Research Article

Notes on Lipschitz Properties of Nonlinear Scalarization Functions with Applications

Fang Lu and Chun-Rong Chen

College of Mathematics and Statistics, Chongqing University, Chongqing 401331, China

Correspondence should be addressed to Chun-Rong Chen; chencr1981@163.com

Received 15 January 2014; Accepted 22 March 2014; Published 27 April 2014

Academic Editor: Ryan Loxton

Copyright © 2014 F. Lu and C.-R. Chen. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

Various kinds of nonlinear scalarization functions play important roles in vector optimization. Among them, the one commonly known as the Gerstewitz function is good at scalarizing. In linear normed spaces, the globally Lipschitz property of such function is deduced via primal and dual spaces approaches, respectively. The equivalence of both expressions for globally Lipschitz constants obtained by primal and dual spaces approaches is established. In particular, when the ordering cone is polyhedral, the expression for calculating Lipschitz constant is given. As direct applications of the Lipschitz property, several sufficient conditions for Hölder continuity of both single-valued and set-valued solution mappings to parametric vector equilibrium problems are obtained using the nonlinear scalarization approach.

1. Introduction

In the development of vector optimization, the theory and the methods of scalarization have always played important roles [1–5]. The linear scalarization is historically the first method proposed and the most widely known and used. Besides this, the nonlinear scalarization is also fully developed. Several well-known nonlinear scalarization functions were introduced, such as the Hiriart-Urruty function [6] and the Gerstewitz (Tammer) function [7, 8]. Among them, the function $\xi_q$ (see Definition 1) commonly known as the Gerstewitz function in vector optimization [7, 9, 10] is a powerful tool, which was introduced in [11] and has further been mentioned in [12, 13]. It has many good properties, such as continuity, sublinearity, convexity, monotonicity, and Lipschitz property. These properties have been fully exploited in the literature [5, 7–10, 14–17] to deal with various problems with vector objectives, such as existence and continuity of solutions, optimality conditions, gap functions, duality, vector variational principles, well posedness, vector minimax inequalities, and vector network equilibrium problems.

However, as far as we know, the locally and globally Lipschitz properties of $\xi_q$ have not been noticed until recently. Tammer and Zălinescu [8] studied Lipschitz continuity properties of such kind of functions and gave some applications for deriving necessary optimality conditions for vector optimization problems. For other close works about this aspect, one can refer to Durea and Tammer [14] and Nam and Zălinescu [18]. Chen and Li [15, 16] deduced the globally Lipschitz property of $\xi_q$ by the dual space approach and applied it to discussing Hölder continuity of solutions to parametric vector (quasi)equilibrium problems. Motivated by the work reported in [8, 15, 16], in this paper we further discuss the globally Lipschitz property of $\xi_q$ in linear normed spaces via the primal space approach (see Proposition 7). The equivalence that $L' = L$ means that the property of strong duality (i.e., $\inf(P) = L' = L = \sup(D)$) holds between primal and dual problems. Furthermore, the above discussions are extended to general Gerstewitz function $\varphi_{-\nu}$, and exact characterizations to the globally Lipschitz property for $\varphi_{-\nu}$ are discussed, which would further complete the theory of [8, 18]. In addition, when the ordering cone is...
polyhedral, the expression for calculating Lipschitz constant is also given.

Vector equilibrium problems (VEPs, for short), also known as generalized Ky Fan inequalities recently, contain many important models as special cases, such as vector variational inequalities, vector complementarity problems, and vector optimization problems (see, e.g., [9, 19, 20] and the references therein). As a significant topic of stability analysis, Hölder or Lipschitz continuity of solutions to parametric VEPs is of considerable interest. Recently, in this field, Hölder continuity of both single-valued and set-valued solution mappings to parametric VEPs have been intensively studied in [16, 17, 21–32], respectively. In addition, recent and related papers about stability of multifunctions published in [33, 34] are worth noticing.

Scalarization approaches have been used as efficient methods to study semicontinuity and Hölder continuity of parametric VEPs. Among them, scalarizing approaches were applied by using linear functionals [27, 35] or nonlinear scalarization functions [15, 16, 24]. We notice that nonlinear scalarization methods by virtue of several nonlinear scalarization functions used Gerstewitz-like scalarization functions to study both semicontinuity and Hölder continuity of solutions to parametric VEPs. Motivated by the work reported in [25, 31, 38], this paper also aims to give some applications of the properties of \( \xi \) to the Hölder continuity of solutions for parametric VEPs. To our aim, the nonlinear scalarization function \( \xi \) as a fundamental tool will play key roles such that, its globally Lipschitz property, monotonicity, and sublinearity will be fully exploited. The results obtained are new and generalizations of known ones [25, 31] for the corresponding scalar cases, and our approach is totally based on the techniques of nonlinear scalarization.

The rest of the paper is organized as follows. In Section 2, we first summarize basic properties of the nonlinear scalarization function \( \xi \), then discuss the globally Lipschitz property of \( \xi \) in linear normed spaces via the primal space approach, establish the equivalence that \( L' = L \), and finally, extend the discussions to the general case \( \varphi \). In Section 3, as applications of the Lipschitz property of \( \xi \), we study Hölder continuity of both single-valued and set-valued solution mappings to parametric VEPs based on the nonlinear scalarization approach. The last section gives some conclusions.

2. Lipschitz Properties of Nonlinear Scalarization Functions

In this section, we first recall the nonlinear scalarization function \( \xi \) in vector optimization. Its main properties, especially, the globally Lipschitz property, are summarized.

Let \( X \) and \( Y \) be linear normed spaces, and let \( K \subset Y \) be a pointed, closed, and convex cone with nonempty interior \( \text{int} \, K \). Let \( Y^* \) be the topological dual space of \( Y \), \( \langle \cdot, \cdot \rangle \) the natural pairing between \( Y \) and \( Y^* \), and \( K^* \) the dual cone of \( K \); that is, \( K^* := \{ \lambda \in Y^* : \langle \lambda, k \rangle \geq 0, \forall k \in K \} \).

**Definition 1** (see [7, 9, 10]). Given a fixed point \( q \in \text{int} \, K \), the nonlinear scalarization function \( \xi \) : \( Y \to R \) is defined by

\[
\xi_q(y) := \min \{ t \in R \mid y \in tq - K \}.
\]

In the special case of \( Y = R^d \), \( K = R^d_+ \), and \( q = (1, \ldots, 1) \in \text{int} \, R^d_+ \), the function \( \xi_q \) can be expressed in the equivalent form \( \xi_q(y) = \max_{1 \leq i \leq d} \{ |y_i| \} \), \( \forall y = (y_1, \ldots, y_d) \in R^d \).

It is well known that \( \xi_q \) is continuous, positively homogeneous, subadditive, and convex on \( Y \), and it is monotone (i.e., \( y^2 - y^1 \in K \Rightarrow \xi_q(y^1) < \xi_q(y^2) \)) and strictly monotone (i.e., \( y^2 - y^1 \in \text{int} \, K \Rightarrow \xi_q(y^1) < \xi_q(y^2) \)) (see [9]). Note, however, that the function \( \xi_q \) is not strongly monotone (i.e., \( y^2 - y^1 \in K \setminus \{0_Y\} \Rightarrow \xi_q(y^1) < \xi_q(y^2) \)).

**Proposition 2** (see [7, 9]). For any fixed \( q \in \text{int} \, K \), \( y \in \text{int} \, Y \), and \( r \in R \),

(i) \( \xi_q(y) < r \iff y \in rq - \text{int} \, K \) (i.e., \( \xi_q(y) \geq r \iff y \notin rq - \text{int} \, K \));

(ii) \( \xi_q(y) \leq r \iff y \in rK - \text{int} \, K \); (i.e., \( \xi_q(y) > r \iff y \notin rK - \text{int} \, K \));

(iii) \( \xi_q(y) = r \iff y \in rK - \partial K \), where \( \partial K \) denotes the topological boundary of \( K \);

(iv) \( \xi_q(rq) = r \), in particular, \( \xi_q(0_Y) = 0 \);

(v) \( \xi_q(y + rq) = \xi_q(y) + r \) (translation property).

Let \( q \in \text{int} \, K \) be a fixed point. The set

\[
K^q := \{ \lambda \in K^* : \langle \lambda, q \rangle = 1 \}
\]

is a weak* compact set of \( Y^* \). Clearly, \( K^q \) is a weak* base of \( K^* \); that is, \( K^q \) is convex and weak* compact such that \( 0_Y \notin K^q \) and \( K^* = \bigcup_{q \in K} K^q \). Notice that [14, Lemma 2.4] we have \( K^q = \partial \xi_q(0_Y) \), where \( \partial \xi_q(0_Y) \) denotes the classical (Fenchel) subdifferential of \( \xi_q \) at \( y = 0_Y \). Generally, for every \( y \in Y \), \( \partial \xi_q(y) = \{ \lambda \in K^q : \langle \lambda, y \rangle = \xi_q(y) \} \subset K^q \).

**Proposition 3** (see [15, 16]). Let \( q \in \text{int} \, K \). Then for any \( y \in Y \),

\[
\xi_q(y) = \max_{\lambda \in K^q} \langle \lambda, y \rangle
\]

Remark that the form of \( (K, q) \)-max scalarizing function \( \phi(y) := \max_{\lambda \in K^q} \langle \lambda, y \rangle \) is also widely developed and has many applications, such as stability analysis of vector equilibrium problems [24] and optimality conditions for vector optimization problems [39].

Another famous scalarizing function is the oriented distance function \( \Delta(y) := d_{-K}(y) - d_{Y \setminus (-K)}(y) \) introduced in [6], where \( d_{\chi}(y) := \inf \{ \| y - a \| : a \in A \} \) denotes the distance from \( y \in Y \) to the set \( A \subset Y \). It is well known that this function has very good general properties (see [3, Proposition 3.2]), especially, which is always 1-Lipschitz.
Remark 4. The oriented distance function \( \Delta \) is not applicable for studying Hölder continuity of solution mappings to parametric vector equilibrium problems (PVEP) and (PVAEP) in this paper, unlike the Gerstewitz function \( \xi \) in the next section. This is because the properties like Proposition 2 (ii) and (v) with \( r \neq 0 \) are not satisfied by the function \( \Delta \), while they play important roles in our study.

Let \( B(x_0, \delta) \) be the closed ball centered at \( x_0 \) and radius \( \delta > 0 \). It is said that \( g : X \to Y \) is Lipschitz around \( x_0 \in X \) if there exist \( L > 0 \) and \( \delta > 0 \) such that \( \| g(x) - g(x') \| \leq L \| x - x' \|, \forall x, x' \in B(x_0, \delta) \), and \( g \) is locally Lipschitz on \( X \subset Y \) if and only if \( g \) is Lipschitz around each \( x_0 \in A \). \( g \) is called (globally) Lipschitz on \( A \) if and only if \( \| g(x) - g(x') \| \leq L \| x - x' \|, \forall x, x' \in A \).

Proposition 5 (see [15, 16]). \( \xi \) is globally Lipschitz on \( Y \), and its Lipschitz constant is \( L := \sup_{\lambda \in K} \| \lambda \| \in [1/\| q \|, +\infty[ \). In particular, under the scalar case of \( Y = \mathbb{R} \) and \( K = \mathbb{R}_+ \), the Lipschitz constant of \( \xi \) is \( L = (1/q) \) (\( q > 0 \)).

Remark 6. Let \( q \in int K \) and \( a \in Y \). The nonlinear scalarization function \( \xi_{q,a} : Y \to \mathbb{R} \) is defined as \( \xi_{q,a}(y) := \min \{ t \in \mathbb{R} \mid y \in a + tK \} \). It is easy to see that \( \xi_{q,a} \) is still globally Lipschitz on \( Y \) with Lipschitz constant \( L = \sup_{\lambda \in K} \| \lambda \| \), because \( \xi_{q,a}(y) = \max_{\| \lambda \| \leq 1} \lambda(a - y) \).

Assume that \( Y \) is a separated locally convex space, and \( K \subset Y \) is a proper, closed, and convex cone with \( \text{int} K \neq \emptyset \). Let \( q \in \text{int} K \). Then it follows from the proof of [8, Theorem 3.1(iii)] (see inequality (6) therein) that \( \xi \) is Lipschitz on \( Y \), namely,

\[
\left| \xi_q(y) - \xi_q(y') \right| \leq p_v(y - y'), \quad \forall y, y' \in Y, \tag{3}
\]

where \( p_v : Y \to \mathbb{R} \) is the Minkowski functional associated with \( V \) and \( V \subset Y \) is a symmetric closed and convex neighborhood of \( y_0 \) such that \( q + V \subset K \).

When \( Y \) is a linear normed space and \( V := \tau_{K,q}B \) for some \( \tau_{K,q} > 0 \) (\( B \) denotes the closed unit ball), because

\[
p_v(x) := \inf \left\{ \alpha > 0 \mid \frac{x}{\alpha} \in V \right\} = \inf \left\{ \alpha > 0 \mid \frac{x}{\alpha} \in \tau_{K,q}B \right\} = \frac{\| x \|}{\tau_{K,q}}, \tag{4}
\]

we get from (3) that, for all \( y, y' \in Y \),

\[
\left| \xi_q(y) - \xi_q(y') \right| \leq \frac{\| y - y' \|}{\tau_{K,q}}. \tag{5}
\]

Whence,

\[
\left| \xi_q(y) - \xi_q(y') \right| \leq \frac{\| y - y' \|}{\tau_{K,q}^\text{max}}, \quad \forall y, y' \in Y, \tag{6}
\]

where \( \tau_{K,q}^\text{max} := \sup \{ \tau > 0 \mid q + \tau B \subset K \} = \sup \{ \tau > 0 \mid B(q, \tau) \subset K \} \). Therefore, the following conclusion holds for \( Y \) a linear normed space.

Proposition 7. \( \xi_q \) is globally Lipschitz on \( Y \) with Lipschitz constant \( L' := 1/\tau_{K,q}^\text{max} \).

Note that \( L' = 1/\tau_{K,q}^\text{max} = \inf \{ 1/\tau > 0 \mid q + \tau B \subset K \} \).

Tammer and Zălinescu [8] recently have studied the Lipschitz property of the Gerstewitz function under more general settings than ours, using the primal space approach, which is different from the dual space approach adopted by us [15, 16]. In this paper, we limit our discussions in linear normed spaces to get more exact characterizations and more clear geometrical interpretations.

Whether the two Lipschitz constants \( L = L' \) hold, we show it as follows.

Proposition 8. Consider \( \sup_{\lambda \in K} \| \lambda \| = 1/\tau_{K,q}^\text{max} \) (i.e., \( L = L' \)).

Proof. Let \( L := \sup_{\lambda \in K} \| \lambda \| = 1/\tau_{K,q}^\text{max} \). Firstly, we prove that \( L \leq L' \).

By the definition of \( \tau_{K,q}^\text{max} \), for any given \( \epsilon > 0 \), we have \( q + (\tau_{K,q}^\text{max} + \epsilon)B \notin K \). Thus there exists \( \tilde{B} \in \mathbb{B} \) such that \( q + (\tau_{K,q}^\text{max} + \epsilon)\tilde{B} \notin K \). It follows from [2, Lemma 3.21(q)] that \( K = \{ k \in Y \mid \langle \lambda, k \rangle \geq 0, \forall \lambda \in K^* \} \), as \( K \) is a closed and convex cone.

Whence, there is an \( \tilde{\lambda} \in K^* \) satisfying \( \langle \lambda, q + (\tau_{K,q}^\text{max} + \epsilon)\tilde{B} \rangle < 0 \). Obviously, \( \tilde{\lambda} \neq 0 \). As \( q \in \text{int} K \), \( \langle \lambda, q \rangle > 0 \). Without loss of generality, we may assume that \( \langle \lambda, q \rangle = 1 \); that is, \( \tilde{\lambda} \in K^* \).

Hence, we can deduce that

\[
0 > \langle \tilde{\lambda}, q + (\tau_{K,q}^\text{max} + \epsilon)\tilde{B} \rangle = \langle \tilde{\lambda}, q \rangle + \langle \tilde{\lambda}, (\tau_{K,q}^\text{max} + \epsilon)\tilde{B} \rangle = 1 + \langle \tau_{K,q}^\text{max} + \epsilon \rangle \langle \tilde{\lambda}, \tilde{B} \rangle \geq 1 - \langle \tau_{K,q}^\text{max} + \epsilon \rangle \| \tilde{\lambda} \| \cdot \| \tilde{B} \|
\]

Thus,

\[
\frac{1}{\tau_{K,q}^\text{max} + \epsilon} < \| \tilde{\lambda} \| \leq L. \tag{8}
\]

By the arbitrariness of \( \epsilon > 0 \), we obtain that \( L' \leq L \).

Secondly, we prove that \( L \geq L' \).

Take any \( \lambda \in K^* \) and \( \tau \in \Gamma := \{ t > 0 \mid q + t\mathbb{B} \subset K \} \). Then, \( \langle \lambda, q + t\mathbb{B} \rangle \geq 0 \); namely, for any \( b \in \mathbb{B} \), \( \langle \lambda, q + t\mathbb{B} \rangle \geq 0 \).

As \( \langle \lambda, q \rangle = 1 \) and \( \tau > 0 \), we deduce that \( 1/\tau + \langle \lambda, b \rangle \geq 0 \).

Hence, \( 1/\tau \geq -\langle \lambda, b \rangle = \langle \lambda, -b \rangle \). By the symmetry of \( \mathbb{B} \) and the arbitrariness of \( b \in \mathbb{B} \), we have \( 1/\tau \geq -\langle \lambda, b \rangle, \forall b \in \mathbb{B} \).

Whence, we get

\[
\inf_{\tau \in \Gamma} \frac{1}{\tau} \geq \sup_{\lambda \in K^*, b \in \mathbb{B}} \langle \lambda, b \rangle = \sup_{\lambda \in K^*} \| \lambda \|, \tag{9}
\]

where the last equality holds by the Cauchy-Schwarz inequality and that \( b \in \mathbb{B} \). Note that \( L' = 1/\tau_{K,q}^\text{max} = \inf \{ 1/\tau > 0 \mid q + \tau B \subset K \} \). Thus, we obtain that \( L' \geq L \).
Based on the above analysis, the equivalence that $L = L'$ holds.

It is obvious that $r^\max_{Ka} = \text{dist}(q, bdK)$, where the latter one denotes the distance from $q \in \text{int} K$ to the boundary of $K$; that is, $\text{dist}(q, bdK) := \inf_{x \in bdK} ||q - x||$. Thus, we also have that $L' = 1/\text{dist}(q, bdK)$ (see also [14, Lemma 2.4]). However, the expression of the form $L' = 1/r^\max_{Ka} = \inf_{1/\tau > 0} |q + \tau B | \subset K$ has its advantage: it is convenient to describe the dual characterization concerning $L = \sup_{x \in K} ||\lambda||$ (see also Remark 13).

Due to the observation above, we could give another proof of the second part of Proposition 8 (i.e., the weak duality between the problems $(P)$ and $(D)$: $\inf(P) = L' \geq L = \sup(D)$; see Remark 13 below). As $q + \text{dist}(q, bdK) \in K$, then, for any $b \in K$, $q + \text{dist}(q, bdK) b \in K$. It follows from the symmetry of $K$ and the arbitrariness of $b \in K$ that $q/\text{dist}(q, bdK) - b \in K$. Take any $\lambda \in K^\circ$. Then $0 \leq \langle \lambda, q/\text{dist}(q, bdK) - b \rangle = \langle \lambda, q \rangle/\text{dist}(q, bdK) - \langle \lambda, b \rangle = 1/\text{dist}(q, bdK) - \langle \lambda, b \rangle$. Hence $L' = 1/\text{dist}(q, bdK) \geq \sup_{x \in \text{int} K} ||\lambda|| = L$. □

We give several examples to illustrate Proposition 8.

Example 9 (see [16, Example 2.1]). Let $Y = \mathbb{R}^2$ and $K = \{(y_1, y_2) \in \mathbb{R}^2 \mid 1/4 y_1 \leq y_2 \leq 2 y_1\}$. Obviously, $K^* = \{\lambda = (\lambda_1, \lambda_2) \in \mathbb{R}^2 \mid \lambda_1 \geq -1/2, \lambda_2 \geq -4\lambda_1\}$, and $K \subsetneq K^*$. Take $q = (2, 3) \in \text{int} K$. Then, $K^q = \{\lambda = (\lambda_1, \lambda_2) \in \mathbb{R}^2 \mid 2\lambda_1 + \lambda_2 = 1, \lambda_1 \in [-1/10, 2]\}$. We have calculated that the Lipschitz constant $L = \sup_{x \in \text{int} K} ||\lambda|| = ||2, -3|| = \sqrt{13}$. Now we calculate another Lipschitz constant $L' = 1/r^\max_{K,q}$. Note that the distance from a point $(x_0, y_0) \in \mathbb{R}^2$ to the line $y = kx + b$ (resp., $Ax + By + C = 0$) is

$$d = \frac{|kx_0 - y_0 + b|}{\sqrt{k^2 + 1}} \quad \text{(resp., } d = \frac{|Ax_0 + By_0 + C|}{\sqrt{A^2 + B^2}})$$

(10)

It is easy to verify that

$$r^\max_{K,q} = \left\|(2, 3) - \left(\frac{8}{\sqrt{5}}, \frac{16}{\sqrt{5}}\right)\right\| = \frac{2 \times 3 - 3}{\sqrt{2^2 + 1}} = \frac{\sqrt{3}}{5}.$$

Thus, we also get that the Lipschitz constant $L' = 1/r^\max_{K,q} = \sqrt{5}$.

Example 10. Let $Y = \mathbb{R}^2$ and $K = \{(y_1, y_2) \in \mathbb{R}^2 \mid y_2 \geq -y_1, y_2 \geq 0\}$. It is clear that $K^* = \{\lambda = (\lambda_1, \lambda_2) \in \mathbb{R}^2 \mid \lambda_1 \geq 0, \lambda_2 \geq \lambda_1\} \subsetneq K$.

**Case i.** Take $q = (-1, 2) \in \text{int} K \setminus K^*$. Then, $K^q = \{\lambda = (\lambda_1, \lambda_2) \in \mathbb{R}^2 \mid \lambda_1 + 2\lambda_2 = 1, \lambda_1 \in [0, 1]\}$. Thus, $L = \sup_{x \in \text{int} K} ||\lambda|| = ||1, 1|| = \sqrt{2}$. Moreover, $r^\max_{K,q} = ||(-1, 2) - (-3/2, 3/2)|| = |-1| - 2/\sqrt{(-1)^2 + 1} = \sqrt{2}/2$, and hence $L' = 1/r^\max_{K,q} = \sqrt{2}$.

**Case ii.** Take $q = (1, 1/2) \in \text{int} K \setminus K^*$. Then, $K^q = \{\lambda = (\lambda_1, \lambda_2) \in \mathbb{R}^2 \mid 3\lambda_1 + \lambda_2 = 2, \lambda_1 \in [0, 2/3]\}$. Thus, $L = \sup_{x \in \text{int} K} ||\lambda|| = ||0, 2|| = 2$. Moreover, $r^\max_{K,q} = ||(1, 1/2) - (1, 0)|| = 1/2$, and hence $L' = 1/r^\max_{K,q} = 2$.

Case iii. Take $q = (1/2, 3/2) \in \text{int} K \cap K^*$. Then, $K^q = \{\lambda = (\lambda_1, \lambda_2) \in \mathbb{R}^2 \mid \lambda_1 + 3\lambda_2 = 2, \lambda_1 \in [0, 1/2]\}$. Thus, $L = \sup_{x \in \text{int} K} ||\lambda|| = ||1/2, 1/2|| = \sqrt{1/2}/2$. Moreover, $r^\max_{K,q} = ||(1/2, 3/2) - (-1/2, 1/2)|| = 1 - x/2 - 3/2/\sqrt{(-1)^2 + 1} = 1/2$, and hence $L' = 1/r^\max_{K,q} = \sqrt{2}/2$.

**Example 11.** Let $Y = \mathbb{R}^2$ and $K = \mathbb{R}^2$. Then $K^* = K$. Take $q = (q_1, q_2) \in \text{int} K$. We have $K^q = \{\lambda = (\lambda_1, \lambda_2) \in \mathbb{R}^2 \mid \lambda_1 q_1 + \lambda_2 q_2 = 1, \lambda_1 \in [0, 1/q_1]\}$. Thus, $L = \sup_{x \in \text{int} K} ||\lambda|| = \max\{1/q_1, 1/q_2\}$. Meanwhile, $r^\max_{K,q} = \min\{q_1, q_2\}$, so $L' = 1/r^\max_{K,q} = 1/\min\{q_1, q_2\}$.

Now we calculate $r^\max_{K,q}$ when $K$ has the following explicit structure (it is said to be a polyhedral convex cone in [40, Definition 2.1.7]), which contains the above examples as special cases. Herein we use the notation $x^T y$, instead of $(x, y)$, to denote the standard inner product on $\mathbb{R}^n$.

**Proposition 12.** If the polyhedral ordering cone $K \subset \mathbb{R}^n$ with nonempty interior is described by linear inequalities, $K := \{x \in \mathbb{R}^n \mid a^i x \leq 0, i = 1, \ldots, m\}$, then for given $q \in \text{int} K$, $r^\max_{K,q} = \min_{1 \leq i \leq m} \{ -a^i q/||a^i|| \}$. Thus, $L = L' = \max_{1 \leq i \leq m} \{ -a^i q/||a^i|| \}$.

**Proof.** Obviously, $K$ is a closed convex cone. By the definition of $r^\max_{K,q}$, we wish to maximize $\tau > 0$ subject to the constraint $\mathbb{B}(q, r) = \{q + r \mid ||r|| \leq \tau\} \subset K$; that is, $a^T q \leq 0, i = 1, \ldots, m$ for all $x \in \mathbb{B}(q, r)$. Therefore, $\mathbb{B}(q, r) \subset K$ if and only if

$$\text{sup} \left\{ a^T q + r \mid ||r|| \leq \tau \right\} \leq 0.$$  

Since $\text{sup} \{ a^T r \mid ||r|| \leq \tau \} = \tau ||a||$, we can rewrite (12) as

$$a^T q + \tau ||a|| \leq 0, \quad i = 1, \ldots, m.$$  

Thus, we have

$$\tau \leq -\frac{a^T q}{||a||} \quad i = 1, \ldots, m.$$  

(14)

Because $q \in \text{int} K$, $a^T q < 0, i = 1, \ldots, m$. Whence, we obtain that

$$r^\max_{K,q} = \max \left\{ \tau > 0 \mid \tau \leq -\frac{a^T q}{||a||} \quad i = 1, \ldots, m \right\}$$

$$= \min_{1 \leq i \leq m} \left\{ -\frac{a^T q}{||a||} \right\} > 0.$$  

(15)

This completes the proof. □

**Remark 13.** Proposition 8 implies that Propositions 5 and 7 (or see [14, Lemma 2.4]) are equivalent. Proposition 8 also shows that the property of strong duality (i.e., $\inf(P) = L'$ would...
$L = \sup(D)$ holds between the following primal problem $(P)$ and dual problem $(D)$ for fixed $q \in \text{int} K$:

\[
(P) \quad \left\{ \begin{array}{l}
\inf \frac{1}{r} \\
\text{s.t.} \quad r > 0,
q + rB \subset K,
\end{array} \right.
\]

\[
(D) \quad \left\{ \begin{array}{l}
\sup \|\lambda\| \\
\text{s.t.} \quad \lambda \in K^*,
\langle \lambda, q \rangle = 1.
\end{array} \right.
\]

(16)

Clearly, the Lipschitz constants $L'$ and $L$ are deduced via two approaches: the primal space and the dual space approaches, respectively.

**Remark 14.** Relative to $L$, the expression for Lipschitz constant $L'$ is natural and exhibits a clear geometrical interpretation. In the setting of linear normed space $Y$, if $r > 0$ is the largest radius such that the closed ball $B(q, r)$ centered at $q \in \text{int} K$ with radius $r$ lies in the ordering cone $K$ of $Y$, that is, $B(q, r) \subset K$, then $1/r$ is the Lipschitz constant $L'$.

Clearly, the value of $r$ coincides with the distance from $q$ to the boundary of $K$. Based on the geometrical interpretation of $L'$, we know that the choice of $q \in \text{int} K$, namely, the location of $q$, will directly confirm the modulus of Lipschitz continuity of $\xi_q$. It is obvious that the Lipschitz constant $L'$ becomes larger whenever $q$ is closer to the boundary of $K$.

As a direct application to the proof of Proposition 8, we give a note on Lipschitz properties of the directional minimal time function [18].

Given a vector $v \in Y$, $v \neq 0$, and a nonempty closed set $\Omega \subset Y$ and $\Omega \neq Y$, the directional minimal time function with direction $v$ and target set $\Omega$ is defined by

\[
T_v(y; \Omega) := \inf \{ t \geq 0 \mid y + tv \in \Omega \}.
\]

(17)

This class of functions is similar to the class of nonlinear scalarization functions that has been extensively used to study vector optimization problems (see [7, 8]):

\[
\varphi_v(y; \Omega) := \inf \{ t \in \mathbb{R} \mid y + tv \in \Omega \}.
\]

(18)

Obviously, $\xi_q$ is a special but popular case of $\varphi_v$, by taking $\Omega := -K$ and $v := -q \in -K$.

Recall that the recession cone of $\Omega$ is given by

\[
\Omega_{\infty} := \{ u \in Y \mid \omega + tu \in \Omega, \forall \omega \in \Omega, \forall t \in \mathbb{R}_+ \}.
\]

(19)

It is known that $\Omega_{\infty}$ is a convex cone, and $\Omega_{\infty}$ is also closed as $\Omega$ is closed (see [8]). From [18, Proposition 2.1] we see that if $v \in \Omega_{\infty}$, then $T_v(y; \Omega) = \max\{\varphi_v(y; \Omega), 0\}, \forall y \in Y$.

We recall a result of the globally Lipschitz property for $T_v$ [18, Proposition 4.1].

**Proposition 15** (see [18]). Suppose that $v \in \text{int} \Omega_{\infty}$. Then $T_v$ is globally Lipschitz on $Y$ with Lipschitz constant $\ell' := \inf\{1/r > 0 \mid B(v, r) \subset \Omega_{\infty}\} = 1/\text{dist}(v, \partial \Omega_{\infty})$.

Notice that $\ell' = 1/r_{\Omega_{\infty}}^\text{max}$, where $r_{\Omega_{\infty}}^\text{max} := \sup\{\tau > 0 \mid v + \tau B \subset \Omega_{\infty}\} = \sup\{\tau > 0 \mid B(v, \tau) \subset \Omega_{\infty}\}$. Based on a similar proof to that of $L = L'$, letting $K := \Omega_{\infty}$, which is a closed and convex cone, and $q := v$, we can get

\[
\ell' \leq \ell := \sup \{ \|\lambda\| \mid \lambda \in \Omega_{\infty}^*, \langle \lambda, v \rangle = 1 \},
\]

(20)

where $\Omega_{\infty}^*$ is the dual cone of $\Omega_{\infty}$. Thus, we have the following equivalent proposition.

**Proposition 16.** If $v \in \text{int} \Omega_{\infty}$, then $T_v$ is globally Lipschitz on $Y$ with Lipschitz constant $\ell$.

**Remark 17.** The relation (20) implies that the strong duality (i.e., $\inf(P') = \ell' = \ell = \sup(D')$) holds between the following primal problem $(P')$ and dual problem $(D')$ for fixed $v \in \text{int} \Omega_{\infty}$:

\[
(P') \quad \left\{ \begin{array}{l}
\inf \frac{1}{r} \\
\text{s.t.} \quad r > 0,
B(v, r) \subset \Omega_{\infty},
\end{array} \right.
\]

\[
(D') \quad \left\{ \begin{array}{l}
\sup \|\lambda\| \\
\text{s.t.} \quad \lambda \in \Omega_{\infty}^*,
\langle \lambda, v \rangle = 1.
\end{array} \right.
\]

(21)

In what follows, we deduce more exact characterizations to the globally Lipschitz property for $\varphi_{\tau}(y; \Omega) := \inf\{ t \in \mathbb{R} \mid y + tv \in \Omega + \Omega \}$, which has been studied in [8, Theorem 3.1]. These results would further complete the theory of [8, 18] and could be applied in many aspects of vector optimization.

**Proposition 18.** If $v \in \text{int} K$ and $\Omega$ satisfies the free-disposal assumption $\Omega - K = \Omega$, then $\varphi_{\tau}$ is globally Lipschitz on $Y$ with Lipschitz constant $L_\varphi := \sup_{\lambda \in K^*} \|\lambda\|$ (or equivalently, $L_\varphi := 1/r_{K^*}^\text{max} = \inf\{1/r > 0 \mid B(v, r) \subset K\} = 1/\text{dist}(v, \partial K)$).

**Proof.** According to [8, Theorem 3.1(ii)], $\varphi_{\tau}$ is finite on $Y$. Moreover, by [8, Theorem 3.1(i)], we get, for any $y, y' \in Y$, $\varphi_{\tau}(y; \Omega) \leq \varphi_{\tau}(y'; \Omega) + \varphi_{\tau}(y - y'; -K)$. Thus, it follows from Proposition 3 that

\[
\varphi_{\ell}(y; \Omega) - \varphi_{\ell}(y'; \Omega) \leq \varphi_{\ell}(y - y'; -K) = \xi_{\ell}(y - y')
\]

\[
= \max_{\lambda \in K^*} \langle \lambda, y - y' \rangle
\]

\[
\leq \sup_{\lambda \in K^*} \|\lambda\| \|y - y'\|
\]

\[
= L_\varphi \|y - y'\|.
\]

(22)

This implies $|\varphi_{\ell}(y; \Omega) - \varphi_{\ell}(y'; \Omega)| \leq L_\varphi \|y - y'\| = L_\varphi \|y - y'\|$ (see Proposition 8), because of the symmetry between $y$ and $y'$.

**Remark 19.** Compared to [8, Theorem 3.1(ii)], the exact expressions for globally Lipschitz constants $L_\varphi$ and $L_\varphi' (L_\varphi = \ell_\varphi' = \ell_\varphi)$ the expressions for globally Lipschitz constants $L_\varphi$ and $L_\varphi' (L_\varphi = \ell_\varphi' = \ell_\varphi)$.
Corollary 20. If \( v \in \text{int}\, K \) and \( \Omega \) satisfies the free-disposal assumption \( \Omega + K = \Omega \), then \( \varphi_{\xi} \) is globally Lipschitz on \( Y \) with Lipschitz constant \( L_{\varphi} \) (or equivalently, \( L_{\varphi}' \)).

Note that \( T_{\ell} \) also holds in Corollary 20, which is new to [18]. This is because the free-disposal condition \( \Omega = \Omega + K \) yields that \( K \subset \Omega_{\text{co}} \), so \( v \in \Omega_{\text{co}} \); then \( T_{\ell} = \max(\varphi_{\xi}, 0) \).

Corollary 21. If \( v \in \text{int}\, \Omega_{\text{co}} \), then \( \varphi_{\xi} \) is globally Lipschitz on \( Y \) with Lipschitz constant \( \ell \) (or equivalently, \( \ell' \)).

Proof. The free-disposal condition \( \Omega = \Omega - K \) shows that \( K \subset -\Omega_{\text{co}} \). As \( \Omega_{\text{co}} \) is a closed and convex cone because \( \Omega \) is closed, hence \( -\Omega_{\text{co}} \) is the largest closed convex cone \( K \) verifying the free-disposal assumption \( \Omega = \Omega - K \). So, the conclusion follows by applying Proposition 18 with \( K := -\Omega_{\text{co}} \).

Similarly, the conclusion also follows by applying Corollary 20 with \( K := \Omega_{\text{co}} \). \( \Box \)

Corollary 22. If \( v \in \text{int}\, \Omega_{\text{co}} \), then \( \varphi_{\xi} \) is globally Lipschitz on \( Y \) with Lipschitz constant \( \ell_{\varphi} := \sup\{\|\varphi_{\xi}\| : \lambda \in \Omega_{\text{co}}, (\lambda, v) = -1\} \) (or equivalently, \( \ell_{\varphi}' := \inf\{1/r > 0 : \exists (v, r) \subset -\Omega_{\text{co}} = 1/\text{dist}(v, bd(-\Omega_{\text{co}})) = 1/\text{min}\{b_{\mathcal{K}}(v, r) : (26) \} \}

Remark that from [8, Corollary 3.4] and [18, Proposition 4.2] we see that the function \( \varphi_{\xi} \) (resp., \( T_{\ell}, \varphi_{\xi} \)) is finite-valued and Lipschitz if and only if \( v \in \text{int}\, \Omega_{\text{co}} \) (resp., \( v \in \text{int}\, \Omega_{\text{co}} \)).

When \( \Omega = -K \) and \( v \in \text{int}\, K \), \( \varphi_{\xi}(y, \Omega) = \xi(y) \), Proposition 18 and Corollary 22 reduce to Proposition 5 or Proposition 7. When \( \Omega = K \) and \( v \in \text{int}\, K \), Corollaries 20 and 21 reduce to the case that \( \xi_{\ell}(y) := \varphi_{\xi}(y, K) = \inf\{t \in \mathbb{R} : y \in -tv + K \} \).

3. Applications to the Hölder Continuity

The globally Lipschitz property of the nonlinear scalarization function \( \xi_{\ell} \) seems to be good at dealing with stability and sensitivity analysis of vector optimization problems, such as [15–17]. In this section, we will give some direct applications of this property to the Hölder Continuity of solutions for parametric vector equilibrium problems. The proofs of the results obtained are applications of the corresponding ones in [25, 31] for the scalar problems, by the usual scalarization function \( \xi_{\ell}' \). The results are new and generalizations of known ones for the corresponding scalar cases.

3.1. A Single-Valued Case. In this subsection, let \((X, d_X)\) be a linear metric space, let \( Y \) be a linear normed space, let \( \Lambda \) and \( \Omega \) be nonempty subsets of metric spaces, and let \( C \subset Y \) be a pointed, closed, and convex cone with \( \text{int}\, C \neq \emptyset \). Let \( K : \Lambda \rightrightarrows X \) be a set-valued mapping with nonempty, closed, and convex values and let \( F : X \times X \times \Omega \rightrightarrows Y \) be a vector-valued mapping.

For the parameters \( \lambda \in \Lambda \) and \( \mu \in \Omega \), we consider the following parametric vector equilibrium problem (PVEP) of finding \( x \in K(\lambda) \) such that

\[
F(x, y, \mu) \notin \text{int}\, C, \quad \forall y \in K(\lambda).
\]

Remark that when \( Y = \mathbb{R} \) and \( C = \mathbb{R}_+ \), the model (PVEP) reduces to the scalar one (PKFI) studied in [25]. Let \( S(\lambda, \mu) \) be the subset of \( K(\lambda) \) of the solutions of (PVEP). For the reference point \((\bar{x}, \bar{y}) \in \Lambda \times \Omega \), we always assume that \( S(\lambda, \mu) \neq \emptyset \) for every \( \lambda \in U(\bar{x}) \) and \( \mu \in U(\bar{y}) \), where \( U(\nu) \) denotes some neighborhood of the reference point \( \nu \).

Definition 23 (classical notion). A set-valued mapping \( G : \Omega \rightrightarrows X \) is said to be \( \ell \cdot \alpha \)-Hölder continuous at \( \mu_0 \), if and only if there is a neighborhood \( U(\mu_0) \) of \( \mu_0 \) such that, \( \forall \mu_1, \mu_2 \in U(\mu_0) \),

\[
G(\mu_1) \subseteq G(\mu_2) + \ell d^\alpha(\mu_1, \mu_2) \mathcal{B}_X,
\]

where \( \ell \geq 0 \) and \( \alpha > 0 \), and \( \mathcal{B}_X \) denotes the unit ball of \( X \).

In particular, when \( X \) is a linear normed space, the vector-valued mapping \( g : \Omega \rightrightarrows X \) is said to be \( \ell \cdot \alpha \)-Hölder continuous at \( \mu_0 \), if and only if \( \|g(\mu_1) - g(\mu_2)\| \leq \ell d^\alpha(\mu_1, \mu_2) \).

We say that \( g \) (or \( G \)) is \( \ell \cdot \alpha \)-Lipschitz continuous at \( \mu_0 \) if and only if \( g(\mu_0) \) is \( \ell \cdot \alpha \)-Hölder continuous at \( \mu_0 \).

Next, we introduce the concept of strong C-convexity for a vector-valued mapping, which extends [25, Definition 2.1] from real-valued to vector-valued case.

Definition 24. Let \((X, d)\) be a linear metric space. A vector-valued mapping \( g : \Omega \rightrightarrows X \) is said to be \( h \cdot \alpha \)-strongly C-convex with respect to \( e \in \text{int}\, C \) on \( X \), if and only if there exists \( e \in \text{int}\, C \) such that

\[
tg(x) + (1-t)g(y) - g(tx + (1-t)y) - ht(1-t)d^\alpha(x, y) \
\leq e \\
\text{for all } x, y \in X \text{ and all } t \in [0,1], \text{ where } h, \alpha > 0. \]

Note that \( e \in \text{int}\, C \) plays the role of the “modulus of strong C-convexity” of the mapping \( g \). Clearly, as in the scalar case, strong C-convexity implies (strict) C-convexity. As shown in [25], the strong C-convexity of \( g \) plays important roles.

Lemma 25. If \( g : \Omega \rightrightarrows Y \) is \( h \cdot \alpha \)-strongly C-convex with respect to \( e \in \text{int}\, C \) on \( X \), then the real-valued function \( x \mapsto \xi_e(g(x)) \) is \( h \cdot \alpha \)-strongly convex on \( X \).

Proof. For any \( x, y \in X \) and all \( t \in [0,1] \), we have

\[
\xi_e(g(tx + (1-t)y)) 
\leq \xi_e(tg(x) + (1-t)g(y)) - ht(1-t)d^\alpha(x, y) 
\leq t\xi_e(g(x)) + (1-t)\xi_e(g(y)) - ht(1-t)d^\alpha(x, y),
\]

(26)
where the first and the third inequalities follow from the monotonicity and sublinearity of $\xi_e$, respectively, and the second equality follows from Proposition 2(v). Whence, the composite function $\xi_e \circ g$ is $h \cdot \alpha$-strongly convex on $X$. \hfill \square

Now we state and prove the following results.

**Theorem 26.** For problem (PVEP), assume that the solutions exist in a neighborhood of considered point $(\overline{\lambda}, \overline{\mu}) \in \Lambda \times \Omega$. Suppose that the following conditions hold.

(i) $K(\cdot)$ is $h \cdot \alpha$-Hölder continuous at $\overline{\lambda}$.

(ii) There exist neighborhoods $U(\overline{\lambda})$ of $\overline{\lambda}$ and $U(\overline{\mu})$ of $\overline{\mu}$ such that, for each $\mu \in U(\overline{\mu})$, $F(\cdot, \cdot, \mu)$ is pseudomonotone on $K(U(\overline{\lambda}))$, where $K(U(\overline{\lambda})) := \bigcup_{u \in U(\overline{\lambda})} K(u)$; that is, $\forall x, y \in K(U(\overline{\lambda})) : x \neq y, F(x, y, \mu) \notin \text{int} C \Rightarrow F(y, x, \mu) \notin C$.

(iii) For any $x \in K(U(\overline{\lambda}))$, $\mu \in U(\overline{\mu})$, $F(x, x, \mu) = -bdC$.

(iv) For any $y \in K(U(\overline{\lambda}))$, $\mu \in U(\overline{\mu})$, $F(x, y, \mu)$ is $\ell$-Lipschitz continuous on $K(U(\overline{\lambda}))$, and for any $x \in K(U(\overline{\lambda}))$, $\mu \in U(\overline{\mu})$, $F(x, y, \mu)$ is $k \cdot \beta$-strongly $C$-convex with respect to $e \in \text{int} C$ on $X$ as well as $\ell'$-Lipschitz continuous on $K(U(\overline{\lambda}))$.

(v) $F(x, y, \cdot)$ is $m \cdot \gamma$-Hölder continuous at $\overline{\mu}$ with respect to $K(U(\overline{\lambda}))$; that is, $\forall \mu_1, \mu_2 \in U(\overline{\mu})$, $\forall x, y \in K(U(\overline{\lambda})) : x \neq y, \|F(x, y, \mu_1) - F(x, y, \mu_2)\| \leq m \hat{d}^\beta(x, y)\delta^\gamma(\mu_1, \mu_2)$.

(vi) $d_\lambda(x(\lambda_1, \mu_1), x(\lambda_2, \mu_2)) \leq \left( \frac{2\gamma_e(\ell + \ell')h}{k} \right)^{1/\beta} d^\alpha(\lambda_1, \lambda_2)^{1/\beta} + \left( \frac{2\gamma_e\gamma_m}{k} \right)^{1/\beta-\alpha} d^\gamma(\mu_1, \mu_2)^{1/\beta-\alpha}$,

where $\gamma_e := \sup_{e \in \text{int} C} \|e\| \in [1/\|e\|, +\infty]$ (or $\gamma_e = 1/\tau_{\text{max}}^\alpha$) is the Lipschitz constant of $\xi_e$ on $Y$.

**Proof.** It follows from Proposition 2(i) that, for given $e \in \text{int} C$,

$$S(\lambda, \mu) := \{ x \in K(\lambda) \mid \xi_e(F(x, y, \mu)) \geq 0, \forall y \in K(\lambda) \}.$$  \hfill (28)

Thus, we could apply Theorem 3.1 of [25] by replacing $f$ therein with $\xi_e \circ F$. Now we need to check all conditions of $\xi_e \circ F$.

First, by virtue of the globally Lipschitz property of $\xi_e$ (see Proposition 5 or 7), on one hand, for $\mu \in U(\overline{\mu})$ and $y \in K(U(\overline{\lambda}))$, $\forall x_1, x_2 \in K(U(\overline{\lambda}))$,

$$[\xi_e(F(x_1, y, \mu)) - \xi_e(F(x_2, y, \mu))] \leq \gamma_e \| F(x_1, y, \mu) - F(x_2, y, \mu) \| \leq \gamma_e \hat{d}_\lambda(x_1, x_2),$$  \hfill (29)

and for $\mu \in U(\overline{\mu})$ and $x \in K(U(\overline{\lambda}))$, $\forall y_1, y_2 \in K(U(\overline{\lambda}))$,

$$[\xi_e(F(x, y_1, \mu)) - \xi_e(F(x, y_2, \mu))] \leq \gamma_e \| F(x, y_1, \mu) - F(x, y_2, \mu) \| \leq \gamma_e \hat{d}_\lambda(y_1, y_2).$$  \hfill (30)

On the other hand, $\forall \mu_1, \mu_2 \in U(\overline{\mu})$, $\forall x, y \in K(U(\overline{\lambda})): x \neq y$,

$$[\xi_e(F(x, y, \mu_1)) - \xi_e(F(x, y, \mu_2))] \leq \gamma_e \| F(x, y, \mu_1) - F(x, y, \mu_2) \| \leq \gamma_e \hat{d}_\lambda(y_1, y_2).$$  \hfill (31)

Letting $f := \xi_e \circ F$, whence the Lipschitz or Hölder constants of [25, Theorem 3.1] are fulfilled with $l := \gamma_e \hat{\ell}$, $l' := \gamma_e \hat{\ell}'$ and $m := \gamma_e \hat{m}$.

Second, using Proposition 2(i)-(ii), condition (ii) implies that

$$\xi_e(F(x, y, \mu)) \geq 0 \Rightarrow \xi_e(F(x, y, \mu)) \leq 0,$$  \hfill (32)

which shows that $\xi_e \circ F(\cdot, \cdot, \cdot)$ is pseudomonotone on $K(U(\overline{\lambda}))$.

In addition, it follows from Proposition 2(iii) and condition (iii) that, for any $x \in K(U(\overline{\lambda}))$, $\mu \in U(\overline{\mu})$, $\xi_e(F(x, x, \mu)) = 0$.

Third, by virtue of Lemma 25, it is clear that $\xi_e \circ F(x, \cdot, \cdot)$ is $k \cdot \beta$-strongly convex on $X$. Thus, all conditions of [25, Theorem 3.1] are satisfied, and hence the conclusion follows. \hfill \square

Apply Theorem 3.2 of [25] by replacing $f$ therein with $\xi_e \circ F$, and combining with similar analysis of Theorem 26, we obtain the following result readily.

**Theorem 27.** Suppose that the conditions of Theorem 26 are satisfied except (ii) and (iv), which are replaced by the following ones, respectively.

(iii) There exist neighborhoods $U(\overline{\lambda})$ of $\overline{\lambda}$ and $U(\overline{\mu})$ of $\overline{\mu}$ such that, for each $\mu \in U(\overline{\mu})$, $F(\cdot, \cdot, \mu)$ is monotone with respect to $\xi_e$ on $K(U(\overline{\lambda}))$; that is, $\forall x, y \in K(U(\overline{\lambda})) : x \neq y, \xi_e(F(y, x, \mu)) \leq \xi_e(F(x, y, \mu))$.

(iv) For any $x \in K(U(\overline{\lambda}))$, $\mu \in U(\overline{\mu})$, $F(x, \cdot, \mu)$ is $k \cdot \beta$-strongly $C$-convex with respect to $e \in \text{int} C$ on $X$ as well as $\ell'$-Lipschitz continuous on $K(U(\overline{\lambda}))$. 


Then for every \((\lambda, \mu) \in U(\lambda) \times U(\mu)\), the solution of (PVEP) is unique, \(x(\lambda, \mu)\), and this function satisfies the Hölder condition
\[
d_X(x(\lambda_1, \mu), x(\lambda_2, \mu)) \leq \left(\frac{4\gamma\ell h}{k}\right)^{1/\beta} \max \{d^{(\beta)}(\lambda_1, \lambda_2), d^{(\beta)}(\mu_1, \mu_2)\},
\]
where \(\gamma, \ell, h, k, \beta\) are positive constants.

Remark 28. When \(Y = \mathbb{R}, C = \mathbb{R}_+, \) and \(e = 1, \gamma = 1\), Theorem 26 (Theorem 27, resp.) reduces to Theorem 3.1 (Theorem 3.2, resp.) of [25]. So we have generalized [25, Theorems 3.1 and 3.2] to vector-valued setting. These results are new in the literature, and the proof is totally based on the technique of nonlinear scalarization.

Remark 29. (a) The monotonicity of \(F(\cdot, \cdot, \mu)\) (condition (ii') of Theorem 27) is stronger than the pseudomonotonicity (condition (ii) of Theorem 26). Under this case, we see that the assumption on the Lipschitz property of \(F(\cdot, y, \mu)\) in (PVEP) is superfluous. (b) If \(K(U(\lambda))\) in condition (v) of Theorem 26 is bounded, then without loss of generality we can take \(\theta = 0\) in assumption (v), since \(d_X(x, y) \leq w\) for some \(w > 0, \forall x, y \in K(U(\lambda)).\) Thus, the condition "\(\beta > \theta\)" in Theorem 26 (Theorem 27, resp.) can be omitted.

3.2. A Set-Valued Case. In this subsection, let \(X, W, Z\) be linear normed spaces, \(A \subset X\) nonempty, and \(A \subset W, \Omega \subset Z\) nonempty and convex subsets. Let \(K: \Lambda \rightharpoonup A\) be nonempty bounded convex values. Let \(Y\) be a linear normed space, and let \(C \subset Y\) be a pointed, closed, and convex cone with int \(C \neq \emptyset\). Let \(F: X \times X \times \Omega \to Y\) be a vector-valued mapping.

For \((\lambda, \mu) \in \Lambda \times \Omega, e \geq 0\) and fixed \(e \in \text{int} C\), in this subsection we mainly consider the following parametric vector approximate equilibrium problem (PVAEP) of finding \(x(\lambda, \mu) \in K(\lambda)\) such that
\[
\bar{F}(y, x, \mu) \notin -C, \quad \forall y \in K(\lambda). \tag{34}
\]
Obviously, it is a special case of the model of finding \(x \in K(\lambda)\) such that
\[
F(y, x, \mu) \notin e + \text{int} C, \quad \forall y \in K(\lambda), \tag{35}
\]
which is the Minty-type dual problem to the Stampacchia-type primal problem (e.g., [41, Section 4]) of finding \(x \in K(\lambda)\) such that
\[
F(y, x, \mu) \notin -e - \text{int} C, \quad \forall y \in K(\lambda). \tag{36}
\]

Stability for parametric variational problems of the Minty type has not received much attention so far. Very recently, Lalitha and Bhatia [38] have studied upper and lower semicontinuity of the solutions as well as the approximate solutions to a parametric quasivariational inequality of the Minty type. Chen and Li [30] have established upper Hölder continuity of the solutions to Minty-type parametric vector quasiequilibrium problems. In this subsection, by using nonlinear scalarization technique, we will study a special Minty-type parametric vector approximate equilibrium problem (PVAEP).

To (PVAEP) combining with Proposition 2(ii), we could introduce the following equivalent scalarization equilibrium problem (SEP) of finding \(\bar{x} \in K(\lambda)\) such that
\[
\xi(\lambda, \mu) := \text{sup} \{F(y, x, \mu) - e \leq 0, \forall y \in K(\lambda)\}. \tag{37}
\]

Set \(S(e, \lambda, \mu) := \{x \in K(\lambda) \mid F(y, x, \mu, e) - e \leq 0, \forall y \in K(\lambda)\}\).

In this subsection, for fixed \(e \in \text{int} C\), we assume that \(S(e, \lambda, \mu) \neq \emptyset\) for small positive \(e, \lambda, \mu\) in a neighborhood of the considered point \((\lambda_0, \mu_0)\). In general, \(S(e, \lambda, \mu)\) may not be a singleton.

As pointed out in [15], we will show how to establish the Hölder continuity of set-valued solution mappings to parametric vector equilibrium problems without using any priori information of the solution sets, by employing nonlinear scalarization approach. In [17], we have given a positive answer to this subject. Now we give another answer herein.

For \(A, B \subset X\), the Pompeiu-Hausdorff distance between \(A\) and \(B\) is defined as
\[
H(A, B) := \max \left\{ \sup_{a \in A} d(a, B), \sup_{b \in B} d(a, B) \right\}, \tag{38}
\]
where \(d(a, B) := \inf_{b \in B} d(a, b)\) and \(d(a, b) = \inf_{a \in A} d(a, b)\).

Theorem 30. For problem (SEP) (i.e., (PVAEP)), let \(S(e, \lambda, \mu)\) be nonempty for small \(e > 0\) and \((\lambda, \mu)\) in a neighborhood of the considered point \((\lambda_0, \mu_0)\). Assume that the following conditions hold.

(i) \(K(\cdot)\) is \(l \cdot \alpha\)-Hölder continuous at \(\lambda_0\); that is, there is a neighborhood \(N\) of \(\lambda_0\) such that, for all \(\lambda_1, \lambda_2 \in N, K(\lambda_1) \subset K(\lambda_2) + \|\lambda_1 - \lambda_2\|\mathbb{B}_X\).

(ii) There is a neighborhood \(U\) of \(\mu_0\) such that, for each \(y \in K(N)\) and \(\mu \in U, F(y, \cdot, \mu)\) is \(C\)-convex on \(K(N)\); that is, for every \(x_1, x_2 \in K(\lambda)\), \(t \in [0, 1], tF(y, x_1, \mu) + (1-t)F(y, x_2, \mu) \subset F(y, tx_1 + (1-t)x_2, \mu) + C\).

(iii) For \(x, y \in K(N), F(y, x, \cdot)\) is \(\bar{F}\)-\(\delta\)-Hölder continuous on \(U\).

(iv) For \(\mu \in U\) and \(x \in K(N), F(\cdot, x, \mu)\) is \(\bar{F}\)-\(\delta\)-Hölder continuous on \(K(N)\).

Then, for any \(\bar{e} > 0, S(\bar{e})\) satisfies the following Hölder property on \([\bar{x}] + \text{co}(X) \times X \times U\):
\[
H \left( S(e_1, \lambda_1, \mu_1), S(e_2, \lambda_2, \mu_2) \right) \leq \bar{K}_1 |e_1 - e_2| + \bar{K}_2 \|\mu_1 - \mu_2\|^{\delta} + \bar{K}_3 \|\lambda_1 - \lambda_2\|^{\delta}, \tag{39}
\]
where \(\bar{K}_1, \bar{K}_2, \bar{K}_3 > 0\) and depends on \(\bar{e}, l, \alpha, \bar{F}, \beta, \bar{\delta}, \gamma, \) and so forth. Herein \(\gamma := \sup_{e \in C} \|\mathbb{E}\| \in [1/\|e\|, +\text{co} (or \gamma = 1/\gamma_{\text{max}})\]
is the Lipschitz constant of \(\xi(\lambda, \mu)\).
Proof. To apply [31, Theorem 2.1] by exchanging the roles of \(x\) and \(y\), we need to check the convexity and the Hölder continuity of \(\xi_e \circ F\). First, by virtue of the globally Lipschitz property of \(\xi_e\), on one hand, for \(x, y \in K(N), \forall \mu_1, \mu_2 \in U\),
\[
\begin{align*}
[\xi_e(F(y, x, \mu_1)) - \xi_e(F(y, x, \mu_2))] & \leq \gamma_e \left\| F(y, x, \mu_1) - F(y, x, \mu_2) \right\| \\
& \leq \gamma_e \left\| \mu_1 - \mu_2 \right\|^\beta.
\end{align*}
\]
(40)

On the other hand, for \(\mu \in U\) and \(x \in K(N), \forall y_1, y_2 \in K(N),
\[
\begin{align*}
[\xi_e(F(y_1, x, \mu)) - \xi_e(F(y_2, x, \mu))] & \leq \gamma_e \left\| F(y_1, x, \mu) - F(y_2, x, \mu) \right\| \\
& \leq \gamma_e \left\| y_1 - y_2 \right\|^\beta.
\end{align*}
\]
(41)

Letting \(f := -\xi_e \circ F\), whence the Hölder constants of [31, Theorem 2.1] are fulfilled with \(h := \gamma \tilde{H}\) and \(q := \gamma \tilde{A}\).

Second, using the monotonicity and sublinearity of \(\xi_e\), for \(y \in K(N)\) and \(\mu \in U\), \(\forall x_1, x_2 \in K(\lambda)\) and \(t \in [0, 1]\),
\[
\begin{align*}
[\xi_e(F(y, tx_1 + (1-t)x_2, \mu)) & \leq \xi_e(tF(y, x_1, \mu) + (1-t)F(y, x_2, \mu)) \\
& \leq t\xi_e(F(y, x_1, \mu)) + (1-t)\xi_e(F(y, x_2, \mu)),
\end{align*}
\]
(42)

which implies that \(\xi_e \circ F\) is convex with respect to the second argument, so \(f := -\xi_e \circ F\) is concave.

Thus, all conditions of [31, Theorem 2.1] by exchanging the roles of \(x\) and \(y\), and \(\mu\) therein are satisfied, and hence the conclusion follows.

Remark 31. When \(Y = \mathbb{R}, C = \mathbb{R}_+, \) and \(e = 1, S_e(\epsilon, \lambda, \mu) = \{x \in K(\lambda) \mid F(y, x, \lambda, \mu) - \epsilon \leq 0, \forall y \in K(\lambda)\}\), Theorem 30 becomes that of Minty type corresponding to [31, Theorem 2.1].

Theorem 30 is new in the literature. In addition, the proof approach via nonlinear scalarization is different from the ones used in previous works [26–30] for set-valued solution mappings.

Similar to the discussion of [31, Corollary 2.3], we readily get the following result when \(K(\lambda) \equiv K, \forall \lambda \in \Lambda\).

Corollary 32. For problem (SEP), assume that \(K(\lambda) = K\), a nonempty bounded convex subset of \(A\) and \(S_e(\epsilon, \mu)\) is nonempty for small \(\epsilon > 0\) and \(\mu\) in a neighborhood of the considered point \(\mu_0\). Assume further that

(i) there is a neighborhood \(U\) of \(\mu_0\) such that, for each \(y \in K\) and \(\mu \in U\), \(F(y, \cdot, \mu)\) is \(C\)-convex on \(K\);

(ii) for \(x, y \in K\), \(F(y, x, \cdot)\) is \(H\)-Hölder continuous on \(U\).

Then, for any \(\tilde{e} > 0\), \(S_e\) satisfies the following Hölder property on \([\tilde{e}, +\infty] \times U\):
\[
H(S_e(\epsilon_1, \mu_1), S_e(\epsilon_2, \mu_2)) \leq \tilde{K}_1 |\epsilon_1 - \epsilon_2| + \tilde{K}_2 \|\mu_1 - \mu_2\|^\beta,
\]
(43)

where \(\tilde{K}_1, \tilde{K}_2 > 0\) and depends on \(\tilde{e}, H, \beta, \gamma_e\), and so forth. Herein \(\gamma_e\) is the Lipschitz constant of \(\xi_e\) on \(Y\).

4. Conclusions

In this paper, motivated by the work of Tammer and Zălinescu [8], we deduce the globally Lipschitz property of \(\xi_e\) in linear normed spaces via the primal space approach. This is different from the dual space approach adopted by our previous works [15, 16]. The equivalence between them is established, and primal–dual interpretations of both expressions for globally Lipschitz constants are explained. Furthermore, exact characterizations to the globally Lipschitz property for general Gerstewitz function \(q_{\mu_0}\), are discussed. We mention that the expression for calculating Lipschitz constant is given when the ordering cone is polyhedral. As simple applications, Hölder continuity of solutions for parametric vector equilibrium problems is also showed. Besides this, the globally Lipschitz property of \(\xi_q\) and \(q_{\mu_0}\) seems to have many potential applications, for example, the stability and sensitivity analysis of vector equilibrium problems [15–17] and optimality conditions for vector optimization problems [8, 14]. For more applications, we will exploit in future research.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

Acknowledgments

This research was supported partially by the National Natural Science Foundation of China (Grant nos. 11301567 and 11171362) and the Fundamental Research Funds for the Central Universities (Grant no. CQDXWL-2012-010). The authors thank the anonymous referees for their valuable comments and suggestions, which helped to improve the paper.

References

Submit your manuscripts at http://www.hindawi.com