Research Article

Bounded Doubly Close-to-Convex Functions

Dorina Răducanu

Faculty of Mathematics and Computer Science, Transilvania University of Brașov, Iuliu Maniu 50, 50091 Brașov, Romania

Correspondence should be addressed to Dorina Răducanu; draducanu@unitbv.ro

Received 11 May 2014; Accepted 22 July 2014; Published 18 August 2014

Academic Editor: Gerd Teschke

Copyright © 2014 Dorina Răducanu. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

We consider a new class \( CC(\alpha, \beta) \) of bounded doubly close-to-convex functions. Coefficient bounds, distortion theorems, and radius of convexity for the class \( CC(\alpha, \beta) \) are investigated. A corresponding class of doubly close-to-starlike functions \( SS(\alpha, \beta) \) is also considered.

1. Introduction

Let \( \mathcal{A} \) denote the class of functions of the form
\[
    f(z) = z + \sum_{n=2}^{\infty} a_n z^n,
\]
which are analytic in the unit disk \( \mathbb{U} = \{ z \in \mathbb{C} : |z| < 1 \} \).

Let \( \mathcal{S}^*, \mathcal{K}, \) and \( \mathcal{C} \) denote the well-known classes of starlike, convex, and close-to-convex functions, respectively. A function \( f \in \mathcal{A} \) is said to be in the class \( \mathcal{C}(\beta) \) of close-to-convex functions of order \( \beta \geq 0 \) (see [1]) if there exist \( g \in \mathcal{K} \) and \( \theta \in \mathbb{R} \) such that
\[
    \left| \arg \left( e^{i\theta} f'(z) \right) - \beta \frac{\pi}{2} \right| \leq \frac{\pi}{2} \quad (z \in \mathbb{U}).
\]

It is clear that \( \mathcal{C}(0) = \mathcal{K} \) and \( \mathcal{C}(1) = \mathcal{C} \).

Denote by \( \mathcal{B} \) the class of analytic functions \( \omega \) in \( \mathbb{U} \) with \( \omega(0) = 0 \) and such that \( |\omega(z)| < 1 \) for all \( z \in \mathbb{U} \).

Suppose that \( f \) and \( g \) are two analytic functions in \( \mathcal{U} \). The function \( f \) is said to be subordinate to the function \( g \), denoted by \( f \prec g \), if there exists a function \( \omega \in \mathcal{B} \) such that \( f(z) = g(\omega(z)), z \in \mathbb{U} \).

Let \( \mathcal{P} \) be the well-known class of analytic functions \( p \) normalized by \( p(0) = 1 \) and having positive real part in \( \mathbb{U} \).

For a fixed \( \alpha > 1/2 \) let \( \mathcal{P}_\alpha \) denote the subclass of \( \mathcal{P} \) defined by
\[
    \mathcal{P}_\alpha = \{ p \in \mathcal{P} : |p(z) - \alpha| < \alpha, z \in \mathbb{U} \}.
\]
The class \( \mathcal{P}_\alpha \) has been investigated by Goel [2] and also by Libera and Livingston [3].

It is easy to observe that when \( \alpha \to \infty \), the class \( \mathcal{P}_\alpha \) reduces to the class \( \mathcal{P} \).

For \( \alpha > 1/2 \), the function
\[
    p_\alpha(z) = \frac{1 + z}{1 - (1 - (1/\alpha)) z} \quad (z \in \mathbb{U})
\]
maps the unit disk \( \mathbb{U} \) onto the domain \( D_\alpha = \{ z \in \mathbb{U} : |z - \alpha| < \alpha \} \). It follows that a function \( p \) is in the class \( \mathcal{P}_\alpha \) if and only if \( p < p_\alpha \).

If
\[
    p_\alpha(z) = 1 + \sum_{n=1}^{\infty} P_n z^n,
\]
then it is easy to check that
\[
    P_1 = 2 - \frac{1}{\alpha}, \quad P_2 = \left( 2 - \frac{1}{\alpha} \right) \left( 1 - \frac{1}{\alpha} \right).
\]

Some properties of the functions belonging to the class \( \mathcal{P}_\alpha \) are listed in the next lemma.

Lemma 1 (see [2, 3]). Let \( p(z) = 1 + \sum_{n=1}^{\infty} P_n z^n \) be in the class \( \mathcal{P}_\alpha \) (\( \alpha > 1/2 \)). Then
\[
    |p_n| \leq 2 - \frac{1}{\alpha} \quad (n \in \{1, 2, \ldots\});
\]
\[
    \frac{1 - r}{1 + (1 - (1/\alpha)) r} \leq |p(z)| \leq \frac{1 + r}{1 - (1 - (1/\alpha)) r};
\]
\[
    \left| \frac{z p'(z)}{p(z)} \right| \leq \frac{(2 - (1/\alpha)) r}{1 - (1/\alpha) r - (1 - (1/\alpha)) r^2};
\]
for \( z \in \mathbb{U} \) and \( |z| = r < 1 \). All the inequalities are sharp.
Let $C\mathcal{C}(\alpha) (\alpha > 1/2)$ denote the class of all functions $f \in \mathcal{A}$ for which there exists a function $g \in \mathcal{K}$ such that $f'/g' < p_\alpha$, where $p_\alpha$ is given by (4), or equivalently
\[ \left| f'(z) - \alpha \right| < \alpha \quad (z \in \mathbb{U}). \] (10)

It is easy to see that when $\alpha \to \infty$ the class $C\mathcal{C}(\alpha)$ reduces to the class $\mathcal{C}$ of close-to-convex functions.

A slightly different class than the class $C\mathcal{C}(\alpha)$ was investigated in [2].

Recently, in [4] the authors considered a new class of analytic functions defined in a similar way to the class $C\mathcal{C}(\alpha)$. For fixed $\beta \geq 0$ and $\gamma \geq 0$ a function $f \in \mathcal{A}$ is called doubly close-to-convex if there exist $g \in C\mathcal{C}(\beta)$ and $\phi \in \mathcal{R}$ such that
\[ \left| \arg \left( e^{i\gamma} f'(z) / g'(z) \right) \right| \leq \gamma \frac{\pi}{2} \quad (z \in \mathbb{U}). \] (11)

Motivated by the ideas from [4] we define a new class of bounded doubly close-to-convex functions.

Definition 2. Let $\alpha > 1/2$ and $\beta > 1/2$ be fixed. A function $f \in \mathcal{A}$ is said to be in the class $C\mathcal{C}(\alpha, \beta)$ if there exist a function $g \in C\mathcal{C}(\alpha)$ such that $f'/g' < p_\beta$, where $p_\beta$ is given by (4) with $\beta$ instead of $\alpha$, or equivalently
\[ \left| f'(z) - \beta \right| < \beta \quad (z \in \mathbb{U}). \] (12)

From the above definition and the definition of the class $C\mathcal{C}(\alpha)$, it follows that $f \in C\mathcal{C}(\alpha, \beta)$ if there exist a function $h \in \mathcal{H}$ and a function $g \in \mathcal{A}$ such that $f'/g' < p_\alpha$ and $f'/g' < p_\beta$, or equivalently
\[ \left| g'(z) - \alpha \right| < \alpha, \quad \left| f'(z) - \beta \right| < \beta \quad (z \in \mathbb{U}). \] (13)

In the next lemma we prove that the new class $C\mathcal{C}(\alpha, \beta)$ is nonempty.

Lemma 3. Let $\alpha > 1/2$ and $\beta > 1/2$. Then, there exists a function $f \in C\mathcal{C}(\alpha, \beta)$.

Proof. Define the following three functions:
\[ f(z) = \int_0^z \left( 1 + e_1 u \right) \left( 1 + e_2 u \right) \left( 1 + (1+e_1)u \right) [1 - (1-(1/\alpha)) e_1 u] \] \[ \times \left[ 1 - (1 - (1/\beta)) e_2 u \right] \] \[ \times [1 - (1 - (1/\beta)) e_2 u]^{-1} du, \] (14)

\[ g(z) = \int_0^z \frac{1 + e_1 u}{(1 + e_3 u)^{2/3}} \left[ 1 - (1 - (1/\alpha)) e_1 u \right] \] \[ \times \left[ 1 - (1 - (1/\beta)) e_2 u \right]^{-1} du, \] (15)

\[ h(z) = \frac{z}{1 + e_3 z}. \]

with $z \in \mathbb{U}$ and $|e_k| = 1, k \in \{1, 2, 3\}$. Since $h \in \mathcal{H}$ (see [5]) and
\[ \frac{g'(z)}{h'(z)} = \frac{1 + e_1 z}{1 - (1 - (1/\alpha)) e_1 z}, \] (16)

it follows that $g \in C\mathcal{C}(\alpha)$. The equality
\[ \frac{f'(z)}{g'(z)} = \frac{1 + e_2 z}{1 - (1 - (1/\beta)) e_2 z} \] (17)

together with $g \in C\mathcal{C}(\alpha)$ shows that the function $f$ defined by (14) belongs to $C\mathcal{C}(\alpha, \beta)$.

In this paper we obtain distortion theorems, radius of convexity, and coefficient bounds for the class $C\mathcal{C}(\alpha, \beta)$. In the last section of the paper a corresponding class of bounded doubly close-to-starlike functions $\delta^* \delta(\alpha, \beta)$ is also considered.

2. Distortion Theorems

In this section distortion theorems for the class $C\mathcal{C}(\alpha, \beta)$ are obtained.

Theorem 4. Let $\alpha > 1/2$ and $\beta > 1/2$. If $f \in C\mathcal{C}(\alpha, \beta)$, then
\[ \frac{(1-r)^2}{(1-r)^2 [1 + (1 - (1/\alpha)) r] [1 + (1 - (1/\beta)) r]} \leq \left| f'(z) \right| \leq \frac{(1-r)^2}{(1-r)^2 [1 - (1 - (1/\alpha)) r] [1 - (1 - (1/\beta)) r]} \] for $z \in \mathbb{U}$ and $|z| = r < 1$.

Proof. Let $f \in C\mathcal{C}(\alpha, \beta)$. Then there exists $g \in C\mathcal{C}(\alpha)$ such that $f'(z) = g'(z) q(z)$, where $q$ belongs to the class $\mathcal{P}_\beta$ defined by (3) with $\beta$ instead of $\alpha$. Since $q \in \mathcal{P}_\beta$, making use of inequalities (8) from Lemma 1, we obtain
\[ \frac{1 - r}{1 + (1 - (1/\beta)) r} \leq \left| g'(z) \right| \leq \frac{1 + r}{1 - (1 - (1/\beta)) r} \] (19)

for $z \in \mathbb{U}$ and $|z| = r < 1$.

The function $g$ belongs to the class $C\mathcal{C}(\alpha)$ and thus, there exists a function $h \in \mathcal{H}$ such that $g'(z) = h'(z) p(z)$, where $p \in \mathcal{P}_\alpha$. Using once more the inequalities (8) from Lemma 1, we have
\[ \frac{1 - r}{1 + (1 - (1/\alpha)) r} \leq \left| g'(z) \right| \leq \frac{1 + r}{1 - (1 - (1/\alpha)) r} \] (20)

for $z \in \mathbb{U}$ and $|z| = r < 1$.

Moreover, $h \in \mathcal{H}$ implies that (see [5, 6])
\[ \frac{1}{1 + (1-r)^2} \leq \left| h'(z) \right| \leq \frac{1}{1 - (1-r)^2} \] (21)

for $z \in \mathbb{U}$ and $|z| = r < 1$.

Combining the inequalities (19), (20), and (21), we obtain the desired inequality (18).
Theorem 5. Let $\alpha > 1/2$ and $\beta > 1/2$. If $f \in C^{(\alpha, \beta)}$, then
\[
|f(z)| \leq 4\alpha\beta \left[ \frac{r}{1-r} + (\alpha + \beta - 1) \log(1-r) \right] \\
- \frac{\alpha\beta}{\alpha - \beta} \left\{ (2\alpha - 1)^2 \log \left[ 1 - \left(1 - \frac{1}{\alpha}\right) r \right] \\
- (2\beta - 1)^2 \log \left[ 1 - \left(1 - \frac{1}{\beta}\right) r \right] \right\},
\]
(22)

\[
|f(z)| \geq 4\alpha\beta \left[ \frac{r}{1+r} - (\alpha + \beta - 1) \log(1+r) \right] \\
+ \frac{\alpha\beta}{\alpha - \beta} \left\{ (2\alpha - 1)^2 \log \left[ 1 + \left(1 - \frac{1}{\alpha}\right) r \right] \\
- (2\beta - 1)^2 \log \left[ 1 + \left(1 - \frac{1}{\beta}\right) r \right] \right\}
\]
(23)

if $\alpha \neq \beta$ and

\[
|f(z)| \leq 4\alpha^2 \left[ \frac{r}{1-r} + (2\alpha - 1) \log(1-r) \right] \\
+ (2\alpha - 1) \left\{ \frac{r}{1 - (1 - (1/\alpha)) r} \\
- 4\alpha^2 \log \left[ 1 - \left(1 - \frac{1}{\alpha}\right) r \right] \right\},
\]
(24)

\[
|f(z)| \geq 4\alpha^2 \left[ \frac{r}{1+r} - (2\alpha - 1) \log(1+r) \right] \\
+ (2\alpha - 1) \left\{ \frac{r}{1 + (1 - (1/\alpha)) r} \\
+ 4\alpha^2 \log \left[ 1 + \left(1 - \frac{1}{\alpha}\right) r \right] \right\}
\]
(25)

if $\alpha = \beta$.

Proof. Let $f \in C^{(\alpha, \beta)}$. Integrating along the straight line segment from origin to $z = re^{i\theta}$ ($0 < r < 1$) the right-hand side of inequality (18) we obtain
\[
|f(z)| \leq \int_0^r \left| f'(p)e^{i\theta} \right| dp
\]
\[
\leq \int_0^r \frac{(1+\rho)^2}{(1-\rho)^2 [1 - (1 - (1/\alpha)) \rho] [1 - (1 - (1/\beta)) \rho]} dp,
\]
(26)

which leads to inequalities (22) and (24).

To prove the lower bound of $|f(z)|$ we proceed in the following way. Let $\delta > 0$ be the radius of the open disk contained entirely in $f(U)$. Consider $z_0$ with $|z_0| = r < 1$ such that $|f(z_0)| = \min_{|z| = r} |f(z)|$. The minimum increases with $r$ and is less than $\delta$. Hence, the linear segment $\Gamma$ which connects the origin with the point $f(z_0)$ will be covered entirely by the values of $f(z)$. Denote by $\gamma$ the arc in $U$ which is mapped by $w = f(z)$ in $\Gamma$. Making use of the left-hand side of inequality (18) we get
\[
|f(z)| \\
\geq \int_0^r \left| f'(z) \right| |dz|
\]
\[
\int_0^r \frac{(1-\rho)^2}{(1+\rho)^2 [1 - (1 - (1/\alpha)) \rho] [1 - (1 - (1/\beta)) \rho]} dp.
\]
(27)

After simple calculations we obtain the inequalities (23) and (25). Thus, the proof of our theorem is completed.

3. Radius of Convexity

In this section we obtain the radius of the disk which is mapped onto a convex domain by the functions belonging to $C^{(\alpha, \beta)}$.

Theorem 6. Let $\alpha > 1/2$ and $\beta > 1/2$. Suppose that $f \in C^{(\alpha, \beta)}$. Then, the function $f$ maps the disk $|z| < r_0$ onto a convex domain, where $r_0$ is the smallest positive root of the equation
\[
(\alpha + \beta - \alpha\beta - 1) r^4 + 4(\alpha\beta - \alpha - \beta + 1) r^3 \\
+ (10\alpha\beta - 5\alpha - 5\beta + 1) r^2 + 4\alpha\beta r - \alpha\beta = 0.
\]
(28)

Proof. Let $f \in C^{(\alpha, \beta)}$. Then, there exists $g \in C^{(\alpha)}$ such that
\[
f'(z) = q(z) g'(z) \quad (z \in U),
\]
(29)

where $q \in P_{\alpha}$. Since $g \in C^{(\alpha)}$, there exists a function $h \in \mathcal{H}$ such that
\[
g'(z) = p(z) h'(z) \quad (z \in U),
\]
(30)

where $p \in P_{\alpha}$. Moreover, since $h \in \mathcal{H}$ it follows that there exists a function $k \in \mathcal{H}$ such that
\[
k(z) = z h'(z) \quad (z \in U).
\]
(31)

Combining the equalities (29), (30), and (31), we get
\[
z f'(z) = q(z) p(z) k(z) \quad (z \in U).
\]
(32)

By taking logarithmic derivative in (32), we obtain
\[
1 + \frac{zf''(z)}{f'(z)} = \frac{zq'(z)}{q(z)} + \frac{zp'(z)}{p(z)} + \frac{z k'(z)}{k(z)}
\]
(33)

which leads to
\[
\Re \left( 1 + \frac{zf''(z)}{f'(z)} \right)
\]
\[
\geq \min \Re \left( \frac{zk'(z)}{k(z)} \right) - \max \left| \frac{zq'(z)}{q(z)} \right| - \max \left| \frac{zp'(z)}{p(z)} \right| .
\]
(34)
For $k \in S^*$, we have (see [7])
\[
\Re \left( \frac{z^k f'(z)}{k(z)} \right) \geq \frac{1-r}{1+r} \tag{35}
\]
with $z \in U$ and $|z| = r < 1$.

Since $p \in P_\alpha$ and $q \in P_\beta$, making use of inequality (9) from Lemma 1, we obtain
\[
\left| \frac{zp'(z)}{p(z)} \right| \leq \frac{2 - (1/\alpha) r}{1 - (1/\alpha) r - (1 - (1/\alpha)) r^2},
\]
\[
\left| \frac{zq'(z)}{q(z)} \right| \leq \frac{2 - (1/\beta) r}{1 - (1/\beta) r - (1 - (1/\beta)) r^2} \tag{36}
\]
with $z \in U$ and $|z| = r < 1$.

Substituting (35) and (36) in (34) we have
\[
\Re \left( 1 + \frac{zf''(z)}{f'(z)} \right) \geq \frac{1-r}{1+r} - \frac{2 - (1/\alpha) r}{1 - (1/\alpha) r - (1 - (1/\alpha)) r^2} - \frac{2 - (1/\beta) r}{1 - (1/\beta) r - (1 - (1/\beta)) r^2}. \tag{37}
\]

It follows that the function $f$ is convex whenever the expression in the right-hand side of (37) is positive. The numerator of this expression can be written as $P(r) = (1 - r)Q(r)$, where
\[
Q(r) = (\alpha + \beta - \alpha \beta - 1) r^4 + 4 (\alpha \beta - \alpha - \beta + 1) r^3 
+ (10 \alpha \beta - 5 \alpha - 5 \beta + 1) r^2 + 4 \alpha \beta r - \alpha \beta. \tag{38}
\]

We observe that $Q(0) = -\alpha \beta < 0$ and $Q(1) = 4(2\alpha - 1)(2\beta - 1) > 0$. It follows that the smallest root $r_0$ of $Q(r) = 0$ and also of $P(r) = 0$ lies between 0 and 1 and, thus, the theorem is proved.

4. Coefficient Estimates

In order to find coefficient estimates for the class $CC(\alpha, \beta)$, we will find first the coefficient estimates for the class $CC(\alpha)$.

**Theorem 7.** Let $\alpha > 1/2$. If the function $f$ given by (1) is in the class $CC(\alpha, \beta)$, then
\[
|a_n| \leq n + \left( \frac{4 - \frac{1}{\alpha} - \frac{1}{\beta}}{2} \right) \frac{n-1}{2} \tag{39}
+ \left( \frac{2 - \frac{1}{\alpha} - \frac{1}{\beta}}{2} \right) \frac{(n-1)(n-2)}{6} \quad (n \in \{2, 3, \ldots\}). \tag{40}
\]

**Proof.** Let $f \in CC(\alpha, \beta)$. Then, there exist
\[
g(z) = z + \sum_{n=2}^{\infty} b_n z^n \in \mathcal{K}(\alpha), \tag{41}
\]
\[
q(z) = 1 + \sum_{n=1}^{\infty} q_n z^n \in P_\beta
\]
such that
\[
f'(z) = g(z) q'(z) \quad (z \in U). \tag{42}
\]

Comparing the coefficients of $z^n$ on both sides of the above equality, we obtain the next relation:
\[
n a_n = n b_{n+1} + 2 b_{n} q_{n-2}
+ \cdots + (n-1) b_{n-1} q_1 \quad (n \in \{2, 3, \ldots\}). \tag{43}
\]

Since $g \in \mathcal{K}(\alpha)$ and $q \in P_\beta$, from (39) and (7), we get
\[
|b_k| \leq 1 + \left( \frac{2 \alpha - 1}{2 \alpha} \right) \frac{(k-1)}{2} \quad (k \in \{2, 3, \ldots\}), \tag{44}
\]
\[
|q_k| \leq 2 - \frac{1}{\beta} \quad (k \in \{1, 2, \ldots\}). \tag{45}
\]

Equating the coefficients of $z^n$ on both sides of (40), we find the following relation between the coefficients:
\[
a_n = n b_{n+1} + 2 b_{n} q_{n-2}
+ \cdots + (n-1) b_{n-1} q_1 \quad (n \in \{2, 3, \ldots\}). \tag{46}
\]

**Theorem 8.** Let $\alpha > 1/2$ and $\beta > 1/2$. If the function $f$ given by (1) is in the class $CC(\alpha, \beta)$, then
\[
|a_n| \leq 1 + \left( \frac{4 - \frac{1}{\alpha} - \frac{1}{\beta}}{2} \right) \frac{n-1}{2} 
+ \left( \frac{2 - \frac{1}{\alpha} - \frac{1}{\beta}}{2} \right) \frac{(n-1)(n-2)}{6} \quad (n \in \{2, 3, \ldots\}). \tag{47}
\]

**Proof.** Let $f \in CC(\alpha, \beta)$. Then, there exist
\[
g(z) = z + \sum_{n=2}^{\infty} b_n z^n \in \mathcal{K}(\alpha), \tag{48}
\]
\[
q(z) = 1 + \sum_{n=1}^{\infty} q_n z^n \in P_\beta
\]
such that
\[
f'(z) = g(z) q'(z) \quad (z \in U). \tag{49}
\]

Comparing the coefficients of $z^n$ on both sides of the above equality, we obtain the next relation:
\[
n a_n = n b_{n+1} + 2 b_{n} q_{n-2}
+ \cdots + (n-1) b_{n-1} q_1 \quad (n \in \{2, 3, \ldots\}). \tag{50}
\]

Since $g \in \mathcal{K}(\alpha)$ and $q \in P_\beta$, from (39) and (7), we get
\[
|b_k| \leq 1 + \left( \frac{2 \alpha - 1}{2 \alpha} \right) \frac{(k-1)}{2} \quad (k \in \{2, 3, \ldots\}), \tag{51}
\]
\[
|q_k| \leq 2 - \frac{1}{\beta} \quad (k \in \{1, 2, \ldots\}). \tag{52}
\]
From (47) in connection with (48), we obtain
\[ |a_n| \leq 1 + \frac{(2\alpha - 1)(n-1)}{2\alpha} + \frac{1}{n} \left[ 2 - \frac{1}{\beta} \right] \left[ 1 + \sum_{k=2}^{n-1} k \left( 1 + \frac{(2\alpha - 1)(k-1)}{2\alpha} \right) \right] \]
\[ = 1 + \frac{(2\alpha - 1)(n-1)}{2\alpha} + \frac{2\beta - 1}{2\beta} (n-1) \]
\[ + \frac{(2\alpha - 1)(2\beta - 1)}{6\alpha \beta} (n-1)(n-2) \]
(49)
which leads to inequality (44).

**5. Maximum Value of** \(|a_3 - (2/3)a_2^2|\)

The problem of finding sharp upper bounds for the functional \(|a_3 - \mu a_2^2|\) for a family of analytic functions is known as the Fekete-Szegö problem. For the classes \(\mathcal{H}\) and \(\mathcal{C}\), the following estimates are known (see, e.g., [8–11]):
\[
\max_{f \in \mathcal{H}} |a_3 - \mu a_2^2| = \max \left\{ \frac{1}{3} |\mu - 1| \right\}, \quad (50)
\]
\[
\max_{f \in \mathcal{C}} |a_3 - \mu a_2^2| = \begin{cases} 
3 - 4\mu, & 0 \leq \mu \leq \frac{1}{3} \\
\frac{4}{3}, & \frac{1}{3} \leq \mu \leq \frac{2}{3} \\
1, & \frac{2}{3} \leq \mu \leq 1.
\end{cases} \quad (51)
\]

In this section, the case \(\mu = 2/3\) of the Fekete-Szegö problem will be considered, first for the class \(\mathcal{C}\) and then, for the class \(\mathcal{C}(\alpha, \beta)\).

In order to prove our results we need the following lemma due to Keogh and Merkes [9].

**Lemma 9.** Let \(\omega(z) = e_{1}z + e_{2}z^2 + \cdots\) be in the class \(\mathcal{B}\) and let \(\lambda \in \mathbb{C}\). Then
\[
|e_{2} - \lambda e_{1}^2| \leq \max \{1, |\lambda|\}. \quad (52)
\]
Equality may be attained for \(\omega(z) = z^2\) and \(\omega(z) = z\).

**Theorem 10.** Let \(\alpha > 1/2\). If \(f \in \mathcal{C}(\alpha)\) is of the form (1), then
\[
|a_3 - \frac{2}{3}a_2^2| \leq 1 - \frac{1}{3\alpha}. \quad (53)
\]

**Proof.** Let \(f \in \mathcal{C}(\alpha)\). Then there exists \(g \in \mathcal{H}\) such that \(f'(z)/g'(z) < p_\alpha(z)\), where \(p_\alpha(z)\) is given by (4). Let \(g(z) = z + b_2z^2 + b_3z^3 + \cdots\). Define
\[
p(z) = \frac{f'(z)}{g'(z)} = 1 + p_1z + p_2z^2 + \cdots \quad (z \in \mathbb{U}). \quad (54)
\]
Since \(p < p_\alpha\), there exists \(\omega(z) = e_{1}z + e_{2}z^2 + \cdots \in \mathcal{B}\) such that
\[
p(z) = p_\alpha(\omega(z)) = 1 + p_1 e_{1}z + (p_2 e_{2} + p_3 e_{1}^2)z^2 + \cdots \quad (z \in \mathbb{U}), \quad (55)
\]
where \(p_1\) and \(p_2\) are given by (6). Combining (54) and (55), after simple calculations, we get
\[
a_2 = b_2 + \frac{p_1}{2}, \quad a_3 = b_3 + \frac{2p_2}{3} + \frac{2}{3}b_2 p_1, \quad (56)
\]
\[
p_1 = p_1 e_{1}, \quad p_2 = p_1 e_{2} + p_2 e_{1}^2, \quad (57)
\]
Substituting (6) and (57) in (56) we obtain
\[
a_2 = b_2 + \left(1 - \frac{1}{2\alpha}\right) e_{1},
\]
\[
a_3 = b_3 + \frac{2}{3} \left(2 - \frac{1}{\alpha}\right) b_2 e_{1} + \frac{1}{3} \left(2 - \frac{1}{\alpha}\right) \left[e_{2} + \left(1 - \frac{1}{\alpha}\right) e_{1}^2\right] \quad (58)
\]
so that
\[
a_3 - \frac{2}{3}a_2^2 = \left(b_3 - \frac{2}{3}b_2^2\right) + \frac{1}{3} \left(2 - \frac{1}{\alpha}\right) \left(e_{2} - \frac{1}{2\alpha} e_{1}^2\right). \quad (59)
\]
Since \(g \in \mathcal{H}\), making use of (50) with \(\mu = 2/3\), we have
\[
\left|b_3 - \frac{2}{3}b_2^2\right| \leq \frac{1}{3}. \quad (60)
\]
In virtue of Lemma 9 and taking into account that \(\alpha > 1/2\), we get
\[
\left|e_{2} - \frac{1}{2\alpha} e_{1}^2\right| \leq 1. \quad (61)
\]
Combining (59), (60), and (61), we obtain
\[
\left|a_3 - \frac{2}{3}a_2^2\right| \leq \left|b_3 - \frac{2}{3}b_2^2\right| + \frac{1}{3} \left(2 - \frac{1}{\alpha}\right) \left|e_{2} - \frac{1}{2\alpha} e_{1}^2\right| \quad (62)
\]
and, thus, the proof is completed.

It is easy to observe that when \(\alpha \to \infty\) inequality (53) reduces to \(|a_3 - (2/3)a_2^2| \leq 1\) which is the same with (51) for \(\mu = 2/3\).

**Theorem 11.** Let \(\alpha > 1/2\) and \(\beta > 1/2\). If \(f\) of the form (1) belongs to the class \(\mathcal{C}(\alpha, \beta)\), then
\[
|a_3 - \frac{2}{3}a_2^2| \leq \frac{1}{3} \left(5 - \frac{1}{\alpha} - \frac{1}{\beta}\right). \quad (63)
\]
Proof. Since \( f \in \mathcal{C}(\alpha, \beta) \), there exists \( g \in \mathcal{C}(\alpha) \) such that \( f'(z)/g'(z) \prec p(\beta)(z) \), where \( p_\beta(z) \) is given by (4) with \( \beta \) instead of \( \alpha \). Let \( g(z) = z + b_2 z^2 + b_3 z^3 + \cdots \) and \( p(z) \) defined by

\[
p(z) = \frac{f'(z)}{g'(z)} = 1 + p_1 z + p_2 z^2 + \cdots \quad (z \in U) .
\]

From \( p < p_\beta \) it follows that there exists \( \omega(z) = e_1 z + e_2 z^2 + \cdots \in \mathcal{B} \) such that \( p(z) = p_\beta(\omega(z)) \).

Using the same method as in the proof of Theorem 10, we obtain

\[
a_3 - \frac{2}{3} a_2^2 = \left( b_3 - \frac{2}{3} b_2^2 \right) + \frac{1}{3} \left( 2 - \frac{1}{\beta} \right) \left( e_2 - \frac{1}{2 \beta} e_2^2 \right) .
\]

Since \( g \in \mathcal{C}(\alpha) \), from (53), we have

\[
\left| b_3 - \frac{2}{3} b_2^2 \right| \leq 1 - \frac{1}{3 \alpha} . \tag{66}
\]

Moreover, for \( \beta > 1/2 \), we get from Lemma 9 that

\[
\left| e_2 - \frac{1}{2 \beta} e_2^2 \right| \leq 1 . \tag{67}
\]

Combining (65), (66), and (67), the inequality (63) follows.

\[ \square \]

6. Bounded Doubly Close-to-Starlike Functions

**Theorem 12.** Let \( \alpha > 1/2 \) and \( \beta > 1/2 \). Then, the following relationships hold:

\[
f(z) \in \mathcal{C}(\alpha) \quad \text{iff} \quad z f'(z) \in \mathcal{S}^* \mathcal{S}(\alpha) ,
\]

\[
f(z) \in \mathcal{C}(\alpha, \beta) \quad \text{iff} \quad z f'(z) \in \mathcal{S}^* \mathcal{S}(\alpha, \beta) ,
\]

\[
f(z) \in \mathcal{S}^* \mathcal{S}(\alpha) \quad \text{iff} \quad \int_0^1 \frac{f(t)}{t} dt \in \mathcal{C}(\alpha) ,
\]

\[
f(z) \in \mathcal{S}^* \mathcal{S}(\alpha, \beta) \quad \text{iff} \quad \int_0^1 \frac{f(t)}{t} dt \in \mathcal{C}(\alpha, \beta) .
\]

Proof. It is well known that a function \( g(z) \in \mathcal{K} \) if and only if \( zg'(z) \in \mathcal{S}^* \).

The definition of the class \( \mathcal{C}(\alpha) \) implies that \( f \in \mathcal{C}(\alpha) \) if and only if there exists \( g \in \mathcal{K} \) such that \( f'(z)/g'(z) < p_\alpha(z) \).

The relation (70) follows from

\[
\frac{zf'(z)}{zg'(z)} = \frac{f'(z)}{g'(z)} < p_\alpha(z) , \quad zg'(z) \in \mathcal{S}^* . \tag{74}
\]

In the same way, \( f \in \mathcal{C}(\alpha, \beta) \) if and only if there exists \( g \in \mathcal{K} \) such that \( f'(z)/g'(z) < p_\beta(z) \). We have

\[
\frac{zf'(z)}{zg'(z)} = \frac{f'(z)}{g'(z)} < p_\beta(z) \tag{75}
\]

and taking into account (70), the relation (71) follows.

The proofs of (72) and (73) are similar and will be omitted.

The condition (71) of Theorem 12 together with Lemma 3 and (14) shows that the function

\[
g(z) = \frac{z(1+e_1 z)(1+e_2 z)}{(1+e_2 z)^2 [1-(1-(1/\alpha))e_1 z][1-(1-(1/\beta)) e_2 z]} \quad (z \in U) , \tag{76}
\]

where \( |e_k| = 1, k \in \{1,2,3\} \) belongs to the class \( \mathcal{S}^* \mathcal{S}(\alpha, \beta) \) and, thus, this class is nonempty.

Combining Theorem 12 with Theorems 4 and 8 the next properties of the class \( \mathcal{S}^* \mathcal{S}(\alpha, \beta) \) can be easily obtained.

**Corollary 13.** Let \( \alpha > 1/2 \) and \( \beta > 1/2 \). If \( f \in \mathcal{S}^* \mathcal{S}(\alpha, \beta) \), then

\[
\frac{r(1-r)^2}{(1+r)^2 [1-(1-(1/\alpha)) r][1-(1-(1/\beta)) r]} \leq |f(z)| \leq \frac{r(1-r)^2}{(1-r)^2 [1-(1-(1/\alpha)) r][1-(1-(1/\beta)) r]} \tag{77}
\]

for \( z \in U \) and \( |z| = r < 1 \).
Corollary 14. Let $\alpha > 1/2$ and $\beta > 1/2$. If $f \in \mathcal{S}_s^*(\alpha, \beta)$ is given by (1), then
\[
|a_n| \leq n + \left(4 - \frac{1}{\alpha} - \frac{1}{\beta}\right) \frac{n(n-1)}{2} + \left(2 - \frac{1}{\alpha}\right)\left(2 - \frac{1}{\beta}\right) \frac{n(n-1)(n-2)}{6} + (2 - \frac{1}{\alpha})(2 - \frac{1}{\beta}) \frac{n(n-1)(n-2)}{6}(n \in \{2, 3, \ldots\}).
\] (78)

Making use of Theorem 12, we can also obtain an upper bound of $|a_3 - (1/2)a_2^2|$ for functions in the class $\mathcal{S}_s^*(\alpha, \beta)$.

Corollary 15. Let $\alpha > 1/2$ and $\beta > 1/2$. If $f \in \mathcal{S}_s^*(\alpha, \beta)$ is given by (1), then
\[
|a_3 - \frac{1}{2}a_2^2| \leq 5 - \frac{1}{\alpha} - \frac{1}{\beta}.
\] (79)

Proof. Let $f \in \mathcal{S}_s^*(\alpha, \beta)$. Then, from (73), the function $F(z) = z + b_2z^2 + b_3z^3 + \cdots$ given by
\[
F(z) = \int_0^z \frac{f(t)}{t} dt
\] (80)
belongs to the class $\mathcal{S}_s^*(\alpha, \beta)$. Comparing the coefficients of $z^2$ and $z^3$ on both sides of the above equality, we obtain
\[
a_2 = 2b_2, \quad a_3 = 3b_3
\] (81)
so that
\[
|a_3 - \frac{1}{2}a_2^2| = 3|b_3 - \frac{2}{3}b_2^2|.
\] (82)

Now, the inequality (79) follows as an application of Theorem 11.

Once again making use of Theorem 12, we have that $f \in \mathcal{S}_s^*(\alpha, \beta)$ if and only if
\[
\frac{zf'(z)}{f(z)} = 1 + \frac{zf''(z)}{f'(z)}(z \in \mathbb{U})
\] (83)
for some $F \in \mathcal{S}_s^*(\alpha, \beta)$. Therefore, a radius of convexity for $\mathcal{S}_s^*(\alpha, \beta)$ will correspond to a radius of starlikeness for $\mathcal{S}_s^*(\alpha, \beta)$.

The next result follows easily from Theorem 6.

Corollary 16. Let $\alpha > 1/2$ and $\beta > 1/2$. Suppose that $f \in \mathcal{S}_s^*(\alpha, \beta)$. Then, the function $f$ maps the disk $|z| < r_0 < 1$ onto a starlike domain, where $r_0$ is the smallest positive root of (28) in Theorem 6.

Conflict of Interests

The author declares that there is no conflict of interests regarding the publication of this paper.

Acknowledgment

The author gratefully thanks the referee for his/her comments.

References
