Research Article

Third-Order Differential Subordination and Superordination Results for Meromorphically Multivalent Functions Associated with the Liu-Srivastava Operator

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There are many articles in the literature dealing with the first-order and the second-order differential subordination and superordination problems for analytic functions in the unit disk, but only a few articles are dealing with the above problems in the third-order case (see, e.g., Antonino and Miller (2011) and Ponnusamy et al. (1992)). The concept of the third-order differential subordination in the unit disk was introduced by Antonino and Miller in (2011). Let Ω be a set in the complex plane $\mathbb{C}$. Also let $\mathfrak{p}$ be analytic in the unit disk $\mathbb{U} = \{z : z \in \mathbb{C} \text{ and } |z| < 1\}$ and suppose that $\psi : \mathbb{C}^4 \times \mathbb{U} \rightarrow \mathbb{C}$. In this paper, we investigate the problem of determining properties of functions $\mathfrak{p}(z)$ that satisfy the following third-order differential superordination: $\Omega \subset \{\psi(\mathfrak{p}(z), z\mathfrak{p}'(z), z^2\mathfrak{p}''(z), z^3\mathfrak{p}'''(z); z) : z \in \mathbb{U}\}$. As applications, we derive some third-order differential subordination and superordination results for meromorphically multivalent functions, which are defined by a family of convolution operators involving the Liu-Srivastava operator. The results are obtained by considering suitable classes of admissible functions.

1. Introduction, Definitions, and Preliminaries

Let $\mathcal{H}(\mathbb{U})$ be the class of functions which are analytic in the open unit disk:

$$\mathbb{U} = \{z : z \in \mathbb{C}, |z| < 1\}. \quad (1)$$

For $n \in \mathbb{N} := \{1, 2, 3, \dots\}$ and $a \in \mathbb{C}$, let

$$\mathcal{H}[a, n] = \left\{ f : f \in \mathcal{H}(\mathbb{U}), \right. \\ \left. f(z) = a + a_n z^n + a_{n+1} z^{n+1} + \dots \right\} \quad (2)$$

and suppose that $\mathcal{H} = \mathcal{H}[1, 1]$.

Let f and F be members of the analytic function class $\mathcal{H}(\mathbb{U})$. The function f is said to be subordinate to F , or F is superordinate to f , if there exists a Schwarz function $\mathfrak{w}(z)$, analytic in $\mathbb{U}$ with

$$\mathfrak{w}(0) = 0, \quad |\mathfrak{w}(z)| < 1 \quad (z \in \mathbb{U}), \quad (3)$$

such that

$$f(z) = F(\mathfrak{w}(z)). \quad (4)$$

In such a case, we write

$$f \prec F \quad \text{or} \quad f(z) \prec F(z). \quad (5)$$

Furthermore, if the function F is univalent in $\mathbb{U}$, then we have the following equivalence (see, for details, [1]):

$$f(z) \prec F(z) \quad (z \in \mathbb{U}) \iff f(0) = F(0), \\ f(\mathbb{U}) \subset F(\mathbb{U}). \quad (6)$$

Let Σ_p denote the class of functions of the form

$$f(z) = \frac{1}{z^p} + \sum_{k=1-p}^{\infty} a_k z^k \quad (p \in \mathbb{N}) \quad (7)$$

which are analytic and multivalent in the punctured unit disk:

$$\mathbb{U}^* = \{z \in \mathbb{C} : 0 < |z| < 1\} = \mathbb{U} \setminus \{0\}. \quad (8)$$

For the function f given by (7) and the function g given by

$$g(z) = \frac{1}{z^p} + \sum_{k=1-p}^{\infty} b_k z^k \quad (p \in \mathbb{N}; z \in \mathbb{U}^*), \quad (9)$$

the Hadamard product (or convolution) $f * g$ of the functions f and g is defined by

$$(f * g)(z) := \frac{1}{z^p} + \sum_{k=1-p}^{\infty} a_k b_k z^k =: (g * f)(z). \quad (10)$$

For parameters $\alpha_i \in \mathbb{C}$ ($i = 1, 2, \dots, q$) and $\beta_j \in \mathbb{C} \setminus \mathbb{Z}_0^-$ ($\mathbb{Z}_0^- = 0, -1, -2, \dots$; $j = 1, 2, \dots, s$), the generalized hypergeometric function ${}_qF_s(\alpha_1, \dots, \alpha_q; \beta_1, \dots, \beta_s; z)$ is defined by (see, for example, [2, 3])

$${}_qF_s(\alpha_1, \dots, \alpha_q; \beta_1, \dots, \beta_s; z) = \sum_{k=0}^{\infty} \frac{(\alpha_1)_k \cdots (\alpha_q)_k}{(\beta_1)_k \cdots (\beta_s)_k} \frac{z^k}{k!} \quad (11)$$

$$(q \leq s + 1; q, s \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}; z \in \mathbb{U}),$$

where $(\nu)_k$ denotes the Pochhammer symbol defined, in terms of Gamma function, by

$$\begin{aligned} (\nu)_k &= \frac{\Gamma(\nu + k)}{\Gamma(\nu)} \\ &= \begin{cases} 1 & (k = 0; \nu \in \mathbb{C} \setminus \{0\}), \\ \nu(\nu + 1) \cdots (\nu + k - 1) & (k \in \mathbb{N}; \nu \in \mathbb{C}). \end{cases} \end{aligned} \quad (12)$$

Recently, Tang et al. [4] introduced a function $h_p^{\lambda, \mu}(\alpha_1, \dots, \alpha_q; \beta_1, \dots, \beta_s; z)$ defined by

$$\begin{aligned} h_p^{\lambda, \mu}(\alpha_1, \dots, \alpha_q; \beta_1, \dots, \beta_s; z) &= (1 - \lambda + \mu) z_q^{-p} F_s(\alpha_1, \dots, \alpha_q; \beta_1, \dots, \beta_s; z) \\ &\quad + (\lambda - \mu) z \left[z_q^{-p} F_s(\alpha_1, \dots, \alpha_q; \beta_1, \dots, \beta_s; z) \right]' \\ &\quad + \lambda \mu z^2 \left[z_q^{-p} F_s(\alpha_1, \dots, \alpha_q; \beta_1, \dots, \beta_s; z) \right]'' \\ &\quad (p \in \mathbb{N}; \lambda, \mu \geq 0; z \in \mathbb{U}). \end{aligned} \quad (13)$$

In particular, when $\lambda = \mu = 0$, we obtain

$$h_p^{0,0}(\alpha_1, \dots, \alpha_q; \beta_1, \dots, \beta_s; z) = h_p(\alpha_1, \dots, \alpha_q; \beta_1, \dots, \beta_s; z), \quad (14)$$

which was introduced and studied by Liu and Srivastava [5].

Corresponding to the function $h_p^{\lambda, \mu}(\alpha_1, \dots, \alpha_q; \beta_1, \dots, \beta_s; z)$ given by (13), we consider a convolution operator

$$H_p^{\lambda, \mu}(\alpha_1, \dots, \alpha_q; \beta_1, \dots, \beta_s) : \Sigma_p \longrightarrow \Sigma_p \quad (15)$$

defined by the following Hadamard product (or convolution):

$$\begin{aligned} H_p^{\lambda, \mu}(\alpha_1, \dots, \alpha_q; \beta_1, \dots, \beta_s) f(z) \\ = h_p^{\lambda, \mu}(\alpha_1, \dots, \alpha_q; \beta_1, \dots, \beta_s; z) * f(z). \end{aligned} \quad (16)$$

For the sake of convenience, we write

$$H_{p,q,s}^{\lambda, \mu}(\beta_1) = H_{p,q,s}^{\lambda, \mu}(\alpha_1) = H_p^{\lambda, \mu}(\alpha_1, \dots, \alpha_q; \beta_1, \dots, \beta_s). \quad (17)$$

It is easily verified from definition (16) that

$$\begin{aligned} z(H_{p,q,s}^{\lambda, \mu}(\beta_1 + 1) f(z))' &= \beta_1 H_{p,q,s}^{\lambda, \mu}(\beta_1) f(z) \\ &\quad - (\beta_1 + p) H_{p,q,s}^{\lambda, \mu}(\beta_1 + 1) f(z), \end{aligned} \quad (18)$$

$$\begin{aligned} z(H_{p,q,s}^{\lambda, \mu}(\alpha_1) f(z))' &= \alpha_1 H_{p,q,s}^{\lambda, \mu}(\alpha_1 + 1) f(z) \\ &\quad - (\alpha_1 + p) H_{p,q,s}^{\lambda, \mu}(\alpha_1) f(z). \end{aligned} \quad (19)$$

We note that, for $\lambda = \mu = 0$, the operator $H_{p,q,s}^{0,0}(\alpha_1)$ reduces to the Liu-Srivastava operator $H_{p,q,s}(\alpha_1)$ (see [5, 6]; see also [7]), while the Liu-Srivastava operator is the meromorphic analogous of the Dziok-Srivastava operator (see [8–10]; see also [11, 12]), which includes (as its special cases) the meromorphic analogous of the Carlson-Shaffer convolution operator $L_p(a, c) = H_{p,2,1}^{0,0}(1, a; c)$ (see [13, 14]), the meromorphic analogous of the Ruscheweyh derivative operator $D^{n+1} = L_p(n + p, 1)$ (see [15]), and the operator

$$J_{\delta,p} = \frac{\delta}{z^{\delta+p}} \int_0^z t^{\delta+p-1} f(t) dt = L_p(\delta, \delta + 1) \quad (\delta > 0) \quad (20)$$

studied by Uralegaddi and Somanatha [16].

Let Ω be any set in $\mathbb{C}$. Also let $\mathfrak{p}$ be analytic in $\mathbb{U}$ and suppose that $\psi : \mathbb{C}^4 \times \mathbb{U} \rightarrow \mathbb{C}$. Recently, Antonino and Miller [17] have extended the theory of second-order differential subordinations in $\mathbb{U}$ introduced by Miller and Mocanu [1] to the third-order case. They determined properties of functions $\mathfrak{p}(z)$ that satisfy the following third-order differential subordination:

$$\{\psi(\mathfrak{p}(z), z\mathfrak{p}'(z), z^2\mathfrak{p}''(z), z^3\mathfrak{p}'''(z); z) : z \in \mathbb{U}\} \subset \Omega. \quad (21)$$

We will now recall some definitions and a theorem due to Antonino and Miller [17], which are required in our next investigations.

Definition 1 (see [17], p. 440, Definition 1). Let $\psi : \mathbb{C}^4 \times \mathbb{U} \rightarrow \mathbb{C}$ and $h(z)$ be univalent in $\mathbb{U}$. If $\mathfrak{p}(z)$ is analytic in $\mathbb{U}$ and satisfies the following third-order differential subordination:

$$\psi(\mathfrak{p}(z), z\mathfrak{p}'(z), z^2\mathfrak{p}''(z), z^3\mathfrak{p}'''(z); z) < h(z), \quad (22)$$

then $p(z)$ is called a solution of the differential subordination. A univalent function $q(z)$ is called a dominant of the solutions of the differential subordination or, more simply, a dominant if $p(z) < q(z)$ for all $p(z)$ satisfying (22). A dominant $\tilde{q}(z)$ that satisfies $\tilde{q}(z) < q(z)$ for all dominants $q(z)$ of (22) is said to be the best dominant.

Definition 2 (see [17], p. 441, Definition 2). Let $\mathcal{Q}$ denote the set of functions q that are analytic and univalent on the set $\bar{\mathbb{U}} \setminus E(q)$, where

$$E(q) = \left\{ \xi : \xi \in \partial\mathbb{U} \text{ and } \lim_{z \rightarrow \xi} q(z) = \infty \right\}, \quad (23)$$

is such that

$$\min |q'(\xi)| = \rho > 0 \quad (24)$$

for $\xi \in \partial\mathbb{U} \setminus E(q)$. Further, let the subclass of $\mathcal{Q}$ for which $q(0) = a$ be denoted by $\mathcal{Q}(a)$ and

$$\mathcal{Q}(1) = \mathcal{Q}_1. \quad (25)$$

Definition 3 (see [17], p. 449, Definition 3). Let Ω be a set in $\mathbb{C}$, $q \in \mathcal{Q}$, and $n \in \mathbb{N} \setminus \{1\}$. The class of admissible functions $\Psi_n[\Omega, q]$ consists of those functions $\psi : \mathbb{C}^4 \times \mathbb{U} \rightarrow \mathbb{C}$ that satisfy the following admissibility condition:

$$\psi(r, s, t, u; z) \notin \Omega \quad (26)$$

whenever

$$\begin{aligned} r &= q(\xi), \quad s = k\xi q'(\xi), \\ \Re\left(\frac{t}{s} + 1\right) &\geq k\Re\left(\frac{\xi q''(\xi)}{q'(\xi)} + 1\right), \\ \Re\left(\frac{u}{s}\right) &\geq k^2\Re\left(\frac{\xi^2 q'''(\xi)}{q'(\xi)}\right), \end{aligned} \quad (27)$$

where $z \in \mathbb{U}$, $\xi \in \partial\mathbb{U} \setminus E(q)$, and $k \geq n$.

Theorem 4 (see [17], p. 449, Theorem 1). Let $p \in \mathcal{H}[a, n]$ with $n \in \mathbb{N} \setminus \{1\}$. Also let $q \in \mathcal{Q}(a)$ and satisfy the following conditions:

$$\Re\left(\frac{\xi q''(\xi)}{q'(\xi)}\right) \geq 0, \quad \left|\frac{z p'(z)}{q'(\xi)}\right| \leq k, \quad (28)$$

where $z \in \mathbb{U}$, $\xi \in \partial\mathbb{U} \setminus E(q)$, and $k \geq n$. If Ω is a set in $\mathbb{C}$, $\psi \in \Psi_n[\Omega, q]$ and

$$\psi(p(z), zp'(z), z^2 p''(z), z^3 p'''(z); z) \in \Omega, \quad (29)$$

then

$$p(z) < q(z). \quad (30)$$

In this paper, following the theory of second-order differential subordinations in the unit disk introduced by Miller and Mocanu [18], we consider the dual problem of

determining properties of functions $p(z)$ that satisfy the following third-order differential superordination:

$$\Omega \subset \left\{ \psi(p(z), zp'(z), z^2 p''(z), z^3 p'''(z); z) : z \in \mathbb{U} \right\}. \quad (31)$$

In other words, we determine the conditions on Ω , Δ , and ψ for which the following implication holds true:

$$\begin{aligned} \Omega \subset \left\{ \psi(p(z), zp'(z), z^2 p''(z), z^3 p'''(z); z) : z \in \mathbb{U} \right\} \\ \implies \Delta \subset p(\mathbb{U}), \end{aligned} \quad (32)$$

where Δ is any set in $\mathbb{C}$.

If either Ω or Δ is a simply connected domain, then (32) can be rephrased in terms of superordination. If $p(z)$ is univalent in $\mathbb{U}$, and if Δ is a simply connected domain with $\Delta \neq \mathbb{C}$, then there is a conformal mapping q of $\mathbb{U}$ onto Δ such that $q(0) = p(0)$. In this case, (32) can be rewritten as follows:

$$\begin{aligned} \Omega \subset \left\{ \psi(p(z), zp'(z), z^2 p''(z), z^3 p'''(z); z) : z \in \mathbb{U} \right\} \\ \implies q(z) < p(z). \end{aligned} \quad (33)$$

If Ω is also a simply connected domain with $\Omega \neq \mathbb{C}$, then there is a conformal mapping h of $\mathbb{U}$ onto Ω such that $h(0) = \psi(p(0), 0, 0, 0; 0)$. In addition, if the function

$$\psi(p(z), zp'(z), z^2 p''(z), z^3 p'''(z); z) \quad (34)$$

is univalent in $\mathbb{U}$, then (33) can be rewritten as

$$\begin{aligned} h(z) < \psi(p(z), zp'(z), z^2 p''(z), z^3 p'''(z); z) \\ \implies q(z) < p(z). \end{aligned} \quad (35)$$

There are three key ingredients in the implication relationship (33): the differential operator ψ , the set Ω , and the “dominating” function q . If two of these entities were given, one would hope to find conditions on the third entity so that (33) would be satisfied. In this paper, we start with a given set Ω and a given function q , and we then determine a set of “admissible” operators ψ so that (33) holds true.

We first introduce the following definition.

Definition 5. Let $\psi : \mathbb{C}^4 \times \mathbb{U} \rightarrow \mathbb{C}$ and the function $h(z)$ be analytic in $\mathbb{U}$. If the functions $p(z)$ and

$$\psi(p(z), zp'(z), z^2 p''(z), z^3 p'''(z); z) \quad (36)$$

are univalent in $\mathbb{U}$ and satisfy the following third-order differential superordination:

$$h(z) < \psi(p(z), zp'(z), z^2 p''(z), z^3 p'''(z); z), \quad (37)$$

then $p(z)$ is called a solution of the differential superordination. An analytic function $q(z)$ is called a subordinated of

the solutions of the differential superordination or more simply a subordinant if $q(z) < p(z)$ for $p(z)$ satisfying (37). A univalent subordinant $\tilde{q}(z)$ that satisfies the condition

$$q(z) < \tilde{q}(z) \quad (38)$$

for all subordinants $q(z)$ of (37) is said to be the best subordinant. We note that the best subordinant is unique up to a rotation of $\mathbb{U}$.

For Ω a set in $\mathbb{C}$, with ψ and p as given in Definition 5, we suppose that (37) is replaced by

$$\Omega \subset \left\{ \psi(p(z), z p'(z), z^2 p''(z), z^3 p'''(z); z) : z \in \mathbb{U} \right\}. \quad (39)$$

Although this more general situation is a “differential containment,” yet we also refer to it as a differential superordination, and the definitions of solution, subordinant, and best subordinant as given above can be extended to this more general case.

We will use the following lemma [[17], p. 445, Lemma D] from the theory of third-order differential subordinations in $\mathbb{U}$ to determine subordinants of the third-order differential superordinations.

Lemma 6 (see [17]). *Let $p \in \mathcal{Q}(a)$, and let $q(z) = a + a_n z^n + \dots$ be analytic in $\mathbb{U}$ with $q(z) \neq a$ and $n \in \mathbb{N} \setminus \{1\}$. If q is not subordinate to p , then there exists points $z_0 = r_0 e^{i\theta_0} \in \mathbb{U}$ and $\xi_0 \in \partial\mathbb{U} \setminus E(p)$, and an $m \geq n$ for which $q(\mathbb{U}_{r_0}) \subset p(\mathbb{U})$,*

- (i) $q(z_0) = p(\xi_0)$,
- (ii) $\Re(\xi_0 p''(\xi_0)/p'(\xi_0)) \geq 0$ and $|z q'(z)/p'(\xi_0)| \leq m$,
- (iii) $z_0 q'(z_0) = m \xi_0 p'(\xi_0)$,
- (iv) $\Re(1 + z_0 q''(z_0)/q'(z_0)) \geq m \Re(1 + \xi_0 p''(\xi_0)/p'(\xi_0))$,
- (v) $\Re(z_0^2 q'''(z_0)/q'(z_0)) \geq m^2 \Re(\xi_0^2 p'''(\xi_0)/p'(\xi_0))$.

2. Admissible Functions and a Fundamental Result

We next define the class of admissible functions referred to in the preceding section.

Definition 7. Let Ω be a set in $\mathbb{C}$, $q \in \mathcal{H}[a, n]$ and $q'(z) \neq 0$. The class of admissible functions $\Psi'_n[\Omega, q]$ consists of those functions $\psi : \mathbb{C}^4 \times \overline{\mathbb{U}} \rightarrow \mathbb{C}$ that satisfy the following admissibility condition:

$$\psi(r, s, t, u; \xi) \in \Omega \quad (40)$$

whenever

$$r = q(z), \quad s = \frac{z q'(z)}{m}, \quad (41)$$

$$\Re\left(\frac{t}{s} + 1\right) \leq \frac{1}{m} \Re\left(\frac{z q''(z)}{q'(z)} + 1\right),$$

$$\Re\left(\frac{u}{s}\right) \leq \frac{1}{m^2} \Re\left(\frac{z^2 q'''(z)}{q'(z)}\right), \quad (42)$$

where $z \in \mathbb{U}$, $\xi \in \partial\mathbb{U}$, and $m \geq n \geq 2$.

If $\psi : \mathbb{C}^4 \times \overline{\mathbb{U}} \rightarrow \mathbb{C}$ and $q \in \mathcal{H}[a, n]$, then the admissibility condition (41) reduces to the following form:

$$\psi\left(q(z), \frac{z q'(z)}{m}; \xi\right) \in \Omega \quad (z \in \mathbb{U}; \xi \in \partial\mathbb{U}; m \geq n \geq 2). \quad (43)$$

If $\psi : \mathbb{C}^3 \times \overline{\mathbb{U}} \rightarrow \mathbb{C}$ and $q \in \mathcal{H}[a, n]$ with $q'(z) \neq 0$, then the admissibility condition (41) reduces to the following form:

$$\psi(r, s, t; \xi) \in \Omega \quad (44)$$

whenever $r = q(z)$, $s = z q'(z)/m$, and

$$\Re\left(\frac{t}{s} + 1\right) \leq \frac{1}{m} \Re\left(\frac{z q''(z)}{q'(z)} + 1\right) \quad (45)$$

$$(z \in \mathbb{U}; \xi \in \partial\mathbb{U}; m \geq n \geq 2).$$

The next theorem is a foundation result in the theory of the third-order differential superordinations in $\mathbb{U}$.

Theorem 8. *Let $q \in \mathcal{H}[a, n]$ and $\psi \in \Psi'_n[\Omega, q]$. If*

$$\psi(p(z), z p'(z), z^2 p''(z), z^3 p'''(z); z) \quad (46)$$

is univalent in $\mathbb{U}$ and $p \in \mathcal{Q}(a)$ satisfy the following conditions:

$$\Re\left(\frac{z q''(z)}{q'(z)}\right) \geq 0, \quad \left|\frac{z p'(z)}{q'(z)}\right| \leq m \quad (47)$$

$$(z \in \mathbb{U}; \xi \in \partial\mathbb{U}; m \geq n \geq 2),$$

then

$$\Omega \subset \left\{ \psi(p(z), z p'(z), z^2 p''(z), z^3 p'''(z); z) : z \in \mathbb{U} \right\} \quad (48)$$

implies that

$$q(z) < p(z). \quad (49)$$

Proof. Suppose that

$$q(z) \not< p(z). \quad (50)$$

Then, by the above lemma, there exists points $z_0 = r_0 e^{i\theta_0} \in \mathbb{U}$ and $\xi_0 \in \partial\mathbb{U} \setminus E(p)$, and an $m \geq n \geq 2$ that satisfy conditions (i)–(v) of the above lemma. Using these conditions with $r = p(\xi_0)$, $s = \xi_0 p'(\xi_0)$, $t = \xi_0^2 p''(\xi_0)$, $u = \xi_0^3 p'''(\xi_0)$, and $\xi = \xi_0$ in Definition 7, we obtain

$$\psi(p(\xi_0), \xi_0 p'(\xi_0), \xi_0^2 p''(\xi_0), \xi_0^3 p'''(\xi_0); \xi_0) \in \Omega, \quad (51)$$

which contradicts (48), so we have

$$q(z) < p(z). \quad (52)$$

□

In the special case when $\Omega \neq \mathbb{C}$ is a simply connected domain and h is a conformal mapping of $\mathbb{U}$ onto Ω , we denote this class $\Psi'_n[h(\mathbb{U}), \mathbf{q}]$ by $\Psi'_n[h, \mathbf{q}]$. The following result is an immediate consequence of Theorem 8.

Theorem 9. Let $\mathbf{q} \in \mathcal{H}[a, n]$. Also let the function h be analytic in $\mathbb{U}$ and suppose that $\psi \in \Psi'_n[h, \mathbf{q}]$. If $\mathbf{p} \in \mathcal{Q}(a)$ satisfies condition (47) and

$$\psi(\mathbf{p}(z), z\mathbf{p}'(z), z^2\mathbf{p}''(z), z^3\mathbf{p}'''(z); z) \quad (53)$$

is univalent in $\mathbb{U}$, then

$$h(z) < \psi(\mathbf{p}(z), z\mathbf{p}'(z), z^2\mathbf{p}''(z), z^3\mathbf{p}'''(z); z) \quad (54)$$

implies that

$$\mathbf{q}(z) < \mathbf{p}(z). \quad (55)$$

Theorems 8 and 9 can only be used to obtain subordinants of the third-order differential superordination of the forms (48) or (54).

Theorem 10. Let the function h be analytic in $\mathbb{U}$ and let $\psi : \mathbb{C}^4 \times \overline{\mathbb{U}} \rightarrow \mathbb{C}$. Suppose that the differential equation

$$\psi(\mathbf{q}(z), z\mathbf{q}'(z), z^2\mathbf{q}''(z), z^3\mathbf{q}'''(z); z) = h(z) \quad (56)$$

has a solution $\mathbf{q} \in \mathcal{Q}(a)$. If $\psi \in \Psi'_n[h, \mathbf{q}]$, $\mathbf{p} \in \mathcal{Q}(a)$, and

$$\psi(\mathbf{p}(z), z\mathbf{p}'(z), z^2\mathbf{p}''(z), z^3\mathbf{p}'''(z); z) \quad (57)$$

is univalent in $\mathbb{U}$, then (54) implies that

$$\mathbf{q}(z) < \mathbf{p}(z) \quad (58)$$

and $\mathbf{q}(z)$ is the best subordinant.

Proof. Since $\psi \in \Psi'_n[h, \mathbf{q}]$, by applying Theorem 9, we deduce that $\mathbf{q}$ is a subordinant of (54). Since $\mathbf{q}$ satisfies (56), it is also a solution of the differential superordination (54). Therefore, all subordinants of (54) will be subordinate to $\mathbf{q}$. It follows that $\mathbf{q}(z)$ will be the best subordinant of (54). $\square$

In the next two sections, by making use of the third-order differential subordination results of Antonino and Miller [17] in the unit disk $\mathbb{U}$ and the third-order differential superordination results in $\mathbb{U}$ obtained in Section 2 (see, for details, Theorems 8, 9, and 10), we determine certain appropriate classes of admissible functions and investigate some third-order differential subordination and differential superordination properties of meromorphically multivalent functions associated with the operator $H_{p,q,s}^{\lambda,\mu}(\beta_1)$ defined by (16). It should be remarked in passing that, in recent years, several authors obtained many interesting results involving various linear and nonlinear convolution operators associated with (second-order) differential subordination and superordination, and the interested reader may refer to several earlier works including (for example) [19] to [20–23].

3. Third-Order Differential Subordination of the Operator $H_{p,q,s}^{\lambda,\mu}(\beta_1)$

We first define the following class of admissible functions, which are required in proving the differential subordination theorem involving the operator $H_{p,q,s}^{\lambda,\mu}(\beta_1)$ defined by (16).

Definition 11. Let Ω be a set in $\mathbb{C}$ and $\mathbf{q} \in \mathcal{Q}_1 \cap \mathcal{H}$. The class of admissible functions $\Phi_H[\Omega, \mathbf{q}]$ consists of those functions $\phi : \mathbb{C}^4 \times \mathbb{U} \rightarrow \mathbb{C}$ that satisfy the following admissibility condition:

$$\phi(a, b, c, d; z) \notin \Omega \quad (59)$$

whenever

$$\begin{aligned} a &= \mathbf{q}(\xi), \quad b = \frac{k\xi\mathbf{q}'(\xi) + \beta_1\mathbf{q}(\xi)}{\beta_1}, \\ \Re\left(\frac{(\beta_1 + 1)(c - a)}{b - a} - (2\beta_1 + 1)\right) &\geq k\Re\left(\frac{\xi\mathbf{q}''(\xi)}{\mathbf{q}'(\xi)} + 1\right), \\ \Re\left(\frac{(\beta_1 + 1)(\beta_1 + 2)(d - 3c + 3b - a)}{b - a}\right) &\geq k^2\Re\left\{\frac{\xi^2\mathbf{q}'''(\xi)}{\mathbf{q}'(\xi)}\right\}, \end{aligned} \quad (60)$$

where $z \in \mathbb{U}$, $\beta_1 \in \mathbb{C} \setminus \{0, -1, -2, \dots\}$, $\xi \in \partial\mathbb{U} \setminus E(\mathbf{q})$, and $k \in \mathbb{N} \setminus \{1\}$.

Theorem 12. Let $\phi \in \Phi_H[\Omega, \mathbf{q}]$. If the functions $f \in \Sigma_p$ and $\mathbf{q} \in \mathcal{Q}_1$ satisfy the following conditions:

$$\Re\left(\frac{\xi\mathbf{q}''(\xi)}{\mathbf{q}'(\xi)}\right) \geq 0, \quad \left|\frac{z^p H_{p,q,s}^{\lambda,\mu}(\beta_1) f(z)}{\mathbf{q}'(\xi)}\right| \leq k, \quad (61)$$

$$\begin{aligned} &\{\phi(z^p H_{p,q,s}^{\lambda,\mu}(\beta_1 + 1) f(z), z^p H_{p,q,s}^{\lambda,\mu}(\beta_1) f(z), \\ &z^p H_{p,q,s}^{\lambda,\mu}(\beta_1 - 1) f(z), \\ &z^p H_{p,q,s}^{\lambda,\mu}(\beta_1 - 2) f(z); z) : z \in \mathbb{U}\} \subset \Omega, \end{aligned} \quad (62)$$

then

$$z^p H_{p,q,s}^{\lambda,\mu}(\beta_1 + 1) f(z) < \mathbf{q}(z). \quad (63)$$

Proof. Define the analytic function $\mathbf{p}(z)$ in $\mathbb{U}$ by

$$\mathbf{p}(z) = z^p H_{p,q,s}^{\lambda,\mu}(\beta_1 + 1) f(z). \quad (64)$$

Then, differentiating (64) with respect to z and using (18), we have

$$z^p H_{p,q,s}^{\lambda,\mu}(\beta_1) f(z) = \frac{z\mathbf{p}'(z) + \beta_1\mathbf{p}(z)}{\beta_1}. \quad (65)$$

Further computations show that

$$\begin{aligned} & z^p H_{p,q,s}^{\lambda,\mu} (\beta_1 - 1) f(z) \\ &= \frac{z^2 \mathbf{p}''(z) + 2(\beta_1 + 1) z \mathbf{p}'(z) + \beta_1 (\beta_1 + 1) \mathbf{p}(z)}{\beta_1 (\beta_1 + 1)}, \end{aligned} \quad (66)$$

$$\begin{aligned} & z^p H_{p,q,s}^{\lambda,\mu} (\beta_1 - 2) f(z) \\ &= (z^3 \mathbf{p}'''(z) + 3(\beta_1 + 2) z^2 \mathbf{p}''(z) \\ &\quad + 3(\beta_1 + 1)(\beta_1 + 2) z \mathbf{p}'(z) \\ &\quad + \beta_1 (\beta_1 + 1)(\beta_1 + 2) \mathbf{p}(z)) \\ &\quad \times (\beta_1 (\beta_1 + 1)(\beta_1 + 2))^{-1}. \end{aligned} \quad (67)$$

We now define the transformation from $\mathbb{C}^4$ to $\mathbb{C}$ by

$$a(r, s, t, u) = r, \quad b(r, s, t, u) = \frac{s + \beta_1 r}{\beta_1}, \quad (68)$$

$$c(r, s, t, u) = \frac{t + 2(\beta_1 + 1)s + \beta_1 (\beta_1 + 1)r}{\beta_1 (\beta_1 + 1)},$$

$$\begin{aligned} & d(r, s, t, u) \\ &= (u + 3(\beta_1 + 2)t + 3(\beta_1 + 1)(\beta_1 + 2)s \\ &\quad + \beta_1 (\beta_1 + 1)(\beta_1 + 2)r) \\ &\quad \times (\beta_1 (\beta_1 + 1)(\beta_1 + 2))^{-1}. \end{aligned} \quad (69)$$

Let

$$\begin{aligned} & \psi(r, s, t, u; z) \\ &= \phi(a, b, c, d; z) \\ &= \phi\left(r, \frac{s + \beta_1 r}{\beta_1}, \frac{t + 2(\beta_1 + 1)s + \beta_1 (\beta_1 + 1)r}{\beta_1 (\beta_1 + 1)}, \right. \\ &\quad \left. \frac{u + 3(\beta_1 + 2)t + 3(\beta_1 + 1)(\beta_1 + 2)s + \beta_1 (\beta_1 + 1)(\beta_1 + 2)r}{\beta_1 (\beta_1 + 1)(\beta_1 + 2)}; z\right). \end{aligned} \quad (70)$$

The proof will make use of Theorem 4. Using (64) to (67), we find from (70) that

$$\begin{aligned} & \psi(\mathbf{p}(z), z \mathbf{p}'(z), z^2 \mathbf{p}''(z), z^3 \mathbf{p}'''(z); z) \\ &= \phi(z^p H_{p,q,s}^{\lambda,\mu} (\beta_1 + 1) f(z), \\ &\quad z^p H_{p,q,s}^{\lambda,\mu} (\beta_1) f(z), z^p H_{p,q,s}^{\lambda,\mu} (\beta_1 - 1) f(z), \\ &\quad z^p H_{p,q,s}^{\lambda,\mu} (\beta_1 - 2) f(z); z). \end{aligned} \quad (71)$$

Hence, clearly, (62) becomes

$$\psi(\mathbf{p}(z), z \mathbf{p}'(z), z^2 \mathbf{p}''(z), z^3 \mathbf{p}'''(z); z) \in \Omega. \quad (72)$$

We note that

$$\begin{aligned} \frac{t}{s} + 1 &= \frac{(\beta_1 + 1)(c - a)}{b - a} - (2\beta_1 + 1), \\ \frac{u}{s} &= \frac{(\beta_1 + 1)(\beta_1 + 2)(d - 3c + 3b - a)}{b - a}. \end{aligned} \quad (73)$$

Thus, the admissibility condition for $\phi \in \Phi_H[\Omega, \mathbf{q}]$ in Definition 11 is equivalent to the admissibility condition for $\psi \in \Psi_2[\Omega, \mathbf{q}]$ as given in Definition 3 with $n = 2$. Therefore, by using (61) and Theorem 4, we have

$$\mathbf{p}(z) \prec \mathbf{q}(z) \quad (74)$$

or, equivalently,

$$z^p H_{p,q,s}^{\lambda,\mu} (\beta_1 + 1) f(z) \prec \mathbf{q}(z), \quad (75)$$

which evidently completes the proof of Theorem 12. $\square$

Our next result is an extension of Theorem 12 to the case where the behavior of $\mathbf{q}(z)$ on $\partial\mathbb{U}$ is not known.

Corollary 13. Let $\Omega \subset \mathbb{C}$ and let the function $\mathbf{q}$ be univalent in $\mathbb{U}$ with $\mathbf{q}(0) = 1$. Suppose also that $\phi \in \Phi_H[\Omega, \mathbf{q}_\rho]$ for some $\rho \in (0, 1)$, where $\mathbf{q}_\rho(z) = \mathbf{q}(\rho z)$. If the functions $f \in \Sigma_p$ and $\mathbf{q}_\rho$ satisfy the following conditions:

$$\Re\left(\frac{\xi \mathbf{q}_\rho''(\xi)}{\mathbf{q}_\rho'(\xi)}\right) \geq 0,$$

$$\left| \frac{z^p H_{p,q,s}^{\lambda,\mu} (\beta_1) f(z)}{\mathbf{q}_\rho'(\xi)} \right| \leq k \quad (z \in \mathbb{U}; \xi \in \partial\mathbb{U} \setminus E(\mathbf{q}_\rho)), \quad (76)$$

$$\phi(z^p H_{p,q,s}^{\lambda,\mu} (\beta_1 + 1) f(z), z^p H_{p,q,s}^{\lambda,\mu} (\beta_1) f(z),$$

$$z^p H_{p,q,s}^{\lambda,\mu} (\beta_1 - 1) f(z),$$

$$z^p H_{p,q,s}^{\lambda,\mu} (\beta_1 - 2) f(z); z) \in \Omega,$$

then

$$z^p H_{p,q,s}^{\lambda,\mu} (\beta_1 + 1) f(z) \prec \mathbf{q}(z). \quad (77)$$

Proof. We note from Theorem 12 that

$$z^p H_{p,q,s}^{\lambda,\mu} (\beta_1 + 1) f(z) \prec \mathbf{q}_\rho(z). \quad (78)$$

The result asserted by Corollary 13 is now deduced from the following subordination property:

$$\mathbf{q}_\rho(z) \prec \mathbf{q}(z). \quad (79)$$

$\square$

If $\Omega \neq \mathbb{C}$ is a simply connected domain, then $\Omega = h(\mathbb{U})$ for some conformal mapping $h(z)$ of $\mathbb{U}$ onto Ω . In this case, the class $\Phi_H[h(\mathbb{U}), \mathbf{q}]$ is written as $\Phi_H[h, \mathbf{q}]$. The following two results are immediate consequences of Theorem 12 and Corollary 13.

Theorem 14. Let $\phi \in \Phi_H[h, \mathbf{q}]$. If the functions $f \in \Sigma_p$ and $\mathbf{q} \in \mathcal{Q}_1$ satisfy the following conditions:

$$\Re \left(\frac{\xi q''(\xi)}{q'(\xi)} \right) \geq 0, \quad \left| \frac{z^p H_{p,q,s}^{\lambda,\mu}(\beta_1) f(z)}{q'(\xi)} \right| \leq k, \quad (80)$$

$$\begin{aligned} & \phi(z^p H_{p,q,s}^{\lambda,\mu}(\beta_1 + 1) f(z), z^p H_{p,q,s}^{\lambda,\mu}(\beta_1) f(z), \\ & z^p H_{p,q,s}^{\lambda,\mu}(\beta_1 - 1) f(z), \\ & z^p H_{p,q,s}^{\lambda,\mu}(\beta_1 - 2) f(z); z) < h(z), \end{aligned} \quad (81)$$

then

$$z^p H_{p,q,s}^{\lambda,\mu}(\beta_1 + 1) f(z) < \mathbf{q}(z). \quad (82)$$

Corollary 15. Let $\Omega \subset \mathbb{C}$ and let the function $\mathbf{q}$ be univalent in $\mathbb{U}$ with $\mathbf{q}(0) = 1$. Suppose also that $\phi \in \Phi_H[h, \mathbf{q}_\rho]$ for some $\rho \in (0, 1)$, where $\mathbf{q}_\rho(z) = \mathbf{q}(\rho z)$. If the functions $f \in \Sigma_p$ and $\mathbf{q}_\rho$ satisfy the following conditions:

$$\Re \left(\frac{\xi q''_\rho(\xi)}{q'_\rho(\xi)} \right) \geq 0,$$

$$\left| \frac{z^p H_{p,q,s}^{\lambda,\mu}(\beta_1) f(z)}{q'_\rho(\xi)} \right| \leq k \quad (z \in \mathbb{U}; \xi \in \partial\mathbb{U} \setminus E(\mathbf{q}_\rho)), \quad (83)$$

$$\begin{aligned} & \phi(z^p H_{p,q,s}^{\lambda,\mu}(\beta_1 + 1) f(z), z^p H_{p,q,s}^{\lambda,\mu}(\beta_1) f(z), \\ & z^p H_{p,q,s}^{\lambda,\mu}(\beta_1 - 1) f(z), \\ & z^p H_{p,q,s}^{\lambda,\mu}(\beta_1 - 2) f(z); z) < h(z), \end{aligned}$$

then

$$z^p H_{p,q,s}^{\lambda,\mu}(\beta_1 + 1) f(z) < \mathbf{q}(z). \quad (84)$$

Our next theorem yields the best dominant of the differential subordination (70).

Theorem 16. Let the function h be univalent in $\mathbb{U}$. Also let $\phi : \mathbb{C}^4 \times \mathbb{U} \rightarrow \mathbb{C}$ and ψ be given by (70). Suppose that the differential equation

$$\psi(q(z), zq'(z), z^2q''(z), z^3q'''(z); z) = h(z) \quad (85)$$

has a solution $\mathbf{q}(z)$ with $\mathbf{q}(0) = 1$, which satisfies condition (61). If the function $f \in \Sigma_p$ satisfies condition (81) and the function

$$\begin{aligned} & \phi(z^p H_{p,q,s}^{\lambda,\mu}(\beta_1 + 1) f(z), z^p H_{p,q,s}^{\lambda,\mu}(\beta_1) f(z), \\ & z^p H_{p,q,s}^{\lambda,\mu}(\beta_1 - 1) f(z), z^p H_{p,q,s}^{\lambda,\mu}(\beta_1 - 2) f(z); z) \end{aligned} \quad (86)$$

is analytic in $\mathbb{U}$, then

$$z^p H_{p,q,s}^{\lambda,\mu}(\beta_1 + 1) f(z) < \mathbf{q}(z) \quad (87)$$

and $\mathbf{q}(z)$ is the best dominant.

Proof. By applying Theorem 12, we deduce that $\mathbf{q}$ is a dominant of (81). Since $\mathbf{q}$ satisfies (85), it is also a solution of (81). Therefore, $\mathbf{q}$ will be dominated by all dominants. Hence $\mathbf{q}$ is the best dominant. $\square$

In view of Definition 11, in the particular case when $\mathbf{q}(z) = 1 + Mz$ ($M > 0$), the class $\Phi_H[\Omega, \mathbf{q}]$ of admissible functions, denoted simply by $\Phi_H[\Omega, M]$, is described below.

Definition 17. Let Ω be a set in $\mathbb{C}$, $\beta_1 \in \mathbb{C} \setminus \{0, -1, -2, \dots\}$, and $M > 0$. The class $\Phi_H[\Omega, M]$ of admissible functions consists of those functions $\phi : \mathbb{C}^4 \times \mathbb{U} \rightarrow \mathbb{C}$ such that

$$\begin{aligned} & \phi \left(1 + Me^{i\theta}, 1 + \frac{k + \beta_1}{\beta_1} Me^{i\theta}, \right. \\ & 1 + \frac{L + (2k + \beta_1)(\beta_1 + 1) Me^{i\theta}}{\beta_1(\beta_1 + 1)}, \\ & \left. 1 + \frac{N + 3(\beta_1 + 2)L + (3k + \beta_1)(\beta_1 + 1)(\beta_1 + 2) Me^{i\theta}}{\beta_1(\beta_1 + 1)(\beta_1 + 2)}; z \right) \\ & \notin \Omega \end{aligned} \quad (88)$$

whenever $z \in \mathbb{U}$, $\Re(Le^{-i\theta}) \geq (k - 1)kM$ and $\Re(Ne^{-i\theta}) \geq 0$ for all $\theta \in \mathbb{R}$ and $k \in \mathbb{N} \setminus \{1\}$.

Corollary 18. Let $\phi \in \Phi_H[\Omega, M]$. If the function $f \in \Sigma_p$ satisfies the following conditions:

$$\begin{aligned} & |z^p H_{p,q,s}^{\lambda,\mu}(\beta_1) f(z)| \leq kM \quad (k \in \mathbb{N} \setminus \{1\}; M > 0), \\ & \phi(z^p H_{p,q,s}^{\lambda,\mu}(\beta_1 + 1) f(z), z^p H_{p,q,s}^{\lambda,\mu}(\beta_1) f(z), \\ & z^p H_{p,q,s}^{\lambda,\mu}(\beta_1 - 1) f(z), \\ & z^p H_{p,q,s}^{\lambda,\mu}(\beta_1 - 2) f(z); z) \in \Omega, \end{aligned} \quad (89)$$

then

$$|z^p H_{p,q,s}^{\lambda,\mu}(\beta_1 + 1) f(z) - 1| < M \quad (M > 0). \quad (90)$$

In the special case when

$$\Omega = \mathbf{q}(\mathbb{U}) = \{\omega : |\omega - 1| < M (M > 0)\}, \quad (91)$$

the class $\Phi_H[\Omega, M]$ is denoted, for brevity, by $\Phi_H[M]$. Corollary 18 can now be rewritten in the following form.

Corollary 19. Let $\phi \in \Phi_H[M]$. If the function $f \in \Sigma_p$ satisfies the following conditions:

$$\begin{aligned} |z^p H_{p,q,s}^{\lambda,\mu}(\beta_1) f(z)| &\leq kM \quad (k \in \mathbb{N} \setminus \{1\}; M > 0), \\ |\phi(z^p H_{p,q,s}^{\lambda,\mu}(\beta_1 + 1) f(z), z^p H_{p,q,s}^{\lambda,\mu}(\beta_1) f(z), \\ &z^p H_{p,q,s}^{\lambda,\mu}(\beta_1 - 1) f(z), \\ &z^p H_{p,q,s}^{\lambda,\mu}(\beta_1 - 2) f(z); z) - 1| < M, \end{aligned} \quad (92)$$

then

$$|z^p H_{p,q,s}^{\lambda,\mu}(\beta_1 + 1) f(z) - 1| < M \quad (M > 0). \quad (93)$$

Corollary 20. Let $\beta_1 \in \mathbb{C} \setminus \{0, -1, -2, \dots\}$ with $\Re(\beta_1) \geq -1/2$ and $M > 0$. If the function $f \in \Sigma_p$ satisfies the following conditions:

$$\begin{aligned} |z^p H_{p,q,s}^{\lambda,\mu}(\beta_1) f(z)| &\leq kM \quad (k \in \mathbb{N} \setminus \{1\}), \\ |z^p H_{p,q,s}^{\lambda,\mu}(\beta_1) f(z) - 1| &< M, \end{aligned} \quad (94)$$

then

$$|z^p H_{p,q,s}^{\lambda,\mu}(\beta_1 + 1) f(z) - 1| < M. \quad (95)$$

Proof. Corollary 20 follows from Corollary 19 by setting

$$\phi(a, b, c, d; z) = b = 1 + \frac{k + \beta_1}{\beta_1} M e^{i\theta}. \quad (96)$$

□

Corollary 21. Let $\beta_1 \in \mathbb{C} \setminus \{0, -1, -2, \dots\}$, $k \in \mathbb{N} \setminus \{1\}$, and $M > 0$. If the function $f \in \Sigma_p$ satisfies the following conditions:

$$\begin{aligned} |z^p H_{p,q,s}^{\lambda,\mu}(\beta_1) f(z)| &\leq kM, \\ |z^p H_{p,q,s}^{\lambda,\mu}(\beta_1 - 2) f(z) - z^p H_{p,q,s}^{\lambda,\mu}(\beta_1 - 1) f(z)| &< \frac{M}{|\beta_1|}, \end{aligned} \quad (97)$$

then

$$|z^p H_{p,q,s}^{\lambda,\mu}(\beta_1 + 1) f(z) - 1| < M. \quad (98)$$

Proof. Let

$$\phi(a, b, c, d; z) = d - c, \quad \Omega = h(\mathbb{U}), \quad (99)$$

where

$$h(z) = \frac{Mz}{|\beta_1|} \quad (M > 0). \quad (100)$$

In order to use Corollary 18, we need to show that $\phi \in \Phi_H[\Omega, M]$; that is, the admissibility condition (88) is satisfied. This follows easily, since

$$\begin{aligned} &\left| \phi \left(1 + M e^{i\theta}, 1 + \frac{k + \beta_1}{\beta_1} M e^{i\theta}, \right. \right. \\ &\quad \left. \left. 1 + \frac{L + (2k + \beta_1)(\beta_1 + 1) M e^{i\theta}}{\beta_1(\beta_1 + 1)}, \right. \right. \\ &\quad \left. \left. 1 + \frac{N + 3(\beta_1 + 2)L + (3k + \beta_1)(\beta_1 + 1)(\beta_1 + 2) M e^{i\theta}}{\beta_1(\beta_1 + 1)(\beta_1 + 2)}; z \right) \right| \\ &= \left| \frac{N + 3(\beta_1 + 2)L + (3k + \beta_1)(\beta_1 + 1)(\beta_1 + 2) M e^{i\theta}}{\beta_1(\beta_1 + 1)(\beta_1 + 2)} \right. \\ &\quad \left. - \frac{L + (2k + \beta_1)(\beta_1 + 1) M e^{i\theta}}{\beta_1(\beta_1 + 1)} \right| \\ &= \left| \frac{N e^{-i\theta} + 2(\beta_1 + 2) L e^{-i\theta} + k(\beta_1 + 1)(\beta_1 + 2) M}{\beta_1(\beta_1 + 1)(\beta_1 + 2) e^{-i\theta}} \right| \\ &\geq \frac{\Re(N e^{-i\theta}) + 2|\beta_1 + 2|\Re(L e^{-i\theta}) + k|\beta_1 + 1||\beta_1 + 2|M}{|\beta_1(\beta_1 + 1)(\beta_1 + 2)|} \\ &\geq \frac{k(2k - 2 + |\beta_1 + 1|)M}{|\beta_1||\beta_1 + 1|} \\ &\geq \frac{M}{|\beta_1|} \end{aligned} \quad (101)$$

whenever $z \in \mathbb{U}$, $\Re(L e^{-i\theta}) \geq (k - 1)kM$, and $\Re(N e^{-i\theta}) \geq 0$ for all $\theta \in \mathbb{R}$ and $k \in \mathbb{N} \setminus \{1\}$. The required result now follows from Corollary 18. □

4. Third-Order Differential Superordination of the Operator $H_{p,q,s}^{\lambda,\mu}(\beta_1)$

In this section, we obtain the third-order differential superordination results for meromorphically multivalent functions associated with the operator $H_{p,q,s}^{\lambda,\mu}(\beta_1)$ defined by (16). Because of this, the class of admissible functions is given in the following definition.

Definition 22. Let Ω be a set in $\mathbb{C}$ and $\mathbf{q} \in \mathcal{H}$ with $\mathbf{q}'(z) \neq 0$. The class of admissible functions $\Phi'_H[\Omega, \mathbf{q}]$ consists of those functions $\phi : \mathbb{C}^4 \times \overline{\mathbb{U}} \rightarrow \mathbb{C}$ that satisfy the following admissibility condition:

$$\phi(a, b, c, d; \xi) \in \Omega \quad (102)$$

whenever

$$\begin{aligned} a &= q(z), \quad b = \frac{zq'(z) + m\beta_1 q(z)}{m\beta_1}, \\ \Re \left(\frac{(\beta_1 + 1)(c - a)}{b - a} - (2\beta_1 + 1) \right) \\ &\leq \frac{1}{m} \Re \left\{ \frac{zq''(z)}{q'(z)} + 1 \right\}, \\ \Re \left(\frac{(\beta_1 + 1)(\beta_1 + 2)(d - 3c + 3b - a)}{b - a} \right) \\ &\leq \frac{1}{m^2} \Re \left\{ \frac{z^2 q'''(z)}{q'(z)} \right\}, \end{aligned} \quad (103)$$

where $z \in \mathbb{U}$, $\beta_1 \in \mathbb{C} \setminus \{0, -1, -2, \dots\}$, $\xi \in \partial\mathbb{U}$, and $m \in \mathbb{N} \setminus \{1\}$.

Theorem 23. Let $\phi \in \Phi'_H[\Omega, q]$. If the functions $f \in \Sigma_p$ and $z^p H_{p,q,s}^{\lambda,\mu}(\beta_1 + 1)f(z) \in \mathcal{Q}_1$ satisfy the following conditions:

$$\begin{aligned} \Re \left(\frac{zq''(z)}{q'(z)} \right) &\geq 0, \quad \left| \frac{z^p H_{p,q,s}^{\lambda,\mu}(\beta_1) f(z)}{q'(z)} \right| \leq m, \\ \phi(z^p H_{p,q,s}^{\lambda,\mu}(\beta_1 + 1)f(z), z^p H_{p,q,s}^{\lambda,\mu}(\beta_1) f(z), \\ z^p H_{p,q,s}^{\lambda,\mu}(\beta_1 - 1)f(z), z^p H_{p,q,s}^{\lambda,\mu}(\beta_1 - 2)f(z); z) \end{aligned} \quad (104)$$

is univalent

$$\begin{aligned} \Omega \subset \{ \phi(z^p H_{p,q,s}^{\lambda,\mu}(\beta_1 + 1)f(z), \\ z^p H_{p,q,s}^{\lambda,\mu}(\beta_1) f(z), z^p H_{p,q,s}^{\lambda,\mu}(\beta_1 - 1)f(z), \\ z^p H_{p,q,s}^{\lambda,\mu}(\beta_1 - 2)f(z); z) : z \in \mathbb{U} \} \end{aligned} \quad (106)$$

implies that

$$q(z) < z^p H_{p,q,s}^{\lambda,\mu}(\beta_1 + 1)f(z). \quad (107)$$

Proof. Let the function $p(z)$ be defined by (64) and ψ by (70). Since $\phi \in \Phi'_H[\Omega, q]$, (71) and (106) yield

$$\Omega \subset \{ \psi(p(z), zp'(z), z^2 p''(z), z^3 p'''(z); z) : z \in \mathbb{U} \}. \quad (108)$$

We see from (68) and (69) that the admissible condition for $\phi \in \Phi'_H[\Omega, q]$ in Definition 22 is equivalent to the admissible condition for ψ as given in Definition 7 with $n = 2$. Hence $\psi \in \Psi'_2[\Omega, q]$, and by using (104) and Theorem 8, we have

$$q(z) < p(z) \quad (109)$$

or, equivalently,

$$q(z) < z^p H_{p,q,s}^{\lambda,\mu}(\beta_1 + 1)f(z), \quad (110)$$

which evidently completes the proof of Theorem 23. $\square$

If $\Omega \neq \mathbb{C}$ is a simply connected domain and $\Omega = h(\mathbb{U})$ for some conformal mapping $h(z)$ of $\mathbb{U}$ onto Ω , then the class $\Phi'_H[h(\mathbb{U}), q]$ is written simply as $\Phi'_H[h, q]$. With proceedings similar as in the preceding section, the following result is an immediate consequence of Theorem 23.

Theorem 24. Let $\phi \in \Phi'_H[h, q]$. Also let the function h be analytic in $\mathbb{U}$. If the functions $f \in \Sigma_p$ and $z^p H_{p,q,s}^{\lambda,\mu}(\beta_1 + 1)f(z) \in \mathcal{Q}_1$ satisfy condition (104) and

$$\begin{aligned} \phi(z^p H_{p,q,s}^{\lambda,\mu}(\beta_1 + 1)f(z), z^p H_{p,q,s}^{\lambda,\mu}(\beta_1) f(z), \\ z^p H_{p,q,s}^{\lambda,\mu}(\beta_1 - 1)f(z), z^p H_{p,q,s}^{\lambda,\mu}(\beta_1 - 2)f(z); z) \end{aligned} \quad (111)$$

is univalent in $\mathbb{U}$, then

$$\begin{aligned} h(z) < \phi(z^p H_{p,q,s}^{\lambda,\mu}(\beta_1 + 1)f(z), z^p H_{p,q,s}^{\lambda,\mu}(\beta_1) f(z), \\ z^p H_{p,q,s}^{\lambda,\mu}(\beta_1 - 1)f(z), \\ z^p H_{p,q,s}^{\lambda,\mu}(\beta_1 - 2)f(z); z) \end{aligned} \quad (112)$$

implies that

$$q(z) < z^p H_{p,q,s}^{\lambda,\mu}(\beta_1 + 1)f(z). \quad (113)$$

Theorems 23 and 24 can only be used to obtain subordinations involving the third-order differential superordination of the forms (106) or (112). The following theorem proves the existence of the best subordinant of (112) for a suitable chosen ϕ .

Theorem 25. Let the function h be analytic in $\mathbb{U}$, and let $\phi : \mathbb{C}^4 \times \overline{\mathbb{U}} \rightarrow \mathbb{C}$ and ψ be given by (70). Suppose that the differential equation

$$\psi(q(z), zq'(z), z^2 q''(z), z^3 q'''(z); z) = h(z) \quad (114)$$

has a solution $q(z) \in \mathcal{Q}_1$. If the functions $f \in \Sigma_p$ and $z^p H_{p,q,s}^{\lambda,\mu}(\beta_1 + 1)f(z) \in \mathcal{Q}_1$ satisfy condition (104) and

$$\begin{aligned} \phi(z^p H_{p,q,s}^{\lambda,\mu}(\beta_1 + 1)f(z), z^p H_{p,q,s}^{\lambda,\mu}(\beta_1) f(z), \\ z^p H_{p,q,s}^{\lambda,\mu}(\beta_1 - 1)f(z), z^p H_{p,q,s}^{\lambda,\mu}(\beta_1 - 2)f(z); z), \end{aligned} \quad (115)$$

is univalent in $\mathbb{U}$, then

$$\begin{aligned} h(z) < \phi(z^p H_{p,q,s}^{\lambda,\mu}(\beta_1 + 1)f(z), z^p H_{p,q,s}^{\lambda,\mu}(\beta_1) f(z), \\ z^p H_{p,q,s}^{\lambda,\mu}(\beta_1 - 1)f(z), \\ z^p H_{p,q,s}^{\lambda,\mu}(\beta_1 - 2)f(z); z) \end{aligned} \quad (116)$$

implies that

$$q(z) < z^p H_{p,q,s}^{\lambda,\mu}(\beta_1 + 1)f(z) \quad (117)$$

and q is the best subordinant.

Proof. The proof of Theorem 25 is similar to that of Theorem 16 and it is being omitted here. $\square$

By combining Theorems 14 and 24, we obtain the following sandwich-type result.

Corollary 26. *Let the functions h_1 and q_1 be analytic functions in $\mathbb{U}$. Also let the function h_2 be univalent in $\mathbb{U}$, $q_2 \in \mathcal{Q}_1$ with $q_1(0) = q_2(0) = 1$ and $\phi \in \Phi_H[h_2, q_2] \cap \Phi'_H[h_1, q_1]$. If the function $f \in \Sigma_p$, $z^p H_{p,q,s}^{\lambda,\mu}(\beta_1 + 1)f(z) \in \mathcal{Q}_1 \cap \mathcal{H}$ and*

$$\begin{aligned} &\phi\left(z^p H_{p,q,s}^{\lambda,\mu}(\beta_1 + 1)f(z), z^p H_{p,q,s}^{\lambda,\mu}(\beta_1)f(z), \right. \\ &\left. z^p H_{p,q,s}^{\lambda,\mu}(\beta_1 - 1)f(z), z^p H_{p,q,s}^{\lambda,\mu}(\beta_1 - 2)f(z); z\right) \end{aligned} \quad (118)$$

is univalent in $\mathbb{U}$, and the conditions (61) and (104) are satisfied, then

$$\begin{aligned} h_1(z) &< \phi\left(z^p H_{p,q,s}^{\lambda,\mu}(\beta_1 + 1)f(z), z^p H_{p,q,s}^{\lambda,\mu}(\beta_1)f(z), \right. \\ &\left. z^p H_{p,q,s}^{\lambda,\mu}(\beta_1 - 1)f(z), \right. \\ &\left. z^p H_{p,q,s}^{\lambda,\mu}(\beta_1 - 2)f(z); z\right) < h_2(z) \end{aligned} \quad (119)$$

implies that

$$q_1(z) < z^p H_{p,q,s}^{\lambda,\mu}(\beta_1 + 1)f(z) < q_2(z). \quad (120)$$

Remark 27. By setting $\lambda = \mu = 0$ in all results of this paper, we can obtain the corresponding results for the well-known Liu-Srivastava operator $H_{p,q,s}(\beta_1)$.

5. Concluding Remarks and Observations

In our present investigation, we have derived several third-order differential subordination and superordination results for meromorphically multivalent functions in the punctured unit disk involving the operator $H_{p,q,s}^{\lambda,\mu}(\beta_1)$ defined by (16) with respect to the parameter $\beta_1 \in \mathbb{C} \setminus \{0, -1, -2, \dots\}$, which is associated with the Liu-Srivastava operator $H_{p,q,s}(\beta_1)$. Our results have been obtained by considering suitable classes of admissible functions. Furthermore, if we use relation (19), we can obtain the corresponding third-order differential subordination and superordination results for the operator $H_{p,q,s}^{\lambda,\mu}(\alpha_1)$ with respect to the parameter $\alpha_1 \in \mathbb{C}$ and here we choose to omit the details involved.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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