Research Article

Existence of Multiple Solutions for Fourth-Order Elliptic Problem

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By using the variant fountain theorem, we study the existence of multiple solutions for a class of superquadratic fourth-order elliptic problem with Navier boundary value condition.

1. Introduction

Consider the following fourth-order boundary value problem:

$$\begin{aligned}\Delta^2 u + c\Delta u &= g(x, u) \quad \text{in } \Omega, \\ u &= \Delta u = 0 \quad \text{on } \partial\Omega,\end{aligned}\tag{1}$$

where Δ^2 denotes the biharmonic operator, $\Omega \subset \mathbb{R}^N$ ($N > 4$) is a bounded domain with smooth boundary, and $g \in C(\overline{\Omega} \times \mathbb{R}, \mathbb{R})$.

The fourth-order elliptic equations which contain a biharmonic operator can describe the static form change of beam or the motion of rigid body. Thus the fourth-order elliptic equations are widely applied in physics, oceanics, aerospace engineering and other engineering. In [1], Lazer and Mckenna considered the biharmonic problem:

$$\begin{aligned}\Delta^2 u + c\Delta u &= d[(u+1)^+ - 1] \quad \text{in } \Omega, \\ u &= \Delta u = 0 \quad \text{on } \partial\Omega,\end{aligned}\tag{2}$$

where $u^+ = \max\{u, 0\}$ and $d \in \mathbb{R}$. They pointed out that this type of nonlinearity furnishes a model to study traveling waves in suspension bridges. Afterwards, in [2], they have proved the existence of $2k - 1$ solutions when $N = 1$ and $d > \lambda_i(\lambda_i - c)$ ($\{\lambda_i\}_{i \geq 1}$ is the sequence of the eigenvalues of $-\Delta$ in $H_0^1(\Omega)$) by the global bifurcation method. In [3] the existence of a negative solution of (2) was proved when

$d > \lambda_1(\lambda_1 - c)$ by using the Leray-Schauder degree. In particular, in [1, 4] the authors observed that problem (2) was interesting also when the nonlinearity $(u+1)^+ - 1$ was replaced by a somewhat more general function $g(\cdot, u)$. In [5], Micheletti and Pistoia used a variational linking theorem to investigate the existence of two solutions for a more general nonlinearity $g(\cdot, u)$. Moreover, by using a variational result, they and Saccon also showed the existence of three solutions for some special $g(\cdot, u)$ (see [6]). Next year, in [7], Micheletti and Saccon obtained two results about the existence of two nontrivial solutions and four nontrivial solutions by the similar variational approach, depending on the position of a suitable parameter with respect to the eigenvalues of the linear part. In recent years, more researchers have used variational approach to investigate the fourth-order elliptic equations. In [8], Xu and Zhang studied the existence of positive solutions of problem (1) when g satisfied the local superlinearity and sublinearity condition and $c < \lambda_1$ by the classical mountain pass theorem. Recently, in [9], Pu et al. used the least action principle, the Ekeland variational principle, and the mountain pass theorem to prove the existence and multiplicity of solutions of (1) when $g(x, u) = a(x)|u|^{s-2}u + f(x, u)$ ($a \in L^\infty(\Omega)$, $s \in (1, 2)$). For other related results, see [8–14] and the references therein. Here, we emphasize that most authors considered the case $c < \lambda_1$.

The variant fountain theorems established in [15] have been used in the study of a class of semilinear elliptic equations (see [16, 17]) and the investigation of the Hamiltonian

system (see [18, 19]). Inspired by [9, 17], we will use the variant fountain theorem to investigate the problem (1). More precisely, we make the following assumptions.

(S₁) There exist constants $d_1 > 0$ and $1 < \nu < (N+4)/(N-4)$ such that

$$|g(x, u)| \leq d_1 (1 + |u|^\nu), \quad \forall (x, u) \in \Omega \times \mathbb{R}. \quad (3)$$

(S₂) $G(x, u) \geq 0$ for all $(x, u) \in \Omega \times \mathbb{R}$ and

$$\liminf_{|u| \rightarrow \infty} \frac{G(x, u)}{|u|^2} = \infty, \quad \text{uniformly for } x \in \Omega. \quad (4)$$

Here, $G(x, u) := \int_0^u g(x, s) ds$ is the primitive of the nonlinearity g .

(S₃) There exist constants $\varrho > (2N/(N+4))\nu$, $L > 0$ and $d_3 > 0$ such that

$$ug(x, u) - 2G(x, u) \geq d_3 |u|^\varrho \quad \forall |u| \geq L, \quad x \in \Omega. \quad (5)$$

Our main result is the following theorem.

Theorem 1. Assume that (S₁)–(S₃) hold and $G(x, u)$ is even in u . Then problem (1) possesses infinitely many solutions.

Remark 2. In Theorem 1, we do not assume $c < \lambda_1$, which is widely used in the investigation of the fourth-order equations. As is known, the so-called global Ambrosetti-Rabinowitz condition (AR-condition for short) is introduced by Ambrosetti and Rabinowitz in [20] and widely used to the existence of infinitely many solutions for superquadratic situation: there is a constant $\alpha > 2$ such that, for all $u \neq 0$ and $x \in \Omega$, the nonlinearity is assumed to satisfy

$$0 < \alpha G(x, u) \leq ug(x, u). \quad (6)$$

In fact, if we choose

$$G(x, u) = H(x) \left(|u|^\mu + (\mu - 2) |u|^{\mu-\varepsilon} \sin^2 \left(\frac{|u|^\varepsilon}{\varepsilon} \right) \right), \quad (7)$$

where $\varepsilon \in (0, \mu - 2)$, $H \in C(\overline{\Omega})$, and $H(x) > 0$ for all $x \in \overline{\Omega}$. Then it is easy to see that G satisfies the conditions (S₁)–(S₃) in Theorem 1 with $\mu = 3$, $\nu = 2$, $\varepsilon = 0.1$, $\varrho = 2.9$, and $N = 5$, but G does not satisfy the AR-condition (6).

Remark 3. By (S₁), we can obtain that there exists a constant $d_2 > 0$ such that

$$|G(x, u)| \leq d_1 (|u| + |u|^{\nu+1}) + d_2, \quad \forall (x, u) \in \Omega \times \mathbb{R}. \quad (8)$$

And by (S₃), there exists a constant $d_4 > 0$ such that

$$ug(x, u) - 2G(x, u) \geq d_3 |u|^\varrho - d_4, \quad \forall (x, u) \in \Omega \times \mathbb{R}. \quad (9)$$

2. Preliminaries

In this section, we will establish the variational setting for our problem and state a variant fountain theorem.

Let $E = H^2(\Omega) \cap H_0^1(\Omega)$ be the Hilbert space equipped with the inner product

$$(u, v)_E = \int_\Omega \Delta u \Delta v \, dx \quad (10)$$

and the norm

$$\|u\|_E = (u, v)_E^{1/2}. \quad (11)$$

A weak solution of problem (1) is a $u \in E$ such that

$$\int_\Omega (\Delta u \Delta v - c \langle \nabla u, \nabla v \rangle) \, dx - \int_\Omega g(x, u) v \, dx = 0 \quad (12)$$

for any $v \in E$. Here and in the sequel, $\langle \cdot, \cdot \rangle$ always denotes the standard inner product in $\mathbb{R}^N$. Let $\Phi : E \rightarrow \mathbb{R}$ be the functional defined by

$$\Phi(u) = \frac{1}{2} \int_\Omega (|\Delta u|^2 - c |\nabla u|^2) \, dx - \int_\Omega G(x, u) \, dx. \quad (13)$$

It is well known that a critical point of the functional Φ in E corresponds to a weak solution of problem (1).

Let λ_i ($i = 1, 2, \dots$) be the eigenvalues of $-\Delta$ in $H_0^1(\Omega)$. Then the eigenvalue problem

$$\begin{aligned} \Delta^2 u + c \Delta u &= \mu u \quad \text{in } \Omega, \\ u &= \Delta u = 0 \quad \text{on } \partial \Omega, \end{aligned} \quad (14)$$

has infinitely many eigenvalues $\mu_i = \lambda_i(\lambda_i - c)$, $i = 1, 2, \dots$.

Define a selfadjoint linear operator $\mathcal{A} : L^2(\Omega) \rightarrow L^2(\Omega)$ by

$$(\mathcal{A}u, v)_2 = \int_\Omega (\Delta u \Delta v - c \langle \nabla u, \nabla v \rangle) \, dx \quad (15)$$

with domain $D(\mathcal{A}) = E$. Here, $(\cdot, \cdot)_2$ denotes the inner product in $L^2(\Omega)$ and in the sequel $L^2(\Omega)$ is simply denoted by L^2 . Then the sequence of eigenvalues of $\mathcal{A}$ is just $\{\mu_i\}$ ($i = 1, 2, \dots$). Denote the corresponding system of eigenfunctions by $\{e_n\}$; it forms an orthogonal basis in L^2 .

Denote

$$n^- = \#\{i \mid \mu_i < 0\}, \quad n^0 = \#\{i \mid \mu_i = 0\}, \quad \bar{n} = n^- + n^0. \quad (16)$$

Here, $\#\{\cdot\}$ denotes the cardinal of a set. Let

$$\begin{aligned} L^- &= \text{span}\{e_1, \dots, e_{n^-}\}, \quad L^0 = \text{span}\{e_{n^-+1}, \dots, e_{\bar{n}}\}, \\ L^+ &= (L^- \oplus L^0)^\perp = \overline{\text{span}\{e_{\bar{n}+1}, \dots\}}. \end{aligned} \quad (17)$$

Decompose L^2 as

$$L^2 = L^- \oplus L^0 \oplus L^+. \quad (18)$$

Then E also possesses the orthogonal decomposition

$$E = E^- \oplus E^0 \oplus E^+ \quad (19)$$

with

$$E^- = L^-, \quad E^0 = L^0, \quad E^+ = E \cap L^+ = \overline{\text{span}\{e_{\bar{n}+1}, \dots\}}. \quad (20)$$

We define on E a new inner product and the associated norm by

$$(u, v) = (\mathcal{A}u^+, v^+) - (\mathcal{A}u^-, v^-) + (u^0, v^0)_2, \quad (21)$$

$$\|u\| = (u, u)^{1/2}.$$

Therefore, Φ can be written as

$$\Phi(u) = \frac{1}{2} (\|u^+\|^2 - \|u^-\|^2) - \Psi(u), \quad (22)$$

where $\Psi(u) = \int_{\Omega} G(x, u) dx$ for all $u = u^- + u^0 + u^+ \in E = E^- \oplus E^0 \oplus E^+$. Then Φ and Ψ are continuously differentiable.

Direct computation shows that

$$\Psi'(u) v = \int_{\Omega} g(x, u) v dx \quad (23)$$

$$\Phi'(u) v = (u^+, v^+) - (u^-, v^-) - \Psi'(u) v$$

for all $u, v \in E$ with $u = u^- + u^0 + u^+$ and $v = v^- + v^0 + v^+$, respectively. It is known that $\Psi' : E \rightarrow E$ is compact.

Denote by $\|\cdot\|_p$ the usual norm of $L^p \equiv L^p(\Omega)$ for all $1 \leq p \leq 2N/(N-4)$; then by the Sobolev embedding theorem, there exists a $\tau_p > 0$ such that

$$\|u\|_p \leq \tau_p \|u\|, \quad \forall u \in E. \quad (24)$$

Noting that the constants ν and ϱ appeared in (S_1) and (S_3) satisfies

$$1 + \nu < \frac{2N}{N-4}, \quad \frac{\varrho}{\varrho - \nu} < \frac{2N}{N-4}. \quad (25)$$

To prove our main result Theorem 1, we need an abstract critical point theorem found in [15].

Let E be a Banach space with the norm $\|\cdot\|$ and $E = \overline{\oplus_{j \in \mathbb{N}} X_j}$ with $\dim X_j < \infty$ for any $j \in \mathbb{N}$. Set $Y_k = \oplus_{j=1}^k X_j$ and $Z_k = \overline{\oplus_{j=k}^{\infty} X_j}$. Consider the following C^1 -functional $\Phi_{\lambda} : E \rightarrow \mathbb{R}$ defined by

$$\Phi_{\lambda}(u) := A(u) - \lambda B(u), \quad \lambda \in [1, 2]. \quad (26)$$

Theorem 4 (see [15, Theorem 2.1]). *Assume that the functional Φ_{λ} defined above satisfies the following:*

- (F_1) Φ_{λ} maps bounded sets to bounded sets for $\lambda \in [1, 2]$, and $\Phi_{\lambda}(-u) = \Phi_{\lambda}(u)$ for all $(\lambda, u) \in [1, 2] \times E$;
- (F_2) $B(u) \geq 0$ for all $u \in E$; moreover, $A(u) \rightarrow \infty$ or $B(u) \rightarrow \infty$ as $\|u\| \rightarrow \infty$;

(F_3) there exist $r_k > \rho_k > 0$ such that

$$\alpha_k(\lambda) := \inf_{u \in Z_k, \|u\| = \rho_k} \Phi_{\lambda}(u) > \beta_k(\lambda) := \max_{u \in Y_k, \|u\| = r_k} \Phi_{\lambda}(u), \quad \forall \lambda \in [1, 2]. \quad (27)$$

Then

$$\alpha_k(\lambda) \leq \zeta_k(\lambda) := \inf_{\gamma \in \Gamma_k} \max_{u \in B_k} \Phi_{\lambda}(\gamma(u)), \quad \forall \lambda \in [1, 2], \quad (28)$$

where $B_k = \{u \in Y_k : \|u\| \leq r_k\}$ and $\Gamma_k := \{\gamma \in C(B_k, E) : \gamma \text{ is odd}, \gamma|_{\partial B_k} = \text{id}\}$. Moreover, for a.e. $\lambda \in [1, 2]$, there exists a sequence $\{u_m^k(\lambda)\}_{m=1}^{\infty}$ such that

$$\sup_m \|u_m^k(\lambda)\| < \infty, \quad \Phi'_{\lambda}(u_m^k(\lambda)) \rightarrow 0, \quad (29)$$

$$\Phi_{\lambda}(u_m^k(\lambda)) \rightarrow \zeta_k(\lambda) \quad \text{as } m \rightarrow \infty.$$

In order to apply this theorem to prove our main result, we define the functionals A , B , and Φ_{λ} on our working space $E = H^2(\Omega) \cap H_0^1(\Omega)$ as follows:

$$A(u) = \frac{1}{2} \|u^+\|^2, \quad B(u) = \frac{1}{2} \|u^-\|^2 + \int_{\Omega} G(x, u) dx, \quad (30)$$

$$\Phi_{\lambda}(u) = A(u) - \lambda B(u) = \frac{1}{2} \|u^+\|^2 - \lambda \left(\frac{1}{2} \|u^-\|^2 + \int_{\Omega} G(x, u) dx \right) \quad (31)$$

for all $u = u^- + u^0 + u^+ \in E = E^- + E^0 + E^+$ and $\lambda \in [1, 2]$. Then $\Phi_{\lambda} \in C^1(E, \mathbb{R})$ for all $\lambda \in [1, 2]$ and

$$\Phi'_{\lambda}(u) v = (u^+, v^+) - \lambda \left((u^-, v^-) + \int_{\Omega} g(x, u) v dx \right). \quad (32)$$

Let $X_j = \text{span}\{e_j\}$, $j = 1, 2, \dots$. Note that Φ_1 is just equal to the functional Φ defined in (22).

3. Proof of Theorem 1

In this section we firstly establish the following two lemmas and then give the proof of Theorem 1.

Lemma 5. *Assume that (S_1) and (S_2) hold. Then $B(u) \geq 0$ for all $u \in E$. Furthermore, $A(u) \rightarrow \infty$ or $B(u) \rightarrow \infty$ as $\|u\| \rightarrow \infty$.*

Proof. Since $G(x, u) \geq 0$, by (30), it is obvious that $B(u) \geq 0$ for all $u \in E$.

By the similar method used in the proof of Lemma 2.6 of [17], for any finite-dimensional subspace $F \subset E$, there exists a constant $\epsilon > 0$ such that

$$m(\{x \in \Omega : |u| \geq \epsilon \|u\|\}) \geq \epsilon, \quad \forall u \in F \setminus \{0\}, \quad (33)$$

where $m(\cdot)$ is the Lebesgue measure in $\mathbb{R}^N$.

Now for the finite-dimensional subspace $E^- \oplus E^0 \subset E$, there exists a constant ϵ corresponding to the one in (33). Let

$$\Lambda_u = \{x \in \Omega : |u| \geq \epsilon \|u\|\}, \quad \forall u \in E^- \oplus E^0 \setminus \{0\}. \quad (34)$$

Then $m(\Lambda_u) \geq \epsilon$. By (S_2) , there exist positive constants d_5 and R_1 such that

$$G(x, u) \geq d_5 |u|^2, \quad \forall x \in \Omega, |u| \geq R_1. \quad (35)$$

Note that

$$|u(x)| \geq R_1, \quad \forall x \in \Lambda_u \quad (36)$$

for any $u \in E^- \oplus E^0$ with $\|u\| \geq R_1/\epsilon$. Combining (35) and (36), for any $u \in E^- \oplus E^0$ with $\|u\| \geq R_1/\epsilon$, we have

$$\begin{aligned} B(u) &= \frac{1}{2} \|u^-\|^2 + \int_{\Omega} G(x, u) dt \\ &\geq \int_{\Lambda_u} G(x, u) dt \geq \int_{\Lambda_u} d_5 |u|^2 dt \\ &\geq d_5 \epsilon^2 \|u\|^2 \cdot m(\Lambda_u) \geq d_5 \epsilon^3 \|u\|^2, \end{aligned} \quad (37)$$

which implies that

$$B(u) \longrightarrow \infty \quad \text{as } \|u\| \longrightarrow \infty \quad \text{on } E^- \oplus E^0. \quad (38)$$

Combining this with $E = E^- \oplus E^0 \oplus E^+$ and (30), we have

$$A(u) \longrightarrow \infty \quad \text{or} \quad B(u) \longrightarrow \infty \quad \text{as } \|u\| \longrightarrow \infty. \quad (39)$$

The proof is completed. $\square$

Lemma 6. Let (S_1) , (S_2) be satisfied. Then there exist a positive integer k_1 and two sequences $r_k > \rho_k \rightarrow \infty$ as $k \rightarrow \infty$ such that

$$\alpha_k(\lambda) := \inf_{u \in Z_k, \|u\| = \rho_k} \Phi_\lambda(u) > 0, \quad \forall k \geq k_1, \quad (40)$$

$$\beta_k(\lambda) := \max_{u \in Y_k, \|u\| = r_k} \Phi_\lambda(u) < 0, \quad \forall k \in \mathbb{N}, \quad (41)$$

where $Y_k = \bigoplus_{j=1}^k X_j = \text{span}\{e_1, e_2, \dots, e_k\}$ and $Z_k = \overline{\bigoplus_{j=k}^\infty X_j} = \overline{\text{span}\{e_k, e_{k+1}, \dots\}}$ for all $k \in \mathbb{N}$.

Proof.

Step 1. We first prove (40).

By virtue of (8) and (31), for any $u \in E^+$

$$\begin{aligned} \Phi_\lambda(u) &\geq \frac{1}{2} \|u\|^2 - 2 \int_{\Omega} G(x, u) dx \\ &\geq \frac{1}{2} \|u\|^2 - 2d_1 (\|u\|_1 + \|u\|_{\nu+1}^{\nu+1}) - 2d_2 \cdot m(\Omega), \quad (42) \\ &\quad \forall \lambda \in [1, 2], \end{aligned}$$

where d_1, d_2 are the constants in (8). Let

$$\iota_{\nu+1}(k) = \sup_{u \in Z_k, \|u\|=1} \|u\|_{\nu+1}, \quad \forall k \in \mathbb{N}. \quad (43)$$

Then

$$\iota_{\nu+1}(k) \longrightarrow 0 \quad \text{as } k \longrightarrow \infty \quad (44)$$

since E is compactly embedded into $L^{\nu+1}$. Note that

$$Z_k \subset E^+, \quad \forall k \geq \bar{n} + 1, \quad (45)$$

where $\bar{n}$ is the integer given in (16). Combining (24), (42), (43), and (45), for $k \geq \bar{n} + 1$, we have

$$\begin{aligned} \Phi_\lambda(u) &\geq \frac{1}{2} \|u\|^2 - 2d_1 \tau_1 \|u\| - 2d_2 \cdot m(\Omega) \\ &\quad - 2d_1 \iota_{\nu+1}^{\nu+1}(k) \|u\|^{\nu+1}, \quad \forall (\lambda, u) \in [1, 2] \times Z_k, \end{aligned} \quad (46)$$

where τ_1 is the constant given in (24). By (44), there exists a positive integer $k_1 \geq \bar{n} + 1$ such that

$$\begin{aligned} \rho_k &:= (16d_1 \iota_{\nu+1}^{\nu+1}(k))^{1/(1-\nu)} \\ &> \max\{16d_1 \tau_1 + 1, 16d_2 \cdot m(\Omega)\}, \quad \forall k \geq k_1 \end{aligned} \quad (47)$$

since $\nu > 1$. Clearly,

$$\rho_k \longrightarrow \infty \quad \text{as } k \longrightarrow \infty. \quad (48)$$

Combining (46) and (47), direct computation shows

$$\alpha_k(\lambda) := \inf_{u \in Z_k, \|u\| = \rho_k} \Phi_\lambda(u) \geq \frac{\rho_k^2}{4} > 0, \quad \forall k \geq k_1. \quad (49)$$

Step 2. We then prove (41).

Note that for any $k \in \mathbb{N}$, Y_k is of finite dimension, so we can choose $M_1 > 0$ sufficiently large such that

$$\|u\| \leq M_1 \left(\int_{\Omega} |u|^2 \right)^{1/2}, \quad \forall u \in Y_k. \quad (50)$$

By (S_2) and (8), for the former M_1 , there exists a $M_2 > 0$ such that

$$G(x, u) \geq M_1^2 |u|^2 - M_2, \quad \forall (t, u) \in [0, T] \times \mathbb{R}^N. \quad (51)$$

Consequently, by (50) and (51), we have

$$\begin{aligned} \Phi_\lambda(u) &\leq \frac{1}{2} \|u^+\|^2 - \frac{1}{2} \|u^-\|^2 - \int_{\Omega} G(x, u) dt \\ &\leq \frac{1}{2} \|u^+\|^2 - \frac{1}{2} \|u^-\|^2 - M_1^2 \int_{\Omega} |u|^2 dt \\ &\quad + M_2 \cdot m(\Omega) \\ &\leq \frac{1}{2} \|u^+\|^2 - \frac{1}{2} \|u^-\|^2 \\ &\quad - M_1^2 \left(\frac{1}{M_1^2} \|u^+\|^2 + \frac{1}{M_1^2} \|u^0\|^2 \right) + M_2 \cdot m(\Omega) \\ &\leq -\frac{1}{2} \|u^+\|^2 - \frac{1}{2} \|u^-\|^2 - \|u^0\|^2 + M_2 \cdot m(\Omega) \\ &\leq -\frac{1}{2} \|u\|^2 + M_2 \cdot m(\Omega) \end{aligned} \quad (52)$$

for all $u = u^- + u^0 + u^+ \in Y_k$. Now for any $k \in \mathbb{N}$, if we choose

$$r_k > \max \left\{ \rho_k, \sqrt{2M_2 \cdot m(\Omega)} \right\}, \quad (53)$$

then (52) implies

$$\beta_k(\lambda) := \max_{u \in Y_k, \|u\|=r_k} \Phi_\lambda(u) < 0, \quad \forall k \in \mathbb{N}. \quad (54)$$

The proof is completed. $\square$

Now we prove our main result Theorem 1.

Proof of Theorem 1. In view of (8), (24), and (31), Φ_λ maps bounded sets to bounded sets uniformly for $\lambda \in [1, 2]$. By virtue of the evenness of $G(x, u)$ in u , it holds that $\Phi_\lambda(-u) = \Phi_\lambda(u)$ for all $(\lambda, u) \in [1, 2] \times E$. Therefore the condition (F_1) of Theorem 4 holds. Lemma 5 shows that the condition (F_2) holds, whereas Lemma 6 implies that condition (F_3) holds for all $k \geq k_1$, where k_1 is given in Lemma 6. Thus, by Theorem 4, for each $k \geq k_1$ and a.e. $\lambda \in [1, 2]$, there exists a sequence $\{u_m^k(\lambda)\}_{m=1}^\infty \subset E$ such that

$$\begin{aligned} \sup_m \|u_m^k(\lambda)\| &< \infty, \quad \Phi'_\lambda(u_m^k(\lambda)) \longrightarrow 0, \\ \Phi_\lambda(u_m^k(\lambda)) &\longrightarrow \zeta_k(\lambda) \\ &\text{as } m \longrightarrow \infty, \end{aligned} \quad (55)$$

where

$$\zeta_k(\lambda) := \inf_{\gamma \in \Gamma_k} \max_{u \in B_k} \Phi_\lambda(\gamma(u)), \quad \forall \lambda \in [1, 2] \quad (56)$$

with $B_k = \{u \in Y_k : \|u\| \leq r_k\}$ and $\Gamma_k := \{\gamma \in C(B_k, E) : \gamma \text{ is odd}, \gamma|_{\partial B_k} = id\}$.

Moreover, by the proof of Lemma 6, we have

$$\zeta_k(\lambda) \in [\bar{\alpha}_k, \bar{\zeta}_k], \quad \forall k \geq k_1, \quad (57)$$

where $\bar{\zeta}_k := \max_{u \in B_k} \Phi_1(u)$ and $\bar{\alpha}_k := \rho_k^2/4 \rightarrow \infty$ as $k \rightarrow \infty$ by (48).

Since the sequence $\{u_m^k(\lambda)\}_{m=1}^\infty$ obtained by (55) is bounded, it is clear that, for each $k \geq k_1$, we can choose $\lambda_n \rightarrow 1$ such that the sequence $\{u_m^k(\lambda_n)\}_{m=1}^\infty$ has a strong convergent subsequence.

In fact, without loss of generality, assume that

$$\begin{aligned} u_m^k(\lambda_n)^- &\longrightarrow u_0^k(\lambda_n)^-, \quad u_m^k(\lambda_n)^0 \longrightarrow u_0^k(\lambda_n)^0, \\ u_m^k(\lambda_n)^+ &\longrightarrow u_0^k(\lambda_n)^+ \\ &\text{as } m \longrightarrow \infty, \end{aligned} \quad (58)$$

$$u_m^k(\lambda_n) \rightharpoonup u_0^k(\lambda_n) \quad \text{as } m \longrightarrow \infty \quad (59)$$

for some $u_0^k(\lambda_n) = u_0^k(\lambda_n)^- + u_0^k(\lambda_n)^0 + u_0^k(\lambda_n)^+ \in E = E^- \oplus E^0 \oplus E^+$ since $\dim(E^- \oplus E^0) < \infty$.

Note that

$$\begin{aligned} \Phi'_{\lambda_n}(u_m^k(\lambda_n)) \\ = u_m^k(\lambda_n)^+ - \lambda_n(u_m^k(\lambda_n)^- + \Psi'(u_m^k(\lambda_n))), \end{aligned} \quad (60)$$

$\forall n \in \mathbb{N}.$

That is,

$$\begin{aligned} u_m^k(\lambda_n)^+ \\ = \Phi'_{\lambda_n}(u_m^k(\lambda_n)) + \lambda_n(u_m^k(\lambda_n)^- + \Psi'(u_m^k(\lambda_n))), \end{aligned} \quad (61)$$

$\forall m \in \mathbb{N}.$

In view of (55), (58), (59), and the compactness of Ψ' , the right-hand side of (61) converges strongly in E and hence $u_m^k(\lambda_n)^+ \rightarrow u_0^k(\lambda_n)^+$ in E . Together with (58), $\{u_m^k(\lambda_n)\}_{m=1}^\infty$ has a strong convergent subsequence in E .

Without loss of generality, we assume

$$\lim_{m \rightarrow \infty} u_m^k(\lambda_n) = u_n^k, \quad \forall n \in \mathbb{N}, \quad k \geq k_1. \quad (62)$$

This together with (55) and (57) yields

$$\begin{aligned} \Phi'_{\lambda_n}(u_n^k) &= 0, \quad \Phi_{\lambda_n}(u_n^k) \in [\bar{\alpha}_k, \bar{\zeta}_k], \\ &\forall n \in \mathbb{N}, \quad k \geq k_1. \end{aligned} \quad (63)$$

Now we claim that the sequence $\{u_n^k\}_{n=1}^\infty$ in (63) is bounded in E and possesses a strong convergent subsequence with the limit $u^k \in E$ for each $k \geq k_1$. For the sake of notational simplicity, throughout the remaining proof of Theorem 1 we always denote $u_n = u_n^k$.

Now we claim that $\{u_n\}$ is bounded in E . Otherwise, going to a subsequence if necessary, we can assume that $\|u_n\| \rightarrow \infty$ as $n \rightarrow \infty$. By (9), we have

$$\begin{aligned} 2\Phi_{\lambda_n}(u_n) - \Phi'_{\lambda_n}(u_n)u_n \\ = \lambda_n \int_{\Omega} [g(x, u_n)u_n - 2G(x, u_n)] dx \\ \geq d_3 \int_{\Omega} |u_n|^q dx - d_4 \cdot m(\Omega), \end{aligned} \quad (64)$$

which yields that

$$\frac{\int_{\Omega} |u_n|^q dx}{\|u_n\|} \longrightarrow 0 \quad \text{as } n \longrightarrow \infty. \quad (65)$$

Write $u_n = u_n^- + u_n^0 + u_n^+ \in E^- \oplus E^0 \oplus E^+$. It follows from (S_1) , (24), (25), (32), and the Hölder inequality that

$$\begin{aligned}
 & \Phi'_{\lambda_n}(u_n)u_n^+ \\
 &= \|u_n^+\|^2 - \lambda_n \int_{\Omega} g(x, u_n) u_n^+ dx \\
 &\geq \|u_n^+\|^2 - 2 \int_{\Omega} |g(x, u_n)| \cdot |u_n^+| dx \\
 &\geq \|u_n^+\|^2 - d_1 \int_{\Omega} |u_n^+| dx - d_1 \int_{\Omega} |u_n|^\nu |u_n^+| dx \\
 &\geq \|u_n^+\|^2 - d_1 \|u_n^+\|_1 \\
 &\quad - d_1 \left(\int_{\Omega} (|u_n|^\nu)^{q/\nu} dx \right)^{\nu/q} \cdot \left(\int_{\Omega} |u_n^+|^{q/(q-\nu)} dx \right)^{(q-\nu)/q} \\
 &\geq \|u_n^+\|^2 - c_1 \|u_n^+\| - c_2 \|u_n\|_q^\nu \cdot \|u_n^+\|
 \end{aligned} \tag{66}$$

for any $n \in \mathbb{N}$. Here and in the sequel, we denote $c_i > 0$ ($i = 1, 2, \dots$) for different positive constants. Since $q > (2N/(N+4))\nu$ and $N \geq 5$, we have $\nu < q$. So, by (65) we get

$$\frac{\|u_n^+\|}{\|u_n\|} \rightarrow 0 \quad \text{as } n \rightarrow \infty. \tag{67}$$

Similarly, we have

$$\frac{\|u_n^-\|}{\|u_n\|} \rightarrow 0 \quad \text{as } n \rightarrow \infty. \tag{68}$$

By (S_3) , there also exist constants $d_6 > 0$ and $d_7 > 0$ such that

$$ug(x, u) - 2G(x, u) \geq d_6 |u| - d_7, \quad \forall (x, u) \in \Omega \times \mathbb{R}. \tag{69}$$

So we get

$$\begin{aligned}
 & 2\Phi_{\lambda_n}(u_n) - \Phi'_{\lambda_n}(u_n)u_n \\
 &= \lambda_n \int_{\Omega} [g(x, u_n)u_n - 2G(x, u_n)] dx \\
 &\geq d_6 \int_{\Omega} |u_n| dx - d_7 \cdot m(\Omega) \\
 &\geq d_6 \int_{\Omega} (|u_n^0| - |u_n^+| - |u_n^-|) dx - d_7 \cdot m(\Omega) \\
 &\geq c_3 \|u_n^0\| - c_4 (\|u_n^-\| + \|u_n^+\|) - c_5
 \end{aligned} \tag{70}$$

keeping in mind that $\dim E^0 < \infty$ and (24). Hence, by (67) and (68), we get

$$\frac{\|u_n^0\|}{\|u_n\|} \rightarrow 0 \quad \text{as } n \rightarrow \infty. \tag{71}$$

Then we arrive at

$$1 = \frac{\|u_n\|}{\|u_n\|} \leq \frac{\|u_n^-\| + \|u_n^0\| + \|u_n^+\|}{\|u_n\|} \rightarrow 0 \quad \text{as } n \rightarrow \infty \tag{72}$$

which is a contradiction. Thus, $\{u_n\}$ is bounded in E . Then the proof that $\{u_n\}$ has a strong convergent subsequence is the same as the preceding proof of $\{u_m^k(\lambda_n)\}_{m=1}^\infty$.

Now for each $k \geq k_1$, by (63), the limit u^k is just a critical point of $\Phi = \Phi_1$ with $\Phi(u^k) \in [\bar{\alpha}_k, \bar{\zeta}_k]$. Since $\bar{\alpha}_k \rightarrow \infty$ as $k \rightarrow \infty$ in (57), we get infinitely many nontrivial critical points of Φ . Therefore, system (1) possesses infinitely many nontrivial solutions. $\square$

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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