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Research Article

Nonlinear Self-Adjoint Classification of a Burgers-KdV Family of Equations

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The concepts of strictly, quasi, weak, and nonlinearly self-adjoint differential equations are revisited. A nonlinear self-adjoint classification of a class of equations with second and third order is carried out.

1. Introduction

Since Ibragimov [1] proposed an extension to the Noether theorem, overcoming the major deficiency of that result, the existence of a Lagrangian, a considerable number of researchers have been applying his ideas for constructing conservation laws for equations without classical Lagrangians.

However, the price for applying what Ibragimov proposed in [1] is the obtainment, *a priori*, of *nonlocal* conservation laws instead of local ones.

In [1, 2], it was introduced the concept of self-adjoint differential equation, latter receiving a new designation in [3, 4], where it was called *strictly self-adjoint equation*. This last one will be adopted in this work. Although such concept was not necessarily new; see [5], the works [1, 2] were the start point of an intense research in this kind of ideas, giving rise to new developments [3, 4, 6] and providing local conservation laws for equations once its symmetries are known.

One of the first papers dealing with some kind of classification was [7]. There, the authors considered a class of fourth-order evolution equations and found the self-adjoint subclasses. Then, in [8], the same class was enlarged by considering nonlinear dispersion as well as source terms.

Weak self-adjointness of some classes of equations was discussed by Gandarias and coauthors in [9–11].

In regard to third-order equations, in [12], a KdV type family was considered. However, at the time of this last reference, the general concept of nonlinear self-adjointness was not already introduced. In [13], a class of third-order dispersive equations was considered from the quasi self-adjoint point of view. More recently, in [14] a general family of dispersive evolution equations was classified with respect to quasi self-adjointness.

Recently [15], we considered a class of time dependent equations up to fifth-order and we obtained necessary and sufficient conditions for determining the nonlinearly self-adjoint subclasses. Nonlinear self-adjointness of equations up to fifth-order can also be found in [16–18].

In [19], a general class of first order (1 + 1) PDE was classified with respect to strictly and quasi self-adjointness. Later, in [20], the subclass of the Riemman, or inviscid Burgers equation, was reconsidered from the point of view of nonlinear self-adjointness. Recently, in the paper [21], the last class was studied incorporating damping and conservation laws were established.

The number of works dealing with systems and self-adjointness is reduced compared with those ones considering scalar equations. To cite some of them; for instance, up to our knowledge, the first paper dealing with some self-adjointness and systems of PDEs was [22]. Nonlinear self-adjointness of

a system of coupled modified KdV equations was studied in [23]. Further examples can be found in [4].

In [24] quasi self-adjointness of a class of wave equation was considered. Sometime ago, in [25], a class of wave equation with dissipation was considered from the nonlinear self-adjoint point of view.

The vast majority of the papers deal with (1 + 1) equations. However, some results considering PDEs with more independent variables have been communicated in the literature. In [26–28], diffusion equations with more than one spatial dimensions were considered. A generalization of Kuramoto-Sivashinsky equation was discussed in [29]. In [30] an extension of KdV equation, the so-called Zakharov-Kuznetsov equation, was studied. All of these papers dealt with nonlinear self-adjointness.

The concepts of self-adjoint differential equations will be better discussed in Section 2. In fact, this is a threefold purpose paper: the first is to provide a review on some works dealing with conservation laws and using the concepts introduced in [1–4, 6]. The second one is to explore the concepts of strict, quasi, weak, and nonlinear self-adjoint differential equations, as the reader can check in Section 2. Although these concepts are commonly, and in fact, powerfully employed for constructing local conservation laws, such concepts have interest by themselves. Finally, it is common to classify equations under certain properties; see, for instance, [31, 32]. Then, in this work we consider, in Section 4, a nonlinear self-adjoint classification of the equation

$$\begin{aligned} u_t &= r(x,t,u)\,u_{xxx} + s(x,t,u)\,u_{xx} \\ &\quad + f(x,t,u)\,u_x + h(x,t,u). \end{aligned} \tag{1}$$

Such equation includes a great number of important equations in mathematical physics. To cite a few number of them, we mention KdV, Burgers, Burgers-KdV, and Riemman equations. More equations belonging to this class will be considered in the next sections.

Moreover, the self-adjoint classification carried out here will be used in [33] for constructing local conservation laws for equations without Lagrangians.

2. Preliminaries

Before presenting the procedure, it is convenient to leave clear that in the present paper we only consider scalar differential equations. In the current section, unless it is explicitly announced, $x = (x^1, \ldots, x^n)$ is an independent variable, while $u = u(x)$ is a dependent one. The set of first order derivatives of u is denoted by $u_{(1)}$ and equal convention is employed for referring to higher order derivatives; for example, $u_{(k)}$ means the set of kth derivatives of u.

We assume the summation over the repeated indices. By differential functions we mean locally analytic functions of a finite number of variables x, u and u derivatives. The highest order of derivatives appearing in a differential function is called its order. The vector space of all differential functions of finite order is denoted by $\mathscr{A}$.

Let us now show the algorithm for constructing conservation laws. Given a PDE

$$F = F\left(x, u, u_{(1)}, \ldots, u_{(m)}\right) = 0, \quad F \in \mathscr{A}. \tag{2}$$

Step 1. We construct the formal Lagrangian $\mathscr{L} = vF$.

Step 2. From the Euler-Lagrange equations, the following system is obtained:

$$F\left(x, u, u_{(1)}, \ldots, u_{(m)}\right) = 0, \tag{3}$$

$$F^*\left(x, u, v, u_{(1)}, v_{(1)}, \ldots, u_{(m)}, v_{(m)}\right) = 0, \tag{4}$$

where the second equation of the system (3)-(4) is called adjoint equation to $F = 0$.

Step 3. A conserved vector for such system is $C = (C^i)$, where

$$\begin{aligned} C^i &= \xi^i \mathscr{L} + W\left[\frac{\partial \mathscr{L}}{\partial u_i} - D_j\left(\frac{\partial \mathscr{L}}{\partial u_{ij}}\right) + D_j D_k\left(\frac{\partial \mathscr{L}}{\partial u_{ijk}}\right) - \cdots\right] \\ &\quad + D_j(W)\left[\frac{\partial \mathscr{L}}{\partial u_{ij}} - D_k\left(\frac{\partial \mathscr{L}}{\partial u_{ijk}}\right) + \cdots\right] + \cdots \end{aligned} \tag{5}$$

and $W = \eta - \xi^i u_i$.

Of course, it is clear that components (5) depend explicitly on the new variable v, which is not a "natural" variable from the original equation. For this reason, the conservation laws provided by the developments [1] are, *a priori*, nonlocal conservation laws and, consequently, the conserved vectors are nonlocal ones.

As it was previously pointed out at the beginning, the points related with conservation laws will be retaken soon, in [33]. However, for those more anxious, we invite them to consult the books [34–37] for the discussion between symmetries and conservation laws. We also guide the interested readers to [38–42] for discussions on conservation laws.

The question is: would it be possible to construct, from the nonlocal conservation laws (5), local ones? This point is essentially related with: would it be possible to replace the nonlocal function v by an expression depending of x, u and eventually derivatives of u? This lead us to the main subject of the paper: "self-adjointness", which will just be revisited. We firstly begin with the following.

Definition 1. Equation (3) is said to be strictly self-adjoint if the equation obtained from the adjoint equation (4) by the substitution $v = u$ is identical with the original equation (3); that is,

$$F^*\big|_{v=u} = \lambda(x, u, \ldots)\,F, \tag{6}$$

for some $\lambda \in \mathscr{A}$.

This concept was first introduced in [1, 2] as self-adjoint differential equations. More recently, in [3, 4], Ibragimov himself changed the designation and he referred to this

concept as *strictly self-adjoint* differential equation. Then we use the last definition proposed by Ibragimov.

Some examples are now welcomed. We start with the following.

Example 2. Consider the Riemman or inviscid Burgers equation:

$$u_t + a(u)\,u_x = 0, \tag{7}$$

where we assume $a'(u) \neq 0$. In this case, the adjoint equation to (7) is

$$v_t + a(u)\,v_x = 0. \tag{8}$$

Clearly, setting $v = u$ into (8), (7) is obtained. Therefore, Riemman equation is strictly self-adjoint. For further details, see [19, 20].

Example 3. Consider now KdV equation:

$$u_t = u_{xxx} + uu_x. \tag{9}$$

Its adjoint equation is

$$v_t = v_{xxx} + uv_x. \tag{10}$$

Then, setting $v = u$ into (10), one obtains (9). Therefore, KdV equation is strictly self-adjoint. For further details, see [1, 16].

Example 4. Consider Harry-Dym equation:

$$u_t = u^3 u_{xxx}. \tag{11}$$

Its adjoint equation is given by [14, 15, 43]

$$\begin{aligned} v_t = {} & u^3 v_{xxx} + 9\left(u^2 v_x + 2uvu_x\right)u_{xx} \\ & + 9u^2 u_x v_{xx} + 18uv_x u_x^2 + 6vu_x^3. \end{aligned} \tag{12}$$

Equation (12) is not strictly self-adjoint, as it can easily be checked directly from (12) setting $v = u$ or consulting [43].

In [43] the concept of quasi self-adjoint differential equation was proposed, which is recalled at the following.

Definition 5. Equation (3) is said to be quasi self-adjoint if the equation obtained from adjoint equation (4) by the substitution $v = \phi(u)$, for a certain ϕ such that $\phi(u) \neq 0$, is identical with the original equation (3); that is,

$$F^*\big|_{v=\phi(u)} = \lambda(x, u, \ldots)\,F, \tag{13}$$

for some $\lambda \in \mathscr{A}$.

Originally, the notion of quasi self-adjointness was slightly different. In fact, in its first formulation [43], it was required that function ϕ satisfies condition $\phi'(u) \neq 0$. However, such condition was relaxed in [4] and we adopt here the last Ibragimov's formulation.

We now analyse our previous examples taking Definition 5 into account.

Example 6. Since (7) is strictly self-adjoint, consequently, it is also quasi self-adjoint. However, let $\phi = \phi(u)$ be a smooth function such that $\phi''(u) \neq 0$. Substituting $v = \phi(u)$ into the left side of (8) the following is obtained:

$$v_t + a(u)v_x\big|_{v=\phi(u)} = \phi'(u)\left[u_t + a(u)\,u_x\right]. \tag{14}$$

This shows that (7) is quasi self-adjoint admitting an arbitrary nonlinear substitution $\phi = \phi(u)$. For further details and discussion, see [19, 20].

Example 7. Consider KdV equation (9) again. Substituting $v = \phi(u)$ into (10), we obtain $v = c_1 u + c_2$, where c_1 and c_2 are arbitrary constants. This lead us to two different substitutions: $v_1 = u$ and $v_2 = 1$. Therefore, KdV equation is quasi self-adjoint. For further details, see [4].

Example 8. Consider Harry-Dym equation (11). As it was already pointed out, it is not strictly self-adjoint. However, in [43] Ibragimov showed that the adjoint equation to (11) is equivalent to the original one if the substitution is taken as follows:

$$v_1 = \frac{1}{u^3}. \tag{15}$$

In [14], Torrisi and Tracinà discovered the new substitution:

$$v_2 = \frac{1}{u^2}. \tag{16}$$

Therefore, (11) is quasi self-adjoint.

Our next definition was formulated by Gandarias in [6].

Definition 9. Equation (3) is said to be weak self-adjoint if the equation obtained from adjoint equation (4) by the substitution $v = \phi(x, u)$ for a certain function ϕ such that $\phi_u \neq 0$ and $\phi_{x^i} \neq 0$, for some x^i, is identical with the original equation (3); that is,

$$F^*\big|_{v=\phi(x,u)} = \lambda(x, u, \ldots)\,F, \tag{17}$$

for some $\lambda \in \mathscr{A}$.

While strictly self-adjointness implies quasi self-adjointness, weak self-adjointness does not imply neither strictly nor quasi self-adjointness. In fact, Definition 9 is stronger than both Definitions 1 and 5. We illustrate now this fact.

Example 10. Consider again (7). Although it is clear that such equation is strictly and quasi self-adjoint, neither $v = u$ nor $v = \phi(u)$ are substitutions satisfying Definition 9. However, let $\varphi = \varphi(z)$ be a smooth real valued function and define $v = \varphi(x - ta(u))$. Then, substituting this v into (10), we arrive at

$$\begin{aligned} & v_t + a(u)\,v_x\big|_{v=\varphi(x-ta(u))} \\ & \quad = -a\varphi'(x - ta(u)) + a\varphi(x - ta(u)) \equiv 0. \end{aligned} \tag{18}$$

Thus, if $\varphi' \neq 0$, it means that $v = \varphi(x - ta(u))$ is a substitution satisfying Definition 9.

Example 11. KdV equation (9) is weak self-adjoint. In fact, one can take the substitution $v = x + tu$ and easily check that (10) is equivalent to (9) with this substitution. For further details, see [4, 16].

Example 12. Considering Harry-Dym equation, it is now clear that it is not strictly self-adjoint, but it is quasi self-adjoint. In [15], we proved that adjoint equation (12) to (11) is also equivalent to itself by considering the substitutions:

$$v_3 = \frac{x}{u^3}, \qquad v_4 = \frac{x^2}{u^3}. \tag{19}$$

While substitutions (15) and (16) show that (11) is quasi self-adjoint, substitutions (19) are enough to prove weak self-adjointness. On the other hand, neither (15) nor (16) are substitutions that satisfy what is required in Definition 9.

Finally, we arrived at the state of the art in this field: nonlinear self-adjointness.

Definition 13. Equation (3) is said to be nonlinearly self-adjoint if the equation obtained from the adjoint equation (4) by the substitution $v = \phi(x,u)$ with a certain function $\phi(x,u) \neq 0$ is identical with the original equation (3); that is,

$$F^*\big|_{v=\phi(x,u)} = \lambda(x,u,\ldots)\,F. \tag{20}$$

for some $\lambda \in \mathscr{A}$.

Definition 13 generalizes all of the previous ones. The substitution required on Definition 13 can be generalized, allowing dependence on the derivatives of function u, that is, a substitution of the type $v = \phi(x,u,u_{(1)},\ldots)$. In the last case, condition (20) is replaced to

$$\begin{aligned} &F^*\big|_{v=\phi(x,u,u_{(1)},\ldots)} \\ &\quad = \lambda(x,u,\ldots)\,F + \lambda^{i_1\cdots i_j}(x,u,u_{(1)},\ldots)\,D_{i_1}\cdots D_{i_j}F, \end{aligned} \tag{21}$$

where

$$D_i = \frac{\partial}{\partial x^i} + u_i\frac{\partial}{\partial u} + u_{ij}\frac{\partial}{\partial u_j} + \cdots \tag{22}$$

are the total derivative operators.

Example 14. It is clear that all of the previous discussed equations provide examples of nonlinear self-adjointness. Let us now give a different example due to Ibragimov [4], with explicit dependence on the differential variables.

Consider the equation

$$u_{xy} - \sin u = 0. \tag{23}$$

Its adjoint equations are given by

$$v_{xy} - v\cos u = 0. \tag{24}$$

Consider the differential function $\phi = u_y$. Then, substituting it into the left side of (24), the following is obtained:

$$v_{xy} - v\cos u\big|_{v=u_y} = D_y\left(u_{xy} - \sin u\right). \tag{25}$$

Finally, we would like to guide the interested reader to [4], which is the real and complete reference on this subject. Therefore, it is an obligatory reading for everyone interested in this field.

3. Nonlinear Self-Adjoint Classification of (1)

Here, we follow Steps 1 and 2 of the algorithm presented at the beginning of Section 2. We start obtaining the formal Lagrangian that in this case is

$$\begin{aligned} \mathscr{L} = v\big(&u_t - r(x,t,u)\,u_{xxx} - s(x,t,u)\,u_{xx} \\ &- f(x,t,u)\,u_x - h(x,t,u)\big). \end{aligned} \tag{26}$$

Since the easiest step was overcome, we move on to Step 2. Consider now Euler-Lagrange operators, given by the formal sums,

$$\begin{aligned} \frac{\delta}{\delta u} &= \frac{\partial}{\partial u} + \sum_{j=1}^{\infty}(-1)^j D_{i_1}\cdots D_{i_j}\frac{\delta}{\delta u_{i_1\cdots i_j}}, \\ \frac{\delta}{\delta v} &= \frac{\partial}{\partial v} + \sum_{j=1}^{\infty}(-1)^j D_{i_1}\cdots D_{i_j}\frac{\partial}{\partial v_{i_1\cdots i_j}}. \end{aligned} \tag{27}$$

Considering the particular Lagrangian (26), our Euler-Lagrange operators can be simplified to

$$\begin{aligned} \frac{\delta}{\delta u} &= \frac{\partial}{\partial u} - D_t\frac{\partial}{\partial u_t} - D_x\frac{\partial}{\partial u_x} + D_x^2\frac{\partial}{\partial u_{xx}} - D_x^3\frac{\partial}{\partial u_{xxx}}, \\ \frac{\delta}{\delta v} &= \frac{\partial}{\partial v}. \end{aligned} \tag{28}$$

Then, we have

$$\begin{aligned} \frac{\delta\mathscr{L}}{\delta v} &= u_t - r(x,t,u)\,u_{xxx} - s(x,t,u)\,u_{xx} \\ &\quad - f(x,t,u)\,u_x - h(x,t,u) \equiv F, \\ \frac{\delta\mathscr{L}}{\delta u} &= -v\left(f_u u_x + s_u u_{xx} + r_u u_{xxx} + h_u\right) \\ &\quad - D_t(v) + D_x(vf) - D_x^2(vs) + D_x^3(vr) \equiv F^*. \end{aligned} \tag{29}$$

In order to avoid a tiring notation, we omit from now the dependence on (x,t,u) of the functions involved in the calculations. Thus, the expression F^* is given by

$$\begin{aligned} F^* = &-v\left(f_u u_x + s_u u_{xx} + r_u u_{xxx} + h_u\right) \\ &- v_t + v_x f + v f_x + v f_u u_x - v_{xx}s - 2v_x s_x \\ &- 2v_x s_u u_x - v s_{xx} - 2v s_{xu}u_x - v s_{uu}u_x^2 - v s_u u_{xx} \\ &+ v_{xxx}r + 3v_{xx}r_x + 3v_{xx}r_u u_x + 3v_x r_{xx} \\ &+ 6v_x r_{xu}u_x + 3v_x r_{uu}u_x^2 + 3v r_{xxu}u_x + 3v r_{xuu}u_x^2 \\ &+ 3v_x r_u u_{xx} + 3v r_{uu}u_x u_{xx} + 3v r_{xu}u_{xx} \\ &+ v r_{xxx} + v r_{uuu}u_x^3 + v r_u u_{xxx}. \end{aligned} \tag{30}$$

Replacing $v = \phi(x,t,u)$ into (30), we obtain

$$\begin{aligned} F^*|_{v=\phi(x,t,u)} = & -\phi_u F \\ & + (-\phi_t - (\phi h)_u + (\phi f)_x - (\phi s)_{xx} + (\phi r)_{xxx}) \\ & + (3(\phi r)_{xxu} - 2(\phi s)_{xu}) u_x \\ & + (3(\phi r)_{xuu} - (\phi s)_{uu}) u_x^2 + (\phi r)_{uuu} u_x^3 \\ & + (3(\phi r)_{xu} - 2(\phi s)_u) u_{xx} + 3(\phi r)_{uu} u_x u_{xx}. \end{aligned} \tag{31}$$

From Definition 13, in order to (1) be nonlinearly self-adjoint, we must have $\lambda = -\phi_u$ and

$$\begin{aligned} & (-\phi_t - (\phi h)_u + (\phi f)_x - (\phi s)_{xx} + (\phi r)_{xxx}) \\ & + (3(\phi r)_{xxu} - 2(\phi s)_{xu}) u_x \\ & + (3(\phi r)_{xuu} - (\phi s)_{uu}) u_x^2 + (\phi r)_{uuu} u_x^3 \\ & + (3(\phi r)_{xu} - 2(\phi s)_u) u_{xx} \\ & + 3(\phi r)_{uu} u_x u_{xx} = 0. \end{aligned} \tag{32}$$

Since the set $\{1, u_x, u_x^2, u_x^3, u_{xx}, u_x u_{xx}\}$ is linearly independent, we obtain the following system of equations:

$$-\phi_t - (\phi h)_u + (\phi f)_x - (\phi s)_{xx} + (\phi r)_{xxx} = 0, \tag{33}$$

$$3(\phi r)_{xxu} - 2(\phi s)_{xu} = 0, \tag{34}$$

$$3(\phi r)_{xuu} - (\phi s)_{uu} = 0, \tag{35}$$

$$(\phi r)_{uuu} = 0, \tag{36}$$

$$3(\phi r)_{xu} - 2(\phi s)_u = 0, \tag{37}$$

$$3(\phi r)_{uu} = 0. \tag{38}$$

From (38) and (35), we conclude that $(\phi s)_{uu} = 0$.

Equations (38) and (37) imply, respectively, (36) and (34). Thus, we arrived at the following system:

$$\begin{gathered} (\phi r)_{uu} = 0, \qquad 3(\phi r)_{xu} - 2(\phi s)_u = 0, \qquad (\phi s)_{uu} = 0, \\ (\phi f)_x - \phi_t - (\phi h)_u - (\phi s)_{xx} + (\phi r)_{xxx} = 0. \end{gathered} \tag{39}$$

4. Examples of Nonlinearly Self-Adjoint Equations of the Type (1)

In this section, we present two examples of nonlinearly self-adjoint equations of type (1). We consider some particular cases of the equations studied in [31]. First, let us consider the equation

$$u_t + p(t) e^{kx} u u_x + q(t) u_{xxx} = 0, \tag{40}$$

where $p(t)$ and $q(t)$ are nonzero functions and $k = constant$. From (39), we obtain

$$q(t)\phi_{uu} = 0, \qquad q(t)\phi_{xu} = 0, \tag{41}$$

$$\left(p(t) e^{kx} u \phi\right)_x + \phi_t + q(t)\phi_{xxx} = 0. \tag{42}$$

From (41), since $q(t)$ is nonzero, we conclude that the function $\phi = \phi(x,t,u)$ is $\phi = A(t)u + B(x,t)$, for certain functions $A = A(t)$ and $B = B(x,t)$. Substituting the expression of ϕ into (42), we have

$$\begin{aligned} & kp(t) e^{kx} A(t) u^2 \\ & \quad + \left[A'(t) + p(t) e^{kx} B_x(x,t) + kp(t) e^{kx} B(x,t)\right] u \\ & \quad + B_t(x,t) + q(t) B_{xxx}(x,t) = 0. \end{aligned} \tag{43}$$

Since the set $\{1, u, u^2\}$ is linearly independent, we have the following system of equations:

$$kp(t) e^{kx} A(t) = 0, \tag{44}$$

$$A'(t) + p(t) e^{kx} B_x(x,t) + kp(t) e^{kx} B(x,t) = 0, \tag{45}$$

$$B_t(x,t) + q(t) B_{xxx}(x,t) = 0. \tag{46}$$

Considering the case $k = 0$ and $p(t)A(t) \neq 0$ in (44). Thus, (45) is simplified to

$$A'(t) + p(t) B_x(x,t) = 0, \tag{47}$$

and, therefore,

$$B(x,t) = -\frac{A'(t)}{p(t)} x + C(t). \tag{48}$$

Equation (48) implies that $B_{xxx}(x,t) = 0$. Then, from (46), we conclude that

$$A(t) = c_1 \int p(t)\, dt + c_2, \qquad B(x,t) = -c_1 x + c_3, \tag{49}$$

where c_1, c_2 and c_3 are arbitrary constants. Therefore, we obtain

$$\phi(x,t,u) = c_1 \left[\left(\int p(t)\, dt\right) u - x\right] + c_2 u + c_3. \tag{50}$$

Whenever $p = q = -1$, and under the change $c_1 \mapsto -c_1$, it is concluded that $\phi = c_1(x + tu) + c_2 u + c_3$, a well-known result on KdV equation; see [4, 16].

Now consider the case $k \neq 0$. Then we must consider three subcases: $A = 0$ and $p \neq 0$, $A \neq 0$ and $p = 0$, and $A = p = 0$.

Case $A = 0$ and $p \neq 0$. From (44)–(46), we conclude that $B(x,t) = b_0 e^{-kx + k^3 Q(t)}$, with $Q'(t) = q(t)$, and then, choose $b_0 = 1$;

$$\phi(x,t,u) = e^{-kx + k^3 Q(t)}. \tag{51}$$

Case $A \neq 0$ *and* $p = 0$. From (45), we easily arrive at $A(t) = c_1$. Then, the substitution is

$$\phi(x,t,u) = c_1 u + B(x,t), \tag{52}$$

where B is a solution of (46). The third subcase $A = p = 0$ is a particular case of this last one taking $c_1 = 0$ into (52).

As a second example, consider the equation

$$u_t + \frac{1}{x} u u_x + g(t) u_{xxx} = 0 \tag{53}$$

with $g(t) \neq 0$. In this case, system (39) reads

$$g(t)\phi_{uu} = 0, \qquad g(t)\phi_{xu} = 0, \tag{54}$$

$$\left(\frac{u}{x}\phi\right)_x + \phi_t + g(t)\phi_{xxx} = 0. \tag{55}$$

From (54), we again obtain

$$\phi = A(t)u + B(x,t). \tag{56}$$

Replacing the function expression in (55) and grouping in terms of u and u^2 we have

$$\begin{aligned} &-\frac{A(t)}{x^2}u^2 + \left(A'(t) + \frac{B_x(x,t)}{x} - \frac{B(x,t)}{x^2}\right)u \\ &\quad + B_t(x,t) + g(t)B_{xxx}(x,t) = 0. \end{aligned} \tag{57}$$

The system

$$-\frac{A(t)}{x^2} = 0, \tag{58}$$

$$A'(t) + \frac{B_x(x,t)}{x} - \frac{B(x,t)}{x^2} = 0, \tag{59}$$

$$B_t(x,t) + g(t)B_{xxx}(x,t) = 0 \tag{60}$$

follows from (57). Equation (58) implies $A(t) = 0$. From (59) and (60), we conclude that

$$\phi(x,t,u) = cx, \tag{61}$$

with $c = constant$.

5. Conclusion

In this paper we discussed about some ideas introduced in [1, 3, 4, 6, 43]. These ideas are very recent and until now the applications are mainly restricted to the obtainment of local conservation laws using the approach suggested in [1]. However, some recent facts show that there is more to self-adjointness than meets the eye.

In fact, in [4], Ibragimov began to consider the concept of approximated nonlinear self-adjointness. Recently [44] explored deeper these concepts. In [45] approximate conservation laws for a nonlinear filtration equation were established.

Nowadays, the fractional calculus seems to be a new branch in Mathematics. In [46] fractional conservation laws using the approach proposed in [1] were presented, which means that the concepts of self-adjoint differential equations must be considered in the sense of fractional differential equations.

Finally, we would like to mention that recently some possible connections between integrable equations, strictly self-adjointness and scale invariance have been reported in [47], although this relation is not clear yet.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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References

[1] N. H. Ibragimov, "A new conservation theorem," *Journal of Mathematical Analysis and Applications*, vol. 333, no. 1, pp. 311–328, 2007.

[2] N. H. Ibragimov, "Integrating factors, adjoint equations and Lagrangians," *Journal of Mathematical Analysis and Applications*, vol. 318, no. 2, pp. 742–757, 2006.

[3] N. H. Ibragimov, "Nonlinear self-adjointness and conservation laws," *Journal of Physics A: Mathematical and Theoretical*, vol. 44, Article ID 432002, 2011.

[4] N. H. Ibragimov, "Nonlinear self-adjointness in constructing conservation laws," *Archives of ALGA*, vol. 7/8, pp. 1–90, 2011.

[5] R. W. Atherton and G. M. Homsy, "On the existence and formulation of variational principles for nonlinear differential equations," *Stuides in Applied Mathematics*, vol. 54, no. 1, pp. 31–60, 1975.

[6] M. L. Gandarias, "Weak self-adjoint differential equations," *Journal of Physics A: Mathematical and Theoretical*, vol. 44, no. 26, Article ID 262001, 2011.

[7] M. S. Bruzón, M. L. Gandarias, and N. H. Ibragimov, "Self-adjoint sub-classes of generalized thin film equations," *Journal of Mathematical Analysis and Applications*, vol. 357, no. 1, pp. 307–313, 2009.

[8] I. L. Freire, "Self-adjoint sub-classes of third and fourth-order evolution equations," *Applied Mathematics and Computation*, vol. 217, no. 22, pp. 9467–9473, 2011.

[9] M. L. Gandarias, M. Redondo, and M. S. Bruzón, "Some weak self-adjoint Hamilton-Jacobi-Bellman equations arising in financial mathematics," *Nonlinear Analysis. Real World Applications*, vol. 13, no. 1, pp. 340–347, 2012.

[10] M. L. Gandarias and M. S. Bruzón, "Some conservation laws for a forced KdV equation," *Nonlinear Analysis. Real World Applications*, vol. 13, no. 6, pp. 2692–2700, 2012.

[11] M. L. Gandarias, "Weak self-adjointness and conservation laws for a porous medium equation," *Communications in Nonlinear Science and Numerical Simulation*, vol. 17, no. 6, pp. 2342–2349, 2012.

[12] A. G. Johnpillai and C. M. Khalique, "Conservation laws of KdV equation with time dependent coefficients," *Communications in Nonlinear Science and Numerical Simulation*, vol. 16, no. 8, pp. 3081–3089, 2011.

[13] M. L. Gandarias and M. S. Bruzón, "Conservation laws for a class of quasi self-adjoint third order equations," *Applied Mathematics and Computation*, vol. 219, no. 2, pp. 668–678, 2012.

[14] M. Torrisi and R. Tracinà, "Quasi self-adjointness of a class of third order nonlinear dispersive equations," *Nonlinear Analysis. Real World Applications*, vol. 14, no. 3, pp. 1496–1502, 2013.

[15] I. L. Freire and J. C. Santos Sampaio, "On the nonlinear self-adjointness and local conservation laws for a class of evolution equations unifying many models," *Communications in Nonlinear Science and Numerical Simulation*, vol. 19, no. 2, pp. 350–360, 2014.

[16] I. L. Freire and J. C. S. Sampaio, "Nonlinear self-adjointness of a generalized fifth-order KdV equation," *Journal of Physics A: Mathematical and Theoretical*, vol. 45, no. 3, Article ID 032001, 2012.

[17] I. L. Freire, "New classes of nonlinearly self-adjoint evolution equations of third- and fifth-order," *Communications in Nonlinear Science and Numerical Simulation*, vol. 18, no. 3, pp. 493–499, 2013.

[18] I. L. Freire and J. C. S. Sampaio, "A review on some results on local conservation laws for certain evolution equations," *TEMA. Tendências em Matemática Aplicada e Computacional*, vol. 14, no. 1, pp. 109–118, 2013.

[19] I. L. Freire, "Conservation laws for self-adjoint first-order evolution equation," *Journal of Nonlinear Mathematical Physics*, vol. 18, no. 2, pp. 279–290, 2011.

[20] I. L. Freire, "New conservation laws for inviscid Burgers equation," *Computational & Applied Mathematics*, vol. 31, no. 3, pp. 559–567, 2012.

[21] M. A. Abdulwahhab, "Conservation laws of inviscid Burgers equation with nonlinear damping," *Communications in Nonlinear Science and Numerical Simulation*, vol. 19, no. 6, pp. 1729–1741, 2014.

[22] N. H. Ibragimov, M. Torrisi, and R. Tracinà, "Self-adjointness and conservation laws of a generalized Burgers equation," *Journal of Physics A: Mathematical and Theoretical*, vol. 44, no. 14, Article ID 145201, 2011.

[23] Y. Bozhkov, S. Dimas, and N. H. Ibragimov, "Conservation laws for a coupled variable-coefficient modified Korteweg–de Vries system in a two-layer fluid model," *Communications in Nonlinear Science and Numerical Simulation*, vol. 18, no. 5, pp. 1127–1135, 2013.

[24] N. H. Ibragimov, M. Torrisi, and R. Tracinà, "Quasi self-adjoint nonlinear wave equations," *Journal of Physics A: Mathematical and Theoretical*, vol. 43, no. 44, Article ID 442001, 2010.

[25] Y. Wang and L. Wei, "Self-adjointness, symmetries, and conservation laws for a class of wave equations incorporating dissipation," *Abstract and Applied Analysis*, vol. 2013, Article ID 407908, 6 pages, 2013.

[26] E. D. Avdonina and N. H. Ibragimov, "Conservation laws of anisotropic heat equations," *Archives of ALGA*, vol. 9, pp. 13–22, 2012.

[27] E. D. Avdonina and N. H. Ibragimov, "Conservation laws and exact solutions for nonlinear diffusion in anisotropic media," *Communications in Nonlinear Science and Numerical Simulation*, vol. 18, no. 10, pp. 2595–2603, 2013.

[28] Y. Bozhkov and K. A. A. Silva, "Nonlinear self-adjointness of a 2D generalized second order evolution equation," *Nonlinear Analysis. Theory, Methods & Applications A*, vol. 75, no. 13, pp. 5069–5078, 2012.

[29] Y. Bozhkov and S. Dimas, "Group classification and conservation laws for a two-dimensional generalized Kuramoto-Sivashinsky equation," *Nonlinear Analysis. Theory, Methods & Applications A*, vol. 84, pp. 117–135, 2013.

[30] R. Tracinà, "On the nonlinear self-adjointness of the Zakharov-Kuznetsov equation," *Communications in Nonlinear Science and Numerical Simulation*, vol. 19, no. 2, pp. 377–382, 2014.

[31] J.-P. Gazeau and P. Winternitz, "Symmetries of variable coefficient Korteweg-de Vries equations," *Journal of Mathematical Physics*, vol. 33, no. 12, pp. 4087–4102, 1992.

[32] F. Güngör, V. I. Lahno, and R. Z. Zhdanov, "Symmetry classification of KdV-type nonlinear evolution equations," *Journal of Mathematical Physics*, vol. 45, no. 6, pp. 2280–2313, 2004.

[33] I. L. Freire and J. C. S. Sampaio, "Conservation laws for a Burgers-KdV family of equations," in preparation, 2014.

[34] G. W. Bluman and S. C. Anco, *Symmetry and Integration Methods for Differential Equations*, Springer, New York, NY, USA, 2002.

[35] G. W. Bluman and S. Kumei, *Symmetries and Differential Equations*, vol. 81 of *Applied Mathematical Sciences*, Springer, New York, NY, USA, 1989.

[36] N. H. Ibragimov, *Transformation Groups Applied to Mathematical Physics*, Mathematics and its Applications (Soviet Series), D. Reidel Publishing, Dordrecht, The Netherlands, 1985, Translated from the Russian.

[37] P. J. Olver, *Applications of Lie Groups to Differential Equations*, Springer, New York, NY, USA, 1986.

[38] P. J. Olver, "Conservation laws and null divergences," *Mathematical Proceedings of the Cambridge Philosophical Society*, vol. 94, no. 3, pp. 529–540, 1983.

[39] P. J. Olver, "Conservation laws of free boundary problems and the classification of conservation laws for water waves," *Transactions of the American Mathematical Society*, vol. 277, no. 1, pp. 353–380, 1983.

[40] R. O. Popovych and A. M. Samoilenko, "Local conservation laws of second-order evolution equations," *Journal of Physics A: Mathematical and Theoretical*, vol. 41, no. 36, Article ID 362002, 2008.

[41] R. O. Popovych and A. Sergyeyev, "Conservation laws and normal forms of evolution equations," *Physics Letters A*, vol. 374, no. 22, pp. 2210–2217, 2010.

[42] A. M. Vinogradov, "Local symmetries and conservation laws," *Acta Applicandae Mathematicae*, vol. 2, no. 1, pp. 21–78, 1984.

[43] N. H. Ibragimov, "Quasi-self-adjoint differential equations," *Archives of ALGA*, vol. 4, pp. 55–60, 2007.

[44] Z.-Y. Zhang, "Approximate nonlinear self-adjointness and approximate conservation laws," *Journal of Physics A: Mathematical and Theoretical*, vol. 46, no. 15, Article ID 155203, 2013.

[45] A. A. Alexandrova, N. H. Ibragimov, and V. O. Lukashchuk, "Group classification and conservation laws of nonlinear filtration equation with a small parameter," *Communications in Nonlinear Science and Numerical Simulation*, vol. 19, no. 2, pp. 364–370, 2014.

[46] S. Yu. Lukashchulk, "Constructing of conservation laws for fractional differential equations," talk presented at 16a Mogran Conference, UFA, Russia, 2013.

[47] P. L. da Silva and I. L. Freire, "Strict self-adjointness and shallow water models," http://arxiv.org/abs/1312.3992.