt{1 + 4\epsilon}) \quad (107)$$

for all $x \in \mathbb{R}$ or

$$|F(x)| \leq \frac{1}{2} (1 - \sqrt{1 - 4\epsilon}) \quad (108)$$

for all $x \in \mathbb{R}$. Since $f(x) = F(x)$ almost every $x \in \mathbb{R}^n$, we get (83), (84), and (85). This completes the proof. $\square$

Conflict of Interests

The author declares that there is no conflict of interests regarding the publication of this paper.

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References

- [1] J. Chung and S.-Y. Chung, "Stability of exponential functional equations with involutions," *Journal of Function Spaces*, vol. 2014, Article ID 619710, 9 pages, 2014.
- [2] J. Chung and J. Chang, "On two functional equations originating from number theory," to appear in *Proceedings of the Indian Academy of Science*.
- [3] K. B. Houston and P. K. Sahoo, "On two functional equations and their solutions," *Applied Mathematics Letters*, vol. 21, no. 9, pp. 974–977, 2008.
- [4] S. Jung and J. Bae, "Some functional equations originating from number theory," *Proceedings of the Indian Academy of Sciences—Mathematical Sciences*, vol. 113, no. 2, pp. 91–98, 2003.
- [5] G. H. Kim and P. K. Sahoo, "Stability of some functional equations related to distance measures-I," *Applied Mathematics Letters*, vol. 24, no. 6, pp. 843–849, 2011.
- [6] J. A. Baker, "The stability of cosine functional equation," *Proceedings of the American Mathematical Society*, vol. 80, no. 3, pp. 411–416, 1980.
- [7] A. Bahyrycz, "On solutions of the second generalization of d'Alembert's functional equation on a restricted domain," *Applied Mathematics and Computation*, vol. 223, pp. 209–215, 2013.
- [8] A. Bahyrycz and J. Brzdęk, "On solutions of the d'Alembert equation on a restricted domain," *Aequationes Mathematicae*, vol. 85, no. 1-2, pp. 169–183, 2013.
- [9] A. Bahyrycz and J. Brzdęk, "A note on d'Alembert's functional equation on a restricted domain," *Aequationes Mathematicae*, 2014.
- [10] J. Brzdęk, "On solutions of a generalization of the Reynolds functional equation," *Demonstratio Mathematica*, vol. 41, no. 4, pp. 859–868, 2008.
- [11] J. Brzdęk, A. Najdecki, and B. Xu, "Two general theorems on superstability of functional equations," *Aequationes mathematicae*, 2014.
- [12] J. Chudziak and J. Tabor, "On the stability of the Gołab-Schinzel functional equation," *Journal of Mathematical Analysis and Applications*, vol. 302, no. 1, pp. 196–200, 2005.
- [13] J. Chung, "On an exponential functional inequality and its distributional version," *Canadian Mathematical Bulletin*, 2014.
- [14] J. G. Dhombres and R. Ger, "Conditional Cauchy equations," *Glasnik Matematički*, vol. 13, no. 33, pp. 39–62, 1978.
- [15] R. Ger and P. Šemrl, "The stability of the exponential equation," *Proceedings of the American Mathematical Society*, vol. 124, no. 3, pp. 779–787, 1996.
- [16] A. Najdecki, "On stability of a functional equation connected with the Reynolds operator," *Journal of Inequalities and Applications*, vol. 2007, Article ID 79816, 3 pages, 2007.

- [17] N. Brillouët-Belluot, J. Brzdęk, and K. Ciepliński, “On some recent developments in Ulam’s type stability,” *Abstract and Applied Analysis*, vol. 2012, Article ID 716936, 41 pages, 2012.
- [18] J. Brzdęk and K. Ciepliński, “Hyperstability and superstability,” *Abstract and Applied Analysis*, vol. 2013, Article ID 401756, 13 pages, 2013.
- [19] D. H. Hyers, G. Isac, and T. M. Rassias, *Stability of Functional Equations in Several Variables*, Progress in Nonlinear Differential Equations and Their Applications, Birkhäuser, Basel, Switzerland, 1998.