Research Article

On the Menelaus and Ceva 6-Figures in the Fibered Projective Planes

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The fibered versions of Menelaus and Ceva 6-figures in the fibered projective plane are given and the conditions to the fibered versions of Menelaus and Ceva 6-figures in the fibered projective plane with base plane, that is, projective plane, are determined.

1. Introduction

In 1965, Zadeh [1] introduced the concept of fuzzy set. Since then, research on the theory and applications of fuzzy sets has been growing steadily. Several basic geometric concepts have been considered by various authors in a fuzzy context [2–4]. An attempt to develop a kind of fuzzy projective geometry deduced from fuzzy vector space was made by Kuijken et al. [5]. This provided a link between the fuzzy versions of classical theories that are very closely related. In these papers, for the most suitable definition of fuzzy geometry, they encountered the following problem. There are two possibilities to fuzzify the points of geometry. Either one assigns a unique degree of membership to each of them or one assigns several degrees of membership to all of them. They explored the second possibility, and it turns out that the geometric structures involved are much richer and defined fibered projective planes and looked at fibered versions of theorems of Desargues and Pappus [6]. The role of the triangular norm in the theory of fibered projective planes and fibered harmonic conjugates and a fibered version of Reidemeister's condition were given in [7]. The authors considered different triangular norms in the theory of fibered projective planes. When the minimum operator is used, this implies that if the underlying projective plane is finite, then a finite number of f -points generate a finite fibered projective plane. However, for other triangular norms, the fibered plane may be infinite. For instance, if we take $a \wedge b = ab$,

the ordinary product, then we see that, as soon as at least one membership degree α is different from 1, we obtain infinitely many membership degrees and therefore infinitely many f -points. Another observation to be made is that not all triangular norms are suitable here. For instance, if we take the Lukasiewicz norm, given by $a \wedge b = \max\{0, a + b - 1\}$, then no point can have a membership degree different from 1. Hence a fibered projective plane with the Lukasiewicz triangular norm is equivalent to a crisp projective plane.

Kaya and Çiftçi introduced the 6-figures of Menelaus and Ceva in Moufang projective planes in [8]. In 1992, in order to state the theorems of Menelaus and Ceva, Klamkin and Liu needed the notion of signed distance in projective plane $\mathbb{P}_2(\mathbb{R})$. In 2008, Kelly [9] generalized the Klamkin and Liu [10] result to projective planes $\mathbb{P}_2(\mathbb{F})$, where $\mathbb{F}$ is the field of characteristic not equal to two. But, in an arbitrary field, distance is not defined. Thus he used the scalars that are associated with the cross-ratio to prove these theorems in projective planes $\mathbb{P}_2(\mathbb{F})$.

In this paper, since the definition of cross-ratio has not extended to fibered projective plane yet, we propose to contribute to fuzzy geometry by looking at the fibered version of Menelaus and Ceva 6-figures in the fibered projective plane which are given and the conditions to the fibered version of Menelaus and Ceva 6-figures in the fibered projective plane with base plane, that is, projective plane, are determined. We believe that such a concept will play an important role to establish a genuine fuzzy geometry.

2. Preliminaries

We first recall some basic notions from fuzzy set theory and fibered geometry. We denote by $\wedge$ a triangular norm on the (real) unit interval $[0, 1]$, that is, a symmetric and associative binary operator satisfying $(a \wedge b) \leq (c \wedge d)$ whenever $a \leq c$ and $b \leq d$, and $a \wedge 1 = a$, for all $a, b, c, d \in [0, 1]$.

Definition 1 (see [6]). Let $\mathcal{P} = (P, B, \circ)$ be any projective plane with point set P and line set B ; that is, P and B are two disjoint sets endowed with a symmetric relation $\circ$ (called the *incidence relation*) such that the graph $(P \cup B, \circ)$ is a bipartite graph with classes P and B and such that two distinct points p, q in $\mathcal{P}$ are incident with exactly one line; every two distinct lines L, M are incident with exactly one point, and every line is incident with at least three points. A set S of *collinear* points is a subset of P , each member of which is incident with a common line L . Dually, one defines a set of *concurrent* lines. We now define fibered points and fibered lines, which are briefly called *f*-points and *f*-lines.

Definition 2 (see [6]). Suppose that $a \in P$ and $\alpha \in]0, 1]$. Then an *f*-point (a, α) is the following fuzzy set on the point set P of $\mathcal{P}$:

$$(a, \alpha) : P \longrightarrow [0, 1] : \begin{cases} a \longrightarrow \alpha, \\ x \longrightarrow 0 \quad \text{if } x \in P \setminus \{a\}. \end{cases} \quad (1)$$

Dually, one defines in the same way the *f*-line (L, β) for $L \in B$ and $\beta \in]0, 1]$. The real number α above is called the membership degree of the *f*-point (a, α) , while the point a is called the base point of it, the same for *f*-lines.

Definition 3 (see [6]). The two *f*-lines (L, α) and (M, β) , with $\alpha \wedge \beta > 0$, intersect in the unique *f*-point $(L \cap M, \alpha \wedge \beta)$. Dually, the *f*-points (a, λ) and (b, μ) , with $\lambda \wedge \mu > 0$, span the unique *f*-line $(\langle a, b \rangle, \lambda \wedge \mu)$.

Definition 4 (see [6]). A (nontrivial) fibered projective plane $\mathcal{FP}$ consists of a set FP of *f*-points of $\mathcal{P}$ and a set FB of *f*-lines of $\mathcal{P}$ such that every point and every line of $\mathcal{P}$ are the base point and base line of at least one *f*-point and *f*-line, respectively (with at least one membership degree different from 1), and such that $\mathcal{FP} = (FP, FB)$ is closed undertaking intersections of *f*-lines and spans of *f*-points. Finally, a set of *f*-points are called *collinear* if each pair of them span the same *f*-line. Dually, a set of *f*-lines are called *concurrent* if each pair of them intersects in the same *f*-point.

Theorem 5 (see [6]). Suppose that one has a fibered projective plane $\mathcal{FP}$ with base plane $\mathcal{P}$, that is, Desarguesian. Choose the three *f*-points (a_1, α_1) , (a_2, α_2) , and (a_3, α_3) in $\mathcal{FP}$ with noncollinear base points, and the three other *f*-points (b_1, β_1) , (b_2, β_2) , and (b_3, β_3) with noncollinear base points, such that the lines $\langle a_i, b_i \rangle$, for $i \in \{1, 2, 3\}$, are concurrent in a point p of $\mathcal{P}$, with $a_i \neq b_i \neq p \neq a_i$. Then the three *f*-lines $(\langle a_i, a_j \rangle, \alpha_i \wedge \alpha_j)$ and $(\langle b_i, b_j \rangle, \beta_i \wedge \beta_j)$ (for $i \neq j$ and $i, j \in \{1, 2, 3\}$) intersect in three collinear *f*-points.

3. Fibered Version of Menelaus and Ceva 6-Figures

The Alexandrian Greek mathematician Menelaus and the seventeenth-century Italian mathematician Ceva are invariably mentioned together. Menelaus' theorem, which involves a test for the collinearity of three points, and Ceva's theorem, which involves a test for the concurrency of three lines, are frequently called the twin theorem. These theorems should have been discovered together, and it is not insignificant that such a long period separates Menelaus and Ceva. During the 1500 years that separate the two there was little development in mathematics. About the year A.D. 100, Menelaus of Alexandria extended a then well-known lemma to spherical triangles in his *Sphaerica*, the high point of Greek trigonometry. It is this lemma for the plane that today bears the name of Menelaus. Of the Ceva brothers, the lesser known Tommaso wrote on the cycloid while Giovanni resurrected the forgotten Menelaus' theorem and published it in 1678 along with the twin theorem now known as Ceva's theorem.

We now recall some observations regarding the triangular norm and the Menelaus and Ceva 6-figures.

In [7], the following theorems give an answer to the question, when does a finite fibered projective plane exist given a particular triangular norm?

Theorem 6 (see [7]). Let $\mathcal{P}$ be a finite crisp projective plane and $\wedge$ a triangular norm. Then there exists some finite nontrivial fibered projective plane $\mathcal{FP}$ with base plane $\mathcal{P}$ if and only if there exists an idempotent element $\alpha \in]0, 1]$; that is, $\alpha \wedge \alpha = \alpha$.

Theorem 7 (see [7]). Let $\mathcal{P}$ be an arbitrary crisp projective plane and $\wedge$ a triangular norm. Then there exists some nontrivial fibered projective plane $\mathcal{FP}$ with base plane $\mathcal{P}$ if and only if one of the following holds:

- (i) there exists an idempotent element $\alpha \in]0, 1]$;
- (ii) there exists an element $\alpha \in [0, 1[$, with the property that $\beta \wedge \beta' > \alpha$, for all $\beta, \beta' \in]\alpha, 1]$.

In [7], the definition and theorems related to *f*-harmonic conjugate in a fibered projective plane whose base plane satisfies Fano's axiom are given.

Definition 8 (see [7]). Suppose that one has a fibered projective plane $\mathcal{FP}$ with base projective plane $\mathcal{P}$. Choose the four *f*-points (a_1, α_1) , (a_2, α_2) , (a_3, α_3) , and (a_4, α_4) in $\mathcal{FP}$, none of the three base points of which are collinear. These *f*-points are called *f*-vertices. The configuration that consists of these four *f*-points, the six *f*-lines $(A_{\{i,j\}}, \beta_{\{i,j\}}) = (\langle a_i, a_j \rangle, \alpha_i \wedge \alpha_j)$, for $i \neq j$, $i, j \in \{1, 2, 3, 4\}$ (which we call *f*-sides), and the three *f*-points $(A_{\{i,j\}} \wedge A_{\{k,l\}}, \alpha_i \wedge \alpha_j \wedge \alpha_k \wedge \alpha_l)$, with $\{i, j, k, l\} = \{1, 2, 3, 4\}$ (the *f*-diagonal points), is called an *f*-complete quadrangle.

Theorem 9 (see [7]). Suppose that one has a fibered projective plane $\mathcal{FP}$ with base plane $\mathcal{P}$. Let a, b , and c be three collinear points in $\mathcal{P}$ and let (a, α) , (b, β) , (c, γ) be three *f*-points of $\mathcal{FP}$. Suppose that the point d on $\langle a, b \rangle$ obtained by intersecting

$\langle a, b \rangle$ with the join of the two diagonal points different from c in any complete quadrangle, where a, b are vertices and c is a diagonal point, is independent of the chosen quadrangle. Then the f -point (d, δ) obtained by intersecting $(\langle a, b \rangle, \alpha \wedge \beta)$ with the f -join of the two f -diagonal points different from (c, γ) in any f -complete quadrangle, where $(a, \alpha), (b, \beta)$ are f -vertices and (c, γ) is an f -diagonal point, is independent of chosen f -complete quadrangle.

The f -point (d, δ) of previous theorem, if it exists, is called the fourth f -conjugate to $((a, \alpha), (b, \beta), (c, \gamma))$.

Theorem 10 (see [7]). Given a triangular norm $\wedge$, then $\wedge$ is the minimum operator if and only if, in any fibered projective plane $\mathcal{FP}$, for every quadruple $((a, \alpha), (b, \beta), (c, \gamma), (d, \delta))$ of f -points with collinear base points, the f -point (d, δ) is the fourth f -conjugate to $((a, \alpha), (b, \beta), (c, \gamma))$ whenever (c, γ) is the fourth f -conjugate to $((a, \alpha), (b, \beta), (d, \delta))$.

The original definitions of Menelaus and Ceva 6-figures are given in [11].

Definition 11 (see [8]). Let $\mathcal{P}$ be a projective plane. A 6-figure in $\mathcal{P}$ is a sequence of six distinct points $(a_1 a_2 a_3, b_1 b_2 b_3)$ such that $a_1 a_2 a_3$ constitutes a nondegenerate triangle with $b_1 \in \langle a_2, a_3 \rangle$, $b_2 \in \langle a_1, a_3 \rangle$, $b_3 \in \langle a_1, a_2 \rangle$. The points $a_1, a_2, a_3, b_1, b_2, b_3$ are called vertices of this 6-figure. Such a configuration is said to be a Menelaus 6-figure or a Ceva 6-figure if b_1, b_2 , and b_3 are collinear or if $\langle a_1, b_1 \rangle, \langle a_2, b_2 \rangle, \langle a_3, b_3 \rangle$ are concurrent, respectively.

We now consider some classical configurations and theorems and extend them to fibered projective planes, for satisfying one of the conditions (i) and (ii) in Theorem 7, suitably.

Definition 12. Let $\mathcal{FP}$ be a fibered projective plane with base plane $\mathcal{P}$. Choose three f -points (a_i, α_i) , $i \in \{1, 2, 3\}$, in $\mathcal{FP}$ with noncollinear base points and the other three f -points (b_k, β_k) , $k \in \{1, 2, 3\}$, with $(b_k, \beta_k) \in (\langle a_i, a_j \rangle, \alpha_i \wedge \alpha_j)$ for $i \neq j \neq k$, $\{i, j, k\} = \{1, 2, 3\}$. If the f -points (b_k, β_k) are f -collinear, the configuration that consists of these six f -points is called an f -Menelaus 6-figure. It is called f -Menelaus line spanned with f -points (b_k, β_k) for $k = \{1, 2, 3\}$.

Theorem 13. Suppose that one has a fibered projective plane $\mathcal{FP}$ with base plane $\mathcal{P}$. Let a_1, a_2, a_3 be three noncollinear points in $\mathcal{P}$ and let $(a_1, \alpha_1), (a_2, \alpha_2), (a_3, \alpha_3)$ be three f -points of $\mathcal{FP}$. Suppose that the point b_3 on $\langle a_1, a_2 \rangle$ is obtained by intersecting $\langle a_1, a_2 \rangle$ with the join of two chosen points b_1 and b_2 where b_1 on $\langle a_2, a_3 \rangle$ and b_2 on $\langle a_1, a_3 \rangle$. Then the f -point (b_3, β_3) obtained by intersecting $(\langle a_1, a_2 \rangle, \alpha_1 \wedge \alpha_2)$ with the f -join of the two f -points (b_1, β_1) and (b_2, β_2) , where $(b_1, \beta_1), (a_2, \alpha_2), (a_3, \alpha_3)$ and $(b_2, \beta_2), (a_1, \alpha_1), (a_3, \alpha_3)$ are f -collinear, is independent of the chosen f -points (b_1, β_1) and (b_2, β_2) .

Proof. Since the three f -points $(a_1, \alpha_1), (a_3, \alpha_3), (b_2, \beta_2)$ and the three f -points $(a_2, \alpha_2), (a_3, \alpha_3), (b_1, \beta_1)$ are f -collinear, $\alpha_1 \wedge \alpha_3 = \alpha_1 \wedge \beta_2 = \alpha_3 \wedge \beta_2$ and $\alpha_2 \wedge \alpha_3 = \alpha_2 \wedge \beta_1 = \alpha_3 \wedge \beta_1$.

One calculates that $\beta_3 = \alpha_1 \wedge \alpha_2 \wedge \alpha_3^2$, and hence the f -point (b_3, β_3) is independent of the choice of the f -points (b_1, β_1) and (b_2, β_2) . $\square$

Theorem 14. Suppose that one has a fibered projective plane $\mathcal{FP}$ with base plane $\mathcal{P}$. Choose the three f -points $(a_1, \alpha_1), (a_2, \alpha_2)$, and (a_3, α_3) in $\mathcal{FP}$, neither of the three base points of which is collinear. Let the f -point (b_3, β_3) be obtained by intersecting $(\langle a_1, a_2 \rangle, \alpha_1 \wedge \alpha_2)$ with the f -join of the two f -points (b_1, β_1) and (b_2, β_2) , where $(b_1, \beta_1), (a_2, \alpha_2), (a_3, \alpha_3)$ and $(b_2, \beta_2), (a_1, \alpha_1), (a_3, \alpha_3)$ are f -collinear. Then the configuration that consists of the six f -points $(a_i, \alpha_i), (b_i, \beta_i)$, $i \in \{1, 2, 3\}$, is an f -Menelaus 6-figure.

Proof. Since the three f -points $(b_1, \beta_1), (b_2, \beta_2)$, and (b_3, β_3) are f -collinear, from Definition 12, the configuration that consists of the six f -points $(a_i, \alpha_i), (b_i, \beta_i)$, $i \in \{1, 2, 3\}$, is an f -Menelaus 6-figure. $\square$

Theorem 15. Let $\mathcal{FP}$ be a fibered projective plane with base plane $\mathcal{P}$. Choose three f -points (a_i, α_i) , $i \in \{1, 2, 3\}$, in $\mathcal{FP}$ with noncollinear base points and with $(b_j, \beta_j) \in (\langle a_i, a_k \rangle, \alpha_i \wedge \alpha_k)$ for $i \neq j \neq k$, $\{i, j, k\} = \{1, 2, 3\}$. If b_1, b_2 , and b_3 in $\mathcal{P}$ are collinear, then the three f -points (b_j, β_j) , $j \in \{1, 2, 3\}$, are f -collinear if and only if $\alpha_1^2 \wedge \alpha_2 \wedge \alpha_3 = \alpha_1 \wedge \alpha_2^2 \wedge \alpha_3 = \alpha_1 \wedge \alpha_2 \wedge \alpha_3^2$.

Proof. A configuration is picked such that three f -points $(a_1, \alpha_1), (a_2, \alpha_2)$, and (a_3, α_3) and $(b_j, \beta_j) \in (\langle a_i, a_k \rangle, \alpha_i \wedge \alpha_k)$ for $i \neq j \neq k$, $\{i, j, k\} = \{1, 2, 3\}$. Suppose the three f -points (b_j, β_j) , $j \in \{1, 2, 3\}$, are f -collinear. Since three f -points (b_j, β_j) are f -collinear and the three f -points $(a_i, \alpha_i), (a_j, \alpha_j)$, and (b_k, β_k) , for $i \neq j \neq k$, $\{i, j, k\} = \{1, 2, 3\}$, are f -collinear $\beta_i \wedge \beta_j = \beta_i \wedge \beta_k = \alpha_i \wedge \alpha_j = \alpha_i \wedge \beta_k = \alpha_j \wedge \beta_k$. Then clearly $\alpha_1^2 \wedge \alpha_2 \wedge \alpha_3 = \alpha_1 \wedge \alpha_2^2 \wedge \alpha_3 = \alpha_1 \wedge \alpha_2 \wedge \alpha_3^2$.

Conversely, if $\alpha_1^2 \wedge \alpha_2 \wedge \alpha_3 = \alpha_1 \wedge \alpha_2^2 \wedge \alpha_3 = \alpha_1 \wedge \alpha_2 \wedge \alpha_3^2$ are satisfied, $\beta_1 \wedge \beta_2 = \beta_1 \wedge \beta_3 = \beta_2 \wedge \beta_3$. Then three f -points $(b_1, \beta_1), (b_2, \beta_2)$, and (b_3, β_3) are f -collinear. $\square$

The Fibered Menelaus Condition (FMC). Let $\mathcal{FP}$ be a fibered projective plane with base plane $\mathcal{P}$. Choose three f -points (a_i, α_i) , $i \in \{1, 2, 3\}$, in $\mathcal{FP}$ with noncollinear base points and the other three f -points (b_k, β_k) , $k \in \{1, 2, 3\}$, with $(b_k, \beta_k) \in (\langle a_i, a_j \rangle, \alpha_i \wedge \alpha_j)$ for $i \neq j \neq k$, $\{i, j, k\} = \{1, 2, 3\}$. The configuration that consists of these six f -points is Menelaus 6-figure if and only if $\alpha_1^2 \wedge \alpha_2 \wedge \alpha_3 = \alpha_1 \wedge \alpha_2^2 \wedge \alpha_3 = \alpha_1 \wedge \alpha_2 \wedge \alpha_3^2$.

Theorem 16. Let $\wedge$ be a triangular norm. Let $\mathcal{FP}$ be any nontrivial fibered projective plane with base plane $\mathcal{P}$ and triangular norm $\wedge$. Then there is at least triangular norm $\wedge$ such that $\mathcal{FP}$ satisfies FMC.

Proof. When the minimum operator for $\wedge$ is used, it implies that α_1, α_2 , and α_3 are idempotent elements in Theorem 15. So FMC is satisfied. Clearly, using the operator defined by $a \wedge b = 1/2$ for all $a, b \geq 1/2$, and $a \wedge b = \min\{a, b\}$, otherwise, is a triangular norm, then also FMC is satisfied. $\square$

Pappus' theorem is an important theorem in geometry. Firstly, Kuijken and Van Maldeghem gave the fibered version of Pappus' theorem using the minimum operator for $\wedge$, [6].

It is known that Pappus' theorem can be proved by using the Menelaus theorem. Now, we give the fibered version of Pappus' theorem by using the f -Menelaus 6-figure.

Theorem 17. *Let $\mathcal{FP}$ be a fibered projective plane with base plane $\mathcal{P}$. The fibered version of the Pappus configuration is obtained by using five f -Menelaus configurations.*

Proof. We choose two triples of f -points (a_i, α_i) , (b_i, β_i) , and (c_i, γ_i) with collinear base points for $i = 1, 2$, and such that neither of the three of the base points a_1 , b_1 , a_2 , b_2 is collinear. Then three intersection f -points are $(a, \alpha) = (\langle a_2, b_1 \rangle, \alpha_2 \wedge \beta_1) \cap (\langle a_1, c_2 \rangle, \alpha_1 \wedge \gamma_2)$, $(b, \beta) = (\langle a_2, b_1 \rangle, \alpha_2 \wedge \beta_1) \cap (\langle b_2, c_1 \rangle, \beta_2 \wedge \gamma_1)$, and $(c, \gamma) = (\langle b_2, c_1 \rangle, \beta_2 \wedge \gamma_1) \cap (\langle a_1, c_2 \rangle, \alpha_1 \wedge \gamma_2)$. Let $(m_1, \mu_1) = (\langle b, c \rangle, \beta \wedge \gamma) \cap (\langle b_1, c_2 \rangle, \beta_1 \wedge \gamma_2)$, $(m_2, \mu_2) = (\langle a, c \rangle, \alpha \wedge \gamma) \cap (\langle a_2, c_1 \rangle, \alpha_2 \wedge \gamma_1)$, and $(m_3, \mu_3) = (\langle a, b \rangle, \alpha \wedge \beta) \cap (\langle a_1, b_2 \rangle, \alpha_1 \wedge \beta_2)$. If the triples of f -collinear points $\{(m_1, \mu_1), (c_2, \gamma_2), (b_1, \beta_1)\}$, $\{(m_2, \mu_2), (a_2, \alpha_2), (c_1, \gamma_1)\}$, $\{(m_3, \mu_3), (b_2, \beta_2), (a_1, \alpha_1)\}$, $\{(a_1, \alpha_1), (c_1, \gamma_1), (b_1, \beta_1)\}$, and $\{(b_2, \beta_2), (a_2, \alpha_2), (c_2, \gamma_2)\}$ are f -Menelaus 6-figure with the three f -points (a, α) , (b, β) , (c, γ) , then $\mu_1 \wedge \gamma_2 = \mu_1 \wedge \beta_1 = \gamma_2 \wedge \beta_1$, $\mu_2 \wedge \alpha_2 = \mu_2 \wedge \gamma_1 = \alpha_2 \wedge \gamma_1$, $\mu_3 \wedge \beta_2 = \mu_3 \wedge \alpha_1 = \alpha_1 \wedge \beta_2$, $\alpha_1 \wedge \gamma_1 = \alpha_1 \wedge \beta_1 = \beta_1 \wedge \gamma_1$, and $\alpha_2 \wedge \gamma_2 = \alpha_2 \wedge \beta_2 = \beta_2 \wedge \gamma_2$. It is easily seen that $\mu_1 \wedge \mu_2 = \mu_1 \wedge \mu_3 = \mu_2 \wedge \mu_3$. The triple of f -points $\{(m_1, \mu_1), (m_2, \mu_2), (m_3, \mu_3)\}$ is f -Menelaus 6-figure with the three f -points (a, α) , (b, β) , (c, γ) . $\square$

Definition 18. Let $\mathcal{FP}$ be a fibered projective plane with base plane $\mathcal{P}$. Choose three f -points (a_i, α_i) , $i \in \{1, 2, 3\}$, in $\mathcal{FP}$ with noncollinear base points and the other three f -points (b_k, β_k) , $k \in \{1, 2, 3\}$, with $(b_k, \beta_k) \in (\langle a_i, a_j \rangle, \alpha_i \wedge \alpha_j)$ for $i \neq j \neq k$, $\{i, j, k\} = \{1, 2, 3\}$. If the f -lines $(\langle a_i, b_i \rangle, \alpha_i \wedge \beta_i)$, $i = 1, 2, 3$, are f -concurrent, the configuration that consists of these six f -points is called an f -Ceva 6-figure. The intersection point of the f -lines $(\langle a_i, b_i \rangle, \alpha_i \wedge \beta_i)$, $i = 1, 2, 3$, is called f -Ceva point.

Theorem 19. *Suppose that one has a fibered projective plane $\mathcal{FP}$ with base plane $\mathcal{P}$. Let a_1, a_2, a_3 be three noncollinear points in $\mathcal{P}$ and let (a_1, α_1) , (a_2, α_2) , (a_3, α_3) be three f -points of $\mathcal{FP}$. Let points b_1 and b_2 be chosen such that b_1 on $\langle a_2, a_3 \rangle$ and b_2 on $\langle a_1, a_3 \rangle$. Suppose that the point b_3 on $\langle a_1, a_2 \rangle$ is obtained by intersecting $\langle a_1, a_2 \rangle$ with the join $(\langle a_1, b_1 \rangle \cap \langle a_2, b_2 \rangle)$ and a_3 . Then the f -point (b_3, β_3) obtained by intersecting $(\langle a_1, a_2 \rangle, \alpha_1 \wedge \alpha_2)$ with the f -join of the two f -points $(\langle a_1, b_1 \rangle, \alpha_1 \wedge \beta_1) \cap (\langle a_2, b_2 \rangle, \alpha_2 \wedge \beta_2)$ and (a_3, α_3) , where (b_1, β_1) , (a_2, α_2) , (a_3, α_3) and (b_2, β_2) , (a_1, α_1) , (a_3, α_3) are f -collinear, is independent of the chosen f -points (b_1, β_1) and (b_2, β_2) .*

Proof. Since three f -points (a_1, α_1) , (a_3, α_3) , (b_2, β_2) and three f -points (a_2, α_2) , (a_3, α_3) , (b_1, β_1) are f -collinear, $\alpha_1 \wedge \alpha_3 = \alpha_1 \wedge \beta_2 = \alpha_3 \wedge \beta_2$ and $\alpha_2 \wedge \alpha_3 = \alpha_2 \wedge \beta_1 = \alpha_3 \wedge \beta_1$. One calculates that $\beta_3 = \alpha_1 \wedge \alpha_2 \wedge \alpha_3^2$, and hence

the f -point (b_3, β_3) is independent of the chosen f -points (b_1, β_1) and (b_2, β_2) . $\square$

Theorem 20. *Let $\mathcal{FP}$ be a fibered projective plane with base plane $\mathcal{P}$. Choose three f -points (a_i, α_i) , $i \in \{1, 2, 3\}$, in $\mathcal{FP}$ with noncollinear base points and with $(b_j, \beta_j) \in (\langle a_i, a_k \rangle, \alpha_i \wedge \alpha_k)$ for $i \neq j \neq k$, $\{i, j, k\} = \{1, 2, 3\}$. If the lines $\langle a_1, b_1 \rangle$, $\langle a_2, b_2 \rangle$, and $\langle a_3, b_3 \rangle$ in $\mathcal{P}$ are concurrent, then three f -lines $(\langle a_i, b_i \rangle, \alpha_i \wedge \beta_i)$, for $i \in \{1, 2, 3\}$, are f -concurrent if and only if $\alpha_1^2 \wedge \alpha_2 \wedge \alpha_3 = \alpha_1 \wedge \alpha_2^2 \wedge \alpha_3 = \alpha_1 \wedge \alpha_2 \wedge \alpha_3^2$.*

Proof. A configuration is picked such that three f -points (a_1, α_1) , (a_2, α_2) , and (a_3, α_3) and $(b_j, \beta_j) \in (\langle a_i, a_k \rangle, \alpha_i \wedge \alpha_k)$ for $i \neq j \neq k$, $\{i, j, k\} = \{1, 2, 3\}$. Suppose that three f -lines $(\langle a_i, b_i \rangle, \alpha_i \wedge \beta_i)$, for $i \in \{1, 2, 3\}$, are f -concurrent. Then three membership degrees in f -concurrent point $\alpha_i \wedge \alpha_j \wedge \beta_i \wedge \beta_j$, $i \neq j$, $\{i, j\} = \{1, 2, 3\}$, are equal. Since three f -points $(b_j, \beta_j) \in (\langle a_i, a_k \rangle, \alpha_i \wedge \alpha_k)$, $\alpha_i \wedge \alpha_j = \alpha_i \wedge \beta_k = \alpha_j \wedge \beta_k$ for $i \neq j \neq k$, $\{i, j, k\} = \{1, 2, 3\}$, are valid, one can easily get $\alpha_1^2 \wedge \alpha_2 \wedge \alpha_3 = \alpha_1 \wedge \alpha_2^2 \wedge \alpha_3 = \alpha_1 \wedge \alpha_2 \wedge \alpha_3^2$.

Conversely, by using f -points (b_j, β_j) , (a_i, α_i) , and (a_k, α_k) that are collinear for $i \neq j \neq k$, $\{i, j, k\} = \{1, 2, 3\}$, in $\alpha_1^2 \wedge \alpha_2 \wedge \alpha_3 = \alpha_1 \wedge \alpha_2^2 \wedge \alpha_3 = \alpha_1 \wedge \alpha_2 \wedge \alpha_3^2$, it is shown that the three values $\alpha_i \wedge \alpha_j \wedge \beta_i \wedge \beta_j$, $i \neq j$, $\{i, j\} = \{1, 2, 3\}$, are equal. $\square$

Corollary 21 (the fibered Ceva condition (FCC)). *Let $\mathcal{FP}$ be a fibered projective plane with base plane $\mathcal{P}$. Choose three f -points (a_i, α_i) , $i \in \{1, 2, 3\}$ in $\mathcal{FP}$ with noncollinear base points and the other three f -points (b_k, β_k) , $k \in \{1, 2, 3\}$ with $(b_k, \beta_k) \in (\langle a_i, a_j \rangle, \alpha_i \wedge \alpha_j)$ for $i \neq j \neq k$, $\{i, j, k\} = \{1, 2, 3\}$. The configuration that consists of these six f -points is Ceva 6-figure if and only if $\alpha_1^2 \wedge \alpha_2 \wedge \alpha_3 = \alpha_1 \wedge \alpha_2^2 \wedge \alpha_3 = \alpha_1 \wedge \alpha_2 \wedge \alpha_3^2$.*

The following theorem shows that f -Ceva 6-figures can be obtained as a corollary of f -Menelaus 6-figures.

Theorem 22. *Let $\mathcal{FP}$ be a fibered projective plane with base plane $\mathcal{P}$. Choose three f -points (a_i, α_i) , $i \in \{1, 2, 3\}$, in $\mathcal{FP}$ with noncollinear base points and the other three f -points (b_k, β_k) , $k \in \{1, 2, 3\}$, with $(b_k, \beta_k) \in (\langle a_i, a_j \rangle, \alpha_i \wedge \alpha_j)$ for $i \neq j \neq k$, $\{i, j, k\} = \{1, 2, 3\}$; the three lines $\langle a_i, b_i \rangle$ are concurrent in $\mathcal{P}$. If the configuration that consists of these six f -points is f -Menelaus 6-figure, it is f -Ceva 6-figure.*

Proof. Let the configuration picked, such that the three f -points (a_1, α_1) , (a_2, α_2) , and (a_3, α_3) and $(b_j, \beta_j) \in (\langle a_i, a_k \rangle, \alpha_i \wedge \alpha_k)$ for $i \neq j \neq k$, $\{i, j, k\} = \{1, 2, 3\}$, be Menelaus 6-figure. Three membership degrees in f -intersection point of three f -lines $(\langle a_i, b_i \rangle, \alpha_i \wedge \beta_i)$ are $\alpha_i \wedge \alpha_j \wedge \beta_i \wedge \beta_j$, $i \neq j$, $\{i, j\} = \{1, 2, 3\}$. It is easily seen that these are equal. So three f -lines $(\langle a_i, b_i \rangle, \alpha_i \wedge \beta_i)$ for $i \in \{1, 2, 3\}$ are f -concurrent. $\square$

The reverse of this theorem is not true in $\mathcal{FP}$.

Fano projective plane, denoted by $PG(2, 2)$, consists of seven points and seven lines. Fano projective plane is only example that is both Menelaus 6-figure and Ceva 6-figure. Even if the base plane $\mathcal{P}$ of $\mathcal{FP}$ is Fano plane, the reverse of the process is not always valid in $\mathcal{FP}$.

Theorem 23. Let $\wedge$ be a triangular norm. Let $\mathcal{FP}$ be any nontrivial fibered projective plane with base plane $\mathcal{P}$, that is, Fano plane. Let three f -points be (a_i, α_i) , $i \in \{1, 2, 3\}$, in $\mathcal{FP}$ with noncollinear base points and the other three f -points (b_k, β_k) , $k \in \{1, 2, 3\}$, with $(b_k, \beta_k) \in (\langle a_i, a_j \rangle, \alpha_i \wedge \alpha_j)$ for $i \neq j \neq k$, $\{i, j, k\} = \{1, 2, 3\}$; the three lines $\langle a_i, b_i \rangle$ are concurrent in $\mathcal{P}$. If the configuration that consists of these six f -points is f -Ceva 6-figure, it cannot be f -Menelaus 6-figure.

Proof. The configuration is picked such that f -points $(a_1, 0.5)$, $(a_2, 0.5)$, $(a_3, 0.5)$ and $(b_1, 0.6)$, $(b_2, 0.7)$, $(b_3, 0.8)$ is f -Ceva 6-figure in $\mathcal{FP}$. But, using the minimum operator for $\wedge$, it is easily seen that the f -points $(b_1, 0.6)$, $(b_2, 0.7)$, and $(b_3, 0.8)$ are not f -collinear. $\square$

4. Conclusion

In this paper, the fibered versions of Menelaus and Ceva 6-figures in the fibered projective plane are defined. Although the conditions of the fibered versions of Menelaus and Ceva 6-figures are similar, the fibered versions of some of their properties in base projective plane cannot hold in fibered projective plane. f -Ceva 6-figure is obtained from f -Menelaus 6-figure automatically. Fano projective plane is both Menelaus 6-figure and Ceva 6-figure. But f -Menelaus 6-figure cannot be obtained from f -Ceva 6-figure in fibered projective plane with Fano plane. We have seen that the triangular norms have important role in the fiber versions of theorems related to theory.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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