Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2014, Article ID 809095, 9 pages
http://dx.doi.org/10.1155/2014/809095

Research Article

New Distributional Global Solutions for the Hunter-Saxton Equation

C. O. R. Sarrico

CMAF, University of Lisbon, Avenida Professor Gama Pinto 2, 1649-003 Lisbon, Portugal

Correspondence should be addressed to C. O. R. Sarrico; corsarrico@gmail.com

Received 7 July 2014; Revised 7 September 2014; Accepted 8 September 2014; Published 14 October 2014

Academic Editor: Ziemowit Popowicz

In the setting of a distributional product, we consider a Riemann problem for the Hunter-Saxton equation $[u_t + ((1/2)u^2)_x]_x = (1/2)u_x^2$ in a convenient space of discontinuous functions. With the help of a consistent extension of the classical solution concept, two classes of discontinuous solutions are obtained: one class of conservative solutions and another of dispersive solutions. A necessary and sufficient condition for the propagation of a distributional profile as a travelling wave is also presented, which allows identifying an interesting set of explicit distributional travelling waves. In the paper, we will show some results we have obtained by applying this framework to other equations and systems.

1. Introduction and Contents

In the present paper, we investigate the Hunter-Saxton equation

$$\left[u_t + \left(\frac{u^2}{2}\right)_x\right]_x = \frac{1}{2}u_x^2, \tag{1}$$

where $x \in \mathbb{R}$ is the space variable, $t \in [0, +\infty[$ is the time variable, and $u(x,t)$ is the unknown variable. We subject $u(x,t)$ to the initial condition

$$u(x,0) = a + (b-a)H(x), \tag{2}$$

where $a, b \in \mathbb{R}$, $a \neq b$, and H stands for the Heaviside function.

This equation models the propagation of the director field

$$n(x,t) = (\cos u(x,t), \sin u(x,t)), \tag{3}$$

in a one-dimensional nematic liquid crystal in which each molecule (long and rigid) has an orientation given by the unit vector $n(x,t)$.

Liquid crystals are a state of matter intermediate between the crystalline solid state and the liquid state (the water is not an example, since it goes directly from solid state to liquid state). One of the most common phases in which a liquid crystal exists is the so-called nematic phase, in which the liquid crystal is invariant under the transformation $n \to -n$ (in (3) only the direction of n is important).

Usually, for a complete description of a liquid crystal, two independent vector fields are needed: one for the fluid flow and another for the orientation of the molecules [1]. In (1), introduced by Hunter and Saxton [2], the orientation of the fluid particles given by (3) is considered independently of any coupling with the fluid flow.

Equation (1) has attracted a lot of attention [3–8] and has many interesting properties. It is completely integrable, has many infinite conserved quantities, and has a Lax pair; it is also bivariational and has a bihamiltonian structure [5]. If the initial condition $u(x,0)$ is smooth and not monotone increasing, the classical solutions $u(x,t)$ break down in finite time. Thus, weaker concepts of solution are needed.

Useful and different concepts of weak global solutions were defined by Hunter and Saxton [2], Hunter and Zheng [6], and Bressan and Constantin [3], with the goal of studying certain Cauchy problems. However, we cannot apply those definitions to the study of the Riemann problem (1), (2). We will adopt a global solution concept, which is a consistent extension of the classic global solution concept and it is defined within the framework of a theory of distributional products. In the setting of this theory, the product of two distributions is a distribution, which depends on the choice of

a certain function α that encodes the indeterminacy inherent to such products. We stress that not only this indeterminacy is in general unavoidable, but also in many situations it has a physical meaning. Concerning this point let us mention [9–12]. Naturally, such an indeterminacy may appear or not in the solutions of our problems (the solutions may depend or not on α). We call such solutions α-solutions. The possibility of their occurrence depends on the physical system: in certain cases we cannot previously know the behavior of the system, possibly due to physical features omitted in the formulation of the model with the goal of simplifying it. Thus, the mathematical indetermination sometimes observed may have this origin. Within our framework we recall some results we have obtained.

For the conservation law

$$u_t + [\phi(u)]_x = \psi(u), \tag{4}$$

where ϕ, ψ are entire functions taking real values on the real axis, we have established [13] necessary and sufficient conditions for the propagation of a travelling wave with a given distributional profile and we have also computed its speed. For example, for LeVeque and Yee equation

$$u_t + u_x = \mu u(1-u)\left(u - \frac{1}{2}\right), \tag{5}$$

where $\mu \neq 0$, we have proved that there exist six travelling waves with the profile $c_1 + (c_2 - c_1)H$ (c_1, c_2 are constants), all of them with speed 1. When $\psi = 0$ and $\phi'' \neq 0$ in (4), we were able to conclude that the only continuous travelling waves are constant states. Thus, if we ask for nonconstant travelling waves for the conservative equation $u_t + [\phi(u)]_x = 0$, with $\phi'' \neq 0$, we have to seek them among distributions that are not continuous functions; for wave profiles that are C^1-functions with one jump discontinuity our methods easily lead to the well-known Rankine-Hugoniot conditions.

Conditions for the propagation of travelling shock-waves with profiles $\beta + m\delta$ and $\beta + mD\delta$ (where β is a continuous function, $m \in \mathbb{R}$, $m \neq 0$, δ stands for the Dirac measure, and D is the usual derivative operator in distributional sense) were also obtained as well as their speeds [14]. For example, for the diffusionless Burgers-Fischer equation

$$u_t + a\left(\frac{1}{2}u^2\right)_x = ru\left(1 - \frac{u}{k}\right), \tag{6}$$

where $a > 0$, $r > 0$, and $k > 0$, the profile $b + m\delta$ (where $b \in \mathbb{R}$) can arise as a travelling wave, if and only if $b = 0$ or $b = k$ with wave speed $ak/2$ in both cases. For Burgers conservative equation, $u_t + ((1/2)u^2)_x = 0$, the profile $\beta + mD\delta$ can emerge as a travelling wave, if and only if $\beta = b$ is a constant function and the wave speed is b.

In the setting of soliton wave collision, we were able to prove that delta waves under collision behave just as classical soliton collisions (as in the Korteweg-de Vries equation) in models ruled by a singular perturbation of Burgers conservative equation [15].

Phenomena of gas dynamics known as "infinitely narrow soliton solutions," discovered by Maslov and his collaborators [16–20], can be obtained directly in distributional form [21].

In a Riemann problem for the 2×2 system of conservation laws

$$u_t + [\phi(u)]_x = 0, \qquad v_t + [\psi(u)v]_x = 0, \tag{7}$$

arising from the so-called generalized pressureless gas dynamics, we were able to show the formation of a delta-shock wave solution only assuming $\phi, \psi : \mathbb{R} \to \mathbb{R}$ continuous [22]. In this case, we arrived, in a more general setting, to the same result of Danilov and Mitrovic [23], who have employed the weak asymptotic method, and also to the same result of Mitrovic et al. [24], who have used a different approach, based on a linearization process.

Also in the Brio system

$$\begin{aligned} u_t + \left(\frac{u^2 + v^2}{2}\right) &= 0, \\ v_t + (uv - v)_x &= 0, \end{aligned} \tag{8}$$

a simplified model for the study of plasmas, we have subjected $u(x,t)$ and $v(x,t)$ to the initial conditions

$$u(x,0) = c_0\delta(x), \qquad v(x,0) = h_0\delta(x), \tag{9}$$

with $c_0, h_0 \in \mathbb{R}$. Under certain assumptions, we have obtained, as solutions, travelling delta-waves with speed $(c_0^2 + h_0^2)/(c_0^2 - h_0^2)$ and certain singular perturbations (which are not measures) propagating with speed 1 [25].

In the present paper and within a convenient space of discontinuous functions, we prove that all solutions of the Riemann problem (1), (2) are of the form

$$u(x,t) = c_1(t) + (b-a)H[x - \gamma_1(t)], \tag{10}$$

or of the form

$$u(x,t) = c_1(t) + \frac{2(b-a)}{2-(b-a)\lambda t}H[x - \gamma_2(t)], \tag{11}$$

where $c_1 : [0, +\infty[\to \mathbb{R}$ is an arbitrary C^1-function with $c_1(0) = a$, $\lambda \in \mathbb{R}$ is an arbitrary real number such that $\lambda(b-a) < 0$, γ_1 is given by

$$\gamma_1(t) = \int_0^t c_1(t)\,dt + \frac{b-a}{2}t, \tag{12}$$

and γ_2 is given by

$$\gamma_2(t) = \int_0^t c_1(t)\,dt - \frac{1}{\lambda}\log[2-(b-a)\lambda t] + \frac{1}{\lambda}\log 2. \tag{13}$$

Thus, we obtain two classes of discontinuous solutions; we will show that the solutions of form (10) are conservative and the solutions of form (11) are dispersive. As a particular case of (10), taking $c_1(t) = a$, we obtain the conservative travelling wave solution

$$u(x,t) = a + (b-a)H\left(x - \frac{a+b}{2}t\right), \tag{14}$$

with the initial profile (2) propagating with speed $(a+b)/2$. This suggests the study of the propagation of travelling waves with a given distributional profile and the evaluation of the speed of each profile. As a consequence, we will see, for instance, the arising of the travelling wave

$$u(x,t) = H\left(x - \frac{1}{2}t\right) - H\left(x - a - \frac{1}{2}t\right) = \begin{cases} 0 & \text{if } x < \frac{1}{2}t \\ 1 & \text{if } \frac{1}{2}t < x < \frac{1}{2}t + a \\ 0 & \text{if } x > \frac{1}{2}t + a, \end{cases} \tag{15}$$

with $a > 0$, which corresponds to the propagation, with speed $1/2$, of the profile $H(x) - H(x-a)$, sometimes called rectangular pulse. We will also see that, for instance, the profile $H(x) - H(x-a) + H(x-b) - H(x-d)$, with $0 < a < b < d$, propagates with speed $1/2$ and the profile $5 + 4\delta$ propagates with speed 3. These examples are a consequence of the identification of an interesting class of explicit distributional travelling waves.

Let us summarize the contents of this paper. In Section 2, we present a survey of our distributional products, displaying all formulas that will be applied in the sequel. We proceed in order to keep computations self-contained. A general view of our distributional products can be seen in [12, Sections 2 and 3]. The details are given in [26] or in some papers that we will mention later. In Section 3, we define the concept of α-solution for (1); we stress that this concept does not depend on approximation processes, but may depend on the α-function, which codifies the indeterminacy inherent to the product of certain distributions. In Section 4, we solve the Riemann problem (1), (2) in a convenient space of solutions. The necessary and sufficient condition for the propagation of a distributional profile is afforded in Section 5, where some examples are given and physically interpreted.

2. Products of Distributions

Let $\mathcal{D}$ be the space of indefinitely differentiable complex-valued functions defined on $\mathbb{R}$, with compact support, and let $\mathcal{D}'$ be the space of Schwartz distributions. In our theory, each function $\alpha \in \mathcal{D}$ with $\int_{-\infty}^{+\infty} \alpha = 1$ affords a general α-product $T_{\dot\alpha} S \in \mathcal{D}'$ of $T, S \in \mathcal{D}'$.

Each α-product is bilinear, and it is transformed as usual by translations; that is,

$$\tau_a \left(T_{\dot\alpha} S\right) = \left(\tau_a T\right)_{\dot\alpha} \left(\tau_a S\right), \tag{16}$$

where τ_a means the usual translation operator in distributional sense. In general, associativity and commutativity do not hold.

Remark 1. Recall that, in the setting of the so-called classical products of distributions, the commutative property is a convention inherent to the definition of such products and the associative property does not hold in general: for instance, $0 = [\delta(x)x](\mathrm{vp}.(1/x)) \neq [\delta(x)][x(\mathrm{vp}.(1/x))] = \delta(x)$ (see the classical monograph of Schwartz [27, pp. 117, 118, 121], where these products are defined).

The general α-product is not consistent with the classical Schwartz products of distributions, but we can single out certain subspaces and define certain modified α-products in order to recover that consistency. This happens with the α-products (18) and (24) below, which will be denoted by the unique symbol $T_{\dot\alpha} S$, because they are mutually compatible (see [12, Sections 2 and 3] for details); these α-products are consistent with the referred Schwartz products of $\mathcal{D}'^p$-distributions with C^p-functions, if the C^p-functions are placed on the right-hand side.

All α-products satisfy the usual rules for derivatives, provided the Leibniz formula is written in the form

$$D\left(T_{\dot\alpha} S\right) = (DT)_{\dot\alpha} S + T_{\dot\alpha} (DS). \tag{17}$$

The first α-product can be evaluated by the formula

$$T_{\dot\alpha} S = T\beta + (T * \alpha) f, \tag{18}$$

for $T \in \mathcal{D}'^p$ and $S = \beta + f \in C^p \oplus \mathcal{D}'_\mu$, where $p \in \{0, 1, 2, \ldots, \infty\}$, $\mathcal{D}'^p$ is the space of distributions of order $\leq p$ in the sense of Schwartz ($\mathcal{D}'^\infty$ means $\mathcal{D}'$), $\mathcal{D}'_\mu$ is the space of distributions whose support has Lebesgue measure zero, $T\beta$ is the usual Schwartz product of a $\mathcal{D}'^p$-distribution with a C^p-function, and $(T * \alpha) f$ is the usual product of a C^∞-function with a distribution. For instance, if β is a continuous function, we have, for each α,

$$\begin{gathered} \delta_{\dot\alpha} \beta = \delta_{\dot\alpha} (\beta + 0) = \delta\beta + (\delta * \alpha) 0 = \beta(0)\delta, \\ \beta_{\dot\alpha} \delta = \beta_{\dot\alpha} (0 + \delta) = \beta 0 + (\beta * \alpha)\delta = [(\beta * \alpha)(0)]\delta, \end{gathered} \tag{19}$$

$$\delta_{\dot\alpha} \delta = \delta_{\dot\alpha} (0 + \delta) = \delta 0 + (\delta * \alpha)\delta = \alpha\delta = \alpha(0)\delta, \tag{20}$$

$$\begin{gathered} \delta_{\dot\alpha} (D\delta) = (\delta * \alpha) D\delta = \alpha D\delta = \alpha(0) D\delta - \alpha'(0)\delta, \\ (D\delta)_{\dot\alpha} \delta = (D\delta * \alpha)\delta = \alpha'\delta = \alpha'(0)\delta, \end{gathered} \tag{21}$$

$$(D\delta)_{\dot\alpha} (D\delta) = (D\delta * \alpha) D\delta = \alpha' D\delta = \alpha'(0) D\delta - \alpha''(0)\delta, \tag{22}$$

$$H_{\dot\alpha} \delta = (H * \alpha)\delta = \left[\int_{-\infty}^{+\infty} \alpha(-\tau) H(\tau)\, d\tau\right]\delta = \left(\int_{-\infty}^{0} \alpha\right)\delta. \tag{23}$$

The second α-product is computed by the formula

$$T_{\dot\alpha} S = D(TF) - (DT)F + (T * \alpha) f, \tag{24}$$

for $T \in \mathcal{D}'^{-1}$ and $S = w + f \in L^1_{\mathrm{loc}} \oplus \mathcal{D}'_\mu$, where $\mathcal{D}'^{-1}$ stands for the space of distributions $T \in \mathcal{D}'$ such that $DT \in \mathcal{D}'^0$, and $F \in C^0$ is such that $DF = w$. Thus, locally, T can be read as a function of bounded variation and F as an absolutely continuous function. In [12, p. 645], we have proved that $T_{\dot\alpha} S$ given by (24) is independent of the choice of the function F

such that $DF = w$. For instance, since $H \in \mathcal{D}'^{-1}$ and $H = H + 0 \in L^1_{\text{loc}} \oplus \mathcal{D}'_\mu$, we have

$$H_{\dot{\alpha}} H = D(HF) - (DH)F + (H * \alpha)0 = DF - \delta F = H, \tag{25}$$

taking $F : \mathbb{R} \to \mathbb{R}$ defined by $F(x) = 0$ for $x \leq 0$ and $F(x) = x$ for $x > 0$. More generally, if $T \in \mathcal{D}'^{-1}$ and $S \in L^1_{\text{loc}}$, then $T_{\dot{\alpha}} S = TS$ (see [28, p. 1002] for a proof). We want to stress that in (18) or (24) the convolution $T * \alpha$ is not to be understood as an approximation of T. Those formulas are to be considered as exact ones.

In general, the support of the α-products cannot be completely localized: indeed, $\operatorname{supp}(T_{\dot{\alpha}} S) \subset \operatorname{supp} S$, as for usual functions, but it may happen that $\operatorname{supp}(T_{\dot{\alpha}} S) \not\subset \operatorname{supp} T$: for instance, if $a, b \in \mathbb{R}$, we have, by (18),

$$(\tau_a \delta)_{\dot{\alpha}} (\tau_b \delta) = [(\tau_a \delta) * \alpha] \tau_b \delta = \alpha(b - a) \tau_b \delta. \tag{26}$$

It is still possible to define many other α-products consistent with the classical products (see, e.g., [28]) but in the sequel they are not needed.

3. The Concept of α-Solution

Let I be an interval of $\mathbb{R}$ with more than one point and let $\mathcal{F}(I)$ be the space of continuously differentiable maps $\tilde{u} : I \to \mathcal{D}'$ in the sense of the usual topology of $\mathcal{D}'$. For $t \in I$ the notation $[\tilde{u}(t)](x)$ is sometimes used to emphasize that the distribution $\tilde{u}(t)$ acts on functions $\xi \in \mathcal{D}$ which depend on x.

Let $\Sigma(I)$ be the space of functions $u : \mathbb{R} \times I \to \mathbb{R}$ such that

(a) for each $t \in I$, $u(x, t) \in L^1_{\text{loc}}(\mathbb{R})$;

(b) $\tilde{u} : I \to \mathcal{D}'$, defined by $[\tilde{u}(t)](x) = u(x, t)$, is in $\mathcal{F}(I)$.

The natural injection $u \mapsto \tilde{u}$ of $\Sigma(I)$ into $\mathcal{F}(I)$ identifies any function of $\Sigma(I)$ with a certain map in $\mathcal{F}(I)$. Since $C^1(\mathbb{R} \times I) \subset \Sigma(I)$, we can write the inclusions

$$C^1(\mathbb{R} \times I) \subset \Sigma(I) \subset \mathcal{F}(I). \tag{27}$$

Thus, identifying u with $\tilde{u}$, (1) can be read as follows:

$$D\left\{\frac{d\tilde{u}}{dt}(t) + \frac{1}{2} D\left[\tilde{u}(t)_{\dot{\alpha}} \tilde{u}(t)\right]\right\} = \frac{1}{2}\left[D\tilde{u}(t)_{\dot{\alpha}} D\tilde{u}(t)\right]. \tag{28}$$

Definition 2. Given α, the function $\tilde{u} \in \mathcal{F}(I)$ will be called an α-solution for (28) on I, if the α-products that appear in this equation are well defined and this equation is satisfied for all $t \in I$.

We have the following results.

Theorem 3. *If u is a classical solution of* (1) *on $\mathbb{R} \times I$, then, for any α, the function $\tilde{u} \in \mathcal{F}(I)$ defined by $[\tilde{u}(t)](x) = u(x, t)$ is an α-solution of* (28) *on I.*

Notice that, by a classical solution of (1) on $\mathbb{R} \times I$, we mean a C^1-function $u(x, t)$ that satisfies (1) on $\mathbb{R} \times I$.

Theorem 4. *If $u : \mathbb{R} \times I \to \mathbb{R}$ is a C^1-function and, for a certain α, the function $\tilde{u} \in \mathcal{F}(I)$ defined by $[\tilde{u}(t)](x) = u(x, t)$ is an α-solution of* (28) *on I, then u is a classical solution of* (1) *on $\mathbb{R} \times I$.*

For the proof it is enough to observe that any C^1-function $u(x, t)$ can be read as a continuously differentiable function $\tilde{u} \in \mathcal{F}(I)$ defined by $[\tilde{u}(t)](x) = u(x, t)$ and to use the consistency of the α-products with the classical Schwartz products of distributions with functions.

Definition 5. Given α, one calls α-solution for (1) on I, to any α-solution $\tilde{u}$ of (28) on I.

As a consequence, an α-solution $\tilde{u}$ in this sense, read as an usual distribution u, affords a consistent extension of the concept of a classical solution for (1). Thus, and for short, we also call the distribution u an α-solution of (1).

4. The Riemann Problem (1), (2)

Let us consider the problem (1), (2) with $(x, t) \in \mathbb{R} \times [0, +\infty[$ and $a \neq b$. When we read this problem in $\mathcal{F}([0, +\infty[)$ having in mind the identification $u \mapsto \tilde{u}$, we must substitute (1) by (28) and condition (2) by the following one:

$$\tilde{u}(0) = a + (b - a) H. \tag{29}$$

We will give, explicitly, all α-solutions for the Cauchy problem (28), (29) which belong to the set $\tilde{U}$ of the maps $\tilde{u} : [0, +\infty[\to \mathcal{D}'$ of the form

$$\tilde{u}(t) = c_1(t) + [c_2(t) - c_1(t)] \tau_{\gamma(t)} H, \tag{30}$$

where $c_1, c_2, \gamma : [0, +\infty[\to \mathbb{R}$ are C^1-functions.

Theorem 6. *Given α, one has the following:*

(I) *if $\alpha(0) = 0$, all α-solutions $\tilde{u} \in \tilde{U}$ of the problem* (28), (29) *are given by*

$$\tilde{u}(t) = c_1(t) + (b - a) \tau_{\gamma(t)} H, \tag{31}$$

where $c_1 : [0, +\infty[\to \mathbb{R}$ is an arbitrary C^1-function with $c_1(0) = a$ and $\gamma : [0, +\infty[\to \mathbb{R}$ is defined by

$$\gamma(t) = \int_0^t c_1(\tau)\, d\tau + \frac{b - a}{2} t; \tag{32}$$

(II) *if $\alpha(0) \neq 0$, the problem* (28), (29) *has α-solutions in $\tilde{U}$ if and only if $\alpha(0)(b - a) < 0$ and, in this case, all α-solutions $\tilde{u} \in \tilde{U}$ are given by*

$$\tilde{u}(t) = c_1(t) + \frac{2(b - a)}{2 - \alpha(0)(b - a) t} \tau_{\gamma(t)} H, \tag{33}$$

where $c_1 : [0, +\infty[\to \mathbb{R}$ is an arbitrary C^1-function with $c_1(0) = a$ and $\gamma : [0, +\infty[\to \mathbb{R}$ is defined by

$$\gamma(t) = \int_0^t c_1(\tau)\, d\tau - \frac{1}{\alpha(0)} \log[2 - \alpha(0)(b - a) t] + \frac{1}{\alpha(0)} \log 2. \tag{34}$$

Proof. Suppose $\tilde{u} \in \tilde{U}$. Then from (30) and (29) we have necessarily

$$c_1(0) + [c_2(0) - c_1(0)]\,\tau_{\gamma(0)}H = a + (b-a)\,H, \tag{35}$$

and taking the derivative we obtain

$$[c_2(0) - c_1(0)]\,\tau_{\gamma(0)}\delta = (b-a)\,\delta. \tag{36}$$

Since $b \neq a$, we have $c_2(0) - c_1(0) \neq 0$, $\gamma(0) = 0$, and so $c_2(0) - c_1(0) = b - a$.

From (30) we have

$$\begin{aligned}\frac{d\tilde{u}}{dt}(t) &= c_1'(t) + [c_2(t) - c_1(t)]'\tau_{\gamma(t)}H \\ &\quad - \gamma'(t)\,[c_2(t) - c_1(t)]\,\tau_{\gamma(t)}\delta,\end{aligned} \tag{37}$$

and, by (25),

$$\begin{aligned}\tilde{u}(t)_{\dot{\alpha}}\,\tilde{u}(t) &= c_1^2(t) + 2c_1(t)\,[c_2(t) - c_1(t)]\,\tau_{\gamma(t)}H \\ &\quad + [c_2(t) - c_1(t)]^2\tau_{\gamma(t)}H \\ &= c_1^2(t) + \left[c_2^2(t) - c_1^2(t)\right]\tau_{\gamma(t)}H, \\ D\left[\tilde{u}(t)_{\dot{\alpha}}\,\tilde{u}(t)\right] &= \left[c_2^2(t) - c_1^2(t)\right]\tau_{\gamma_1(t)}\delta.\end{aligned} \tag{38}$$

By applying (20),

$$\begin{aligned}&[D\tilde{u}(t)]_{\dot{\alpha}}\,[D\tilde{u}(t)] \\ &\quad = \{[c_2(t) - c_1(t)]\tau_{\gamma(t)}\delta\}_{\dot{\alpha}}\left\{[c_2(t) - c_1(t)]\,\tau_{\gamma(t)}\delta\right\} \\ &\quad = [c_2(t) - c_1(t)]^2\alpha(0)\,\tau_{\gamma(t)}\delta,\end{aligned} \tag{39}$$

and (28) turns out to be

$$\begin{aligned}&[c_2(t) - c_1(t)]'\tau_{\gamma(t)}\delta - \gamma'(t)\,[c_2(t) - c_1(t)]\,\tau_{\gamma(t)}D\delta \\ &\quad + \frac{1}{2}\left[c_2^2(t) - c_1^2(t)\right]\tau_{\gamma(t)}D\delta \\ &\quad = \frac{1}{2}[c_2(t) - c_1(t)]^2\alpha(0)\,\tau_{\gamma(t)}\delta.\end{aligned} \tag{40}$$

This equality takes place if and only if the following two equalities are satisfied:

$$[c_2(t) - c_1(t)]' = \frac{1}{2}[c_2(t) - c_1(t)]^2\alpha(0), \tag{41}$$

$$\gamma'(t)\,[c_2(t) - c_1(t)] = \frac{1}{2}\left[c_2^2(t) - c_1^2(t)\right]. \tag{42}$$

Now suppose $\alpha(0) = 0$. Then from (41) we have $c_2(t) - c_1(t) = b - a \neq 0$ and (31) follows from (30). Thus, from (31) and (29), we conclude that $c_1(0) = a$. Also from (42) we have

$$\begin{aligned}\gamma'(t) &= \frac{1}{2}\left[c_1(t) + c_2(t)\right] \\ &= \frac{1}{2}\left[2c_1(t) + b - a\right] = c_1(t) + \frac{b-a}{2},\end{aligned} \tag{43}$$

and (32) follows.

Suppose $\alpha(0) \neq 0$. Then doing $y(t) = c_2(t) - c_1(t)$, we have, from (41),

$$y'(t) = \frac{1}{2}\alpha(0)\,y^2(t) \quad \text{with } y(0) = b - a. \tag{44}$$

This Cauchy problem has a C^1-solution on $[0, +\infty[$ if and only if $\alpha(0)(b-a) < 0$. Clearly, such a solution is unique on $[0, +\infty[$ and is given by

$$y(t) = c_2(t) - c_1(t) = \frac{2(b-a)}{2 - (b-a)\,\alpha(0)\,t}. \tag{45}$$

Since $c_2(t) - c_1(t) \neq 0$ for all $t \in [0, +\infty[$, from (42) and (45), we have

$$\gamma'(t) = c_1(t) + \frac{b-a}{2 - (b-a)\,\alpha(0)\,t}, \tag{46}$$

and (34) follows. Also (33) follows from (30) and (45). Finally $c_1(0) = a$ is a consequence of (29) and (33). □

As a consequence we can conclude that the α-solutions of the problem (1), (2) of the form

$$u(x,t) = c_1(t) + [c_2(t) - c_1(t)]\,H[x - \gamma(t)] \tag{47}$$

can be described as we have done in the Introduction (see (10), (11), (12), and (13)). Thus, just as the usual weak global solutions for the Cauchy problem, the α-solutions of (1), (2) are not unique as well. However, the α-solutions can arise as physically meaningful since the kinetic energy

$$\begin{aligned}E(t) &= \int_{-\infty}^{+\infty} u_x^2(x,t)\,dx = \int_{-\infty}^{+\infty} [D\tilde{u}(t)]_{\dot{\alpha}}\,[D\tilde{u}(t)] \\ &= \left\langle [D\tilde{u}(t)]_{\dot{\alpha}}\,[D\tilde{u}(t)], 1\right\rangle\end{aligned} \tag{48}$$

is finite for each $t \in [0, +\infty[$: for (31) we have, since $\alpha(0) = 0$,

$$\begin{aligned}E(t) &= \int_{-\infty}^{+\infty} [(b-a)\tau_{\gamma(t)}\delta]_{\dot{\alpha}}\left[(b-a)\,\tau_{\gamma(t)}\delta\right] \\ &= \int_{-\infty}^{+\infty} (b-a)^2\alpha(0)\,\tau_{\gamma(t)}\delta = 0,\end{aligned} \tag{49}$$

and for (33) we have, since $\alpha(0)(b-a) < 0$,

$$\begin{aligned}E(t) &= \int_{-\infty}^{+\infty}\left[\frac{2(b-a)}{2 - \alpha(0)(b-a)\,t}\tau_{\gamma(t)}\delta\right]_{\dot{\alpha}}\left[\frac{2(b-a)}{2 - \alpha(0)(b-a)\,t}\tau_{\gamma(t)}\delta\right] \\ &= \int_{-\infty}^{+\infty}\frac{4(b-a)^2}{[2 - \alpha(0)(b-a)\,t]^2}\alpha(0)\,\tau_{\gamma(t)}\delta \\ &= \frac{4(b-a)^2}{[2 - \alpha(0)(b-a)t]^2}\alpha(0).\end{aligned} \tag{50}$$

Hence, the α-solutions (31) are conservative since the kinetic energy is conserved (does not depend on t) and the α-solutions (33) are dispersive since $E(t) \to 0$ as $t \to +\infty$.

Taking $c_1(t) = a$ in (31) we obtain the travelling wave

$$\widetilde{u}(t) = a + (b-a)\,\tau_{\gamma(t)}H, \tag{51}$$

with $\gamma(t) = ((a+b)/2)t$. This α-solution is the unique discontinuous travelling wave solution of (1) belonging to $\widetilde{U}$. In the next section we will present an interesting class of distributional travelling waves for the Hunter-Saxton equation.

5. Travelling Waves

We introduce the following definition for the sake of simplicity.

Definition 7. Let $\gamma : [0,+\infty[\to \mathbb{R}$ be a C^1-function. The profile $T \in \mathscr{D}'$ is said to α-propagate (according to (1)) with the movement $\gamma(t)$ if $\widetilde{u}_T : [0,+\infty[\to \mathscr{D}'$ defined by $\widetilde{u}_T(t) = \tau_{\gamma(t)}T$ is an α-solution of (28). Naturally one calls $\widetilde{u}_T$ an α-travelling wave.

Theorem 8. *Given α, let $\gamma : [0,+\infty[\to \mathbb{R}$ be a C^1-function and let T be a profile in $\mathscr{D}'$. Then T, α-propagates with the movement $\gamma(t)$ if and only if the following three conditions are satisfied:*

(I) *$T_{\dot\alpha}T$ and $DT_{\dot\alpha}DT$ are well defined α-products;*

(II) *the speed $\gamma'(t) = c$ is a constant function for all $t \geq 0$;*

(III) $cD^2T = (1/2)[D^2(T_{\dot\alpha}T) - (DT)_{\dot\alpha}(DT)]$.

Proof. Let us suppose that T, α-propagates with the movement $\gamma(t)$. Then, by Definition 7, $\tau_{\gamma(t)}T$ is an α-solution of (28), which means that

$$\begin{aligned} &D\left\{\frac{d\left(\tau_{\gamma(t)}T\right)}{dt}(t) + \frac{1}{2}D\left[\left(\tau_{\gamma(t)}T\right)_{\dot\alpha}\left(\tau_{\gamma(t)}T\right)\right]\right\} \\ &\quad = \frac{1}{2}\left(D\tau_{\gamma(t)}T\right)_{\dot\alpha}\left(D\tau_{\gamma(t)}T\right), \end{aligned} \tag{52}$$

for all $t \in [0,+\infty[$, which is equivalent to

$$\begin{aligned} &D\left[\tau_{\gamma(t)}DT\left(-\gamma'(t)\right) + \frac{1}{2}D\tau_{\gamma(t)}\left(T_{\dot\alpha}T\right)\right] \\ &\quad = \frac{1}{2}\left(\tau_{\gamma(t)}DT\right)_{\dot\alpha}\left(\tau_{\gamma(t)}DT\right). \end{aligned} \tag{53}$$

Then,

$$\tau_{\gamma(t)}D^2T\left(-\gamma'(t)\right) + \frac{1}{2}\tau_{\gamma(t)}D^2\left(T_{\dot\alpha}T\right) = \frac{1}{2}\tau_{\gamma(t)}\left[(DT)_{\dot\alpha}(DT)\right], \tag{54}$$

and applying the operator $\tau_{-\gamma(t)}$ we conclude that

$$\gamma'(t)\,D^2T = \frac{1}{2}\left[D^2\left(T_{\dot\alpha}T\right) - (DT)_{\dot\alpha}(DT)\right]. \tag{55}$$

Since the right-hand side of this equality is independent of t we conclude that $\gamma'(t) = c$ is a constant function and the statement is proved. □

Supposing $T \in C^2$, and using the consistency of the α-products with the Schwartz products of distributions, (III) turns out to be

$$cT'' = \frac{1}{2}\left[\left(T^2\right)'' - \left(T'\right)^2\right], \tag{56}$$

and we can write

$$2cT'' = \left(T'\right)^2 + 2TT'', \tag{57}$$

which is the ordinary differential equation we obtain when, for the Hunter-Saxton equation, we seek travelling waves solutions $u(x,t) = T(x-ct)$ with $T \in C^2$. Thus, Definition 7 is a consistent extension of the travelling wave classical concept.

An interesting class of α-travelling waves arises from the following result.

Corollary 9. *Let $S \in \mathscr{D}'$ be such that $D^2S \neq 0$ and suppose that, for a certain α, the following two conditions are satisfied:*

(a) *$S_{\dot\alpha}S = kS$ for a certain $k \in \mathbb{R}$;*

(b) $(DS)_{\dot\alpha}(DS) = 0$.

Then, if $\lambda_1, \lambda_2 \in \mathbb{R}$ and $\lambda_2 \neq 0$, the profile $T = \lambda_1 + \lambda_2 S$, α-propagates with speed $c = \lambda_1 + (1/2)\lambda_2 k$.

Proof. By applying Theorem 8 and the assumptions of this corollary, we have

$$\begin{aligned} T_{\dot\alpha}T &= \lambda_1^2 + 2\lambda_1\lambda_2 S + \lambda_2^2\left(S_{\dot\alpha}S\right) = \lambda_1^2 + 2\lambda_1\lambda_2 S + \lambda_2^2 kS \\ &= \lambda_1^2 + \left(2\lambda_1\lambda_2 + \lambda_2^2 k\right)S, \\ (DT)_{\dot\alpha}(DT) &= (\lambda_2 DS)_{\dot\alpha}\left(\lambda_2 DS\right) = \lambda_2^2 (DS)_{\dot\alpha}(DS) = 0, \\ D^2T &= \lambda_2 D^2 S, \end{aligned} \tag{58}$$

and (III) turns out to be

$$c\lambda_2 D^2 S = \frac{1}{2}\left(2\lambda_1\lambda_2 + \lambda_2^2 k\right)D^2 S, \tag{59}$$

which is equivalent to

$$\left(c\lambda_2 - \lambda_1\lambda_2 - \frac{1}{2}\lambda_2^2 k\right)D^2 S = 0, \tag{60}$$

and, since $D^2S \neq 0$, $c = \lambda_1 + (1/2)\lambda_2 k$ follows. □

Example 10. Taking $S = H$, we have $D^2H = D\delta \neq 0$, $H_{\dot\alpha}H = H$,

$$(DH)_{\dot\alpha}(DH) = \delta_{\dot\alpha}\delta = \alpha(0)\,\delta, \tag{61}$$

so that we can apply the above corollary with $k = 1$, $\alpha(0) = 0$. Thus, for instance, taking $\lambda_1 = a$ and $\lambda_2 = b-a$, we conclude that the profile

$$T = a + (b-a)\,H \tag{62}$$

α-propagates with speed $c = (a+b)/2$ for any α such that $\alpha(0) = 0$; we recover the travelling wave (51) which has just this profile and this speed. The director field of the liquid crystal given by (3),

$$n(x,t) = \left(\cos\left(a + (b-a)H\left(x - \frac{a+b}{2}t\right)\right), \sin\left(a + (b-a)H\left(x - \frac{a+b}{2}t\right)\right)\right), \tag{63}$$

can be physically interpreted in the following way: at the instant $t = 0$ the initial director field

$$n(x,0) = \begin{cases} (\cos a, \sin a) & \text{if } x < 0 \\ (\cos b, \sin b) & \text{if } x > 0 \end{cases} \tag{64}$$

has a singularity at $x = 0$; this singularity travels with speed $(a+b)/2$ so that, at any instant $t > 0$,

$$n(x,t) = \begin{cases} (\cos a, \sin a) & \text{if } x < \dfrac{a+b}{2}t \\ (\cos b, \sin b) & \text{if } x > \dfrac{a+b}{2}t. \end{cases} \tag{65}$$

Thus, before the singularity the molecules are all parallel to the direction $(\cos a, \sin a)$ and after the singularity all molecules are parallel to the direction $(\cos b, \sin b)$.

Example 11. Taking $S = H - \tau_a H$ with $a > 0$, we have $D^2 S = D\delta - \tau_a D\delta \neq 0$. Since $S \in \mathscr{D}'^{-1}$ and $S \in L^1_{\text{loc}}$ we also have

$$\begin{aligned} S_{\dot\alpha} S &= (H - \tau_a H)_{\dot\alpha} (H - \tau_a H) \\ &= (H - \tau_a H)(H - \tau_a H) = (H - \tau_a H) = S, \end{aligned} \tag{66}$$

and applying (26) we can write

$$\begin{aligned} (DS)_{\dot\alpha} (DS) &= (\delta - \tau_a \delta)_{\dot\alpha} (\delta - \tau_a \delta) \\ &= \alpha(0)\delta - \alpha(a)\tau_a\delta - \alpha(-a)\delta + \alpha(0)\tau_a\delta. \end{aligned} \tag{67}$$

Thus, we can apply the above corollary with $k = 1$, $\alpha(0) = \alpha(a) = \alpha(-a) = 0$. Taking, for instance, $\lambda_1 = 0$ and $\lambda_2 = 1$, we conclude that the rectangular pulse $T = H - \tau_a H$, α-propagates with speed $c = 1/2$ for all α such that $\alpha(0) = \alpha(a) = \alpha(-a) = 0$. It is also easy to conclude that the profile $H - \tau_a H + \tau_b H - \tau_d H$, with $0 < a < b < d$, that is, the profile that takes the value 1 on $I =]0, a[\cup]b, d[$ and vanishes out of I, α-propagates with speed $c = 1/2$ as well. This result can be easily generalized and the interpretations are analogous to that of Example 10.

Example 12. Taking $S = 1 + \delta$, we have $D^2 S = D^2\delta \neq 0$,

$$\begin{aligned} S_{\dot\alpha} S &= (1+\delta)_{\dot\alpha}(1+\delta) = 1 + (1 * \alpha)\delta + \delta + \alpha(0)\delta \\ &= 1 + 2\delta + \alpha(0)\delta = 1 + [2 + \alpha(0)]\delta, \end{aligned} \tag{68}$$

$$(DS)_{\dot\alpha}(DS) = (D\delta)_{\dot\alpha}(D\delta) = \alpha'(0) D\delta - \alpha''(0)\delta$$

(see (22)), so that we can apply Corollary 9 with $\alpha'(0) = \alpha''(0) = 0$, $\alpha(0) = -1$, and $k = 1$. Thus, for instance, with $\lambda_1 = 1$ and $\lambda_2 = 4$, we conclude that the profile $T = 5 + 4\delta$, α-propagates with speed $c = 3$ for all α such that $\alpha'(0) = \alpha''(0) = 0$ and $\alpha(0) = -1$. This profile corresponds to the travelling wave $u(x,t) = 5 + 4\delta(x - 3t)$ and the director field

$$\begin{aligned} n(x,t) &= (\cos[5 + 4\delta(x-3t)], \sin[5 + 4\delta(x-3t)]) \\ &= (\cos 5 + (\cos 5 - \cos 1)\delta(x-3t), \\ &\quad \sin 5 + (\sin 5 - \sin 1)\delta(x-3t)) \end{aligned} \tag{69}$$

(about this equality, see Remark 13, below) can be interpreted in the following way: at the instant $t = 0$, the initial director field

$$\begin{aligned} n(x,0) = (&\cos 5 + (\cos 5 - \cos 1)\delta(x), \\ &\sin 5 + (\sin 5 - \sin 1)\delta(x)) \end{aligned} \tag{70}$$

has a singularity at $x = 0$; this singularity travels with speed 3 so that at any instant $t > 0$ the director field is given by

$$n(x,t) = (\cos 5, \sin 5) \quad \text{if } x \neq 3t. \tag{71}$$

Thus, the direction of the molecules, out of the singularity, is always the same but the singularity travels along the space with speed 3. Clearly this is physically different from the constant director field $(\cos 5, \sin 5)$.

Remark 13. In the framework of our distributional products, if $\phi : \mathbb{C} \to \mathbb{C}$ is an entire function and $T \in \mathscr{D}'$, we can define the composition $\phi \circ T$ by the formula

$$\phi \circ T = \phi(0) + \phi'(0)T + \frac{\phi''(0)}{2!}T^2 + \cdots + \frac{\phi^{(n)}(0)}{n!}T^n + \cdots, \tag{72}$$

whenever this series converges in $\mathscr{D}'$. Here, T^n $(n \geq 0)$ is defined by the recurrence relation

$$T^0 = 1, \quad T^n = T^{n-1}{}_{\dot\alpha} T, \tag{73}$$

if the α-products are well defined. These definitions are consistent with the usual meaning of $\phi \circ T$ and T^n when $T \in C^0$ (see [14, p. 372]). Thus, if $\alpha(0) = -1$, as in Example 12, we have

$$\phi \circ (5 + 4\delta) = \phi(5 + 4\delta) = \phi(5) + [\phi(5) - \phi(1)]\delta, \tag{74}$$

[14, th. 4.1]. Since $\phi \circ \tau_{\gamma(t)} T = \tau_{\gamma(t)}(\phi \circ T)$, whenever $\phi \circ T$ is well defined [14, p. 372], we have for the travelling wave $\tilde{u}(t) = \tau_{3t}(5 + 4\delta)$, associated with the profile $5 + 4\delta$,

$$\begin{aligned} \phi[\tau_{3t}(5 + 4\delta)] &= \tau_{3t}[\phi \circ (5 + 4\delta)] \\ &= \tau_{3t}\{\phi(5) + [\phi(5) - \phi(1)]\delta\}. \end{aligned} \tag{75}$$

As a consequence, taking $\phi = \cos$ or $\phi = \sin$, equality (69) follows as a rigorous expression!

Remark 14. It is interesting to verify (see [14, th. 4.1]) that for all $a, b \in \mathbb{C}$ the equality

$$\cos^2 [a + b\delta(x)] + \sin^2 [a + b\delta(x)] = 1 \tag{76}$$

is satisfied for all α. A generalized concept of angle is still possible in this setting!

The speeds that we have computed in these examples are independent of α. However, there exist also profiles whose speeds can depend on α.

Example 15. Taking $S = \delta$, we have $D^2S = D^2\delta \neq 0$, $S_{\dot{\alpha}}S = \alpha(0)\delta$, and $(DS)_{\dot{\alpha}}(DS) = \alpha'(0)D\delta - \alpha''(0)\delta$. Thus, with $\alpha'(0) = \alpha''(0) = 0$, we have $k = \alpha(0)$, and, with $\lambda_1 = 0$ and $\lambda_2 = m \neq 0$, we conclude that the profile $T = m\delta$, α-propagates with speed $c = m\alpha(0)/2$. Since $\alpha(0)$ can take any value, the profile δ, α-propagates with an arbitrary speed for any α such that $\alpha'(0) = \alpha''(0) = 0$. As in Example 12 an analogous interpretation of this situation can be made. Notice also that since $m = 2c/\alpha(0)$, for each α such that $\alpha(0) \neq 0$, we can see the profile $T = m\delta$ as a δ-soliton; its shape $T = (2c/\alpha(0))\delta$ depends on the speed c!

Conflict of Interests

The author declares that there is no conflict of interests regarding the publication of this paper.

Acknowledgment

The present research was supported by FCT, PEst-OE/MAT/UI0209/2013.

References

[1] E. G. Virga, *Variational Theories for Liquid Crystals*, Chapman & Hall, London, UK, 1994.

[2] J. K. Hunter and R. Saxton, "Dynamics of director fields," *SIAM Journal on Applied Mathematics*, vol. 51, no. 6, pp. 1498–1521, 1991.

[3] A. Bressan and A. Constantin, "Global solutions of the Hunter-Saxton equation," *SIAM Journal on Mathematical Analysis*, vol. 37, no. 3, pp. 996–1026, 2005.

[4] A. Bressan, H. Holden, and X. Raynaud, "Lipschitz metric for the Hunter-Saxton equation," *Journal de Mathématiques Pures et Appliquées*, vol. 94, no. 1, pp. 68–92, 2010.

[5] J. K. Hunter and Y. X. Zheng, "On a completely integrable nonlinear hyperbolic variational equation," *Physica D. Nonlinear Phenomena*, vol. 79, no. 2-4, pp. 361–386, 1994.

[6] J. K. Hunter and Y. X. Zheng, "On a nonlinear hyperbolic variational equation. I. Global existence of weak solutions," *Archive for Rational Mechanics and Analysis*, vol. 129, no. 4, pp. 305–353, 1995.

[7] J. K. Hunter and Y. X. Zheng, "On a nonlinear hyperbolic variational equation: (II) the zero-viscosity and dispersion limits," *Archive for Rational Mechanics and Analysis*, vol. 129, no. 4, pp. 355–383, 1995.

[8] P. Zhang and Y. Zheng, "Existence and uniqueness of solutions of an asymptotic equation arising from a variational wave equation with general data," *Archive for Rational Mechanics and Analysis*, vol. 155, no. 1, pp. 49–83, 2000.

[9] A. Bressan and F. Rampazzo, "On differential systems with vector valued impulsive controls," *Bolletino della Unione Matematica Italiana B*, vol. 7, no. 2, pp. 641–656, 1988.

[10] J. Colombeau and A. Y. LeRoux, "Multiplications of distributions in elasticity and hydrodynamics," *Journal of Mathematical Physics*, vol. 29, no. 2, pp. 315–319, 1988.

[11] G. Dal Maso, P. Lefloch, and F. Murat, "Definitions and week stability of nonconservative products," *Journal de Mathématiques Pures et Appliquées*, vol. 74, no. 6, pp. 483–548, 1995.

[12] C. O. R. Sarrico, "Distributional products and global solutions for nonconservative inviscid Burgers equation," *Journal of Mathematical Analysis and Applications*, vol. 281, no. 2, pp. 641–656, 2003.

[13] C. O. R. Sarrico, "Products of distributions and singular travelling waves as solutions of advection-reaction equations," *Russian Journal of Mathematical Physics*, vol. 19, no. 2, pp. 244–255, 2012.

[14] C. O. R. Sarrico, "Products of distributions, conservation laws and the propagation of δ'-shock waves," *Chinese Annals of Mathematics—Series B*, vol. 33, no. 3, pp. 367–384, 2012.

[15] C. O. R. Sarrico, "Collision of delta-waves in a turbulent model studied via a distribution product," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 73, no. 9, pp. 2868–2875, 2010.

[16] V. G. Danilov, V. P. Maslov, and V. M. Shelkovich, "Algebras of the singularities of singular solutions of first-order quasilinear strictly hyperbolic systems," *Teoreticheskaya i Matematicheskaya Fizika*, vol. 114, no. 1, pp. 3–55, 1998 (Russian).

[17] V. G. Danilov, V. P. Maslov, and V. M. Shelkovich, "Algebras of the singularities of singular solutions of first-order quasilinear strictly hyperbolic systems," *Theoretical and Mathematical Physics*, vol. 114, no. 1, pp. 1–42, 1998.

[18] Y. V. Egorov, "On the theory of generalized functions," *Uspekhi Matematicheskikh Nauk*, vol. 45, no. 5, pp. 3–40, 1990 (Russian), Russian Mathematical Surveys, vol. 45, no. 5, pp. 1–49, 1990.

[19] V. P. Maslov, "Nonstandard characteristics in asymptotic problems," *Uspekhi Matematicheskikh Nauk*, vol. 38, no. 6, pp. 3–36, 1983 (Russian), Russian Mathematical Surveys, vol. 38, no. 6, pp. 1–42, 1983.

[20] V. P. Maslov and G. A. Omel'yanov, "Asymptotic soliton-form solutions of equations with small dispersion," *Uspekhi Matematicheskikh Nauk*, vol. 36, no. 3, pp. 63–126, 1981 (Russian), translated in Russian Mathematical Surveys, vol. 36, no.3, pp. 73–149, 1981.

[21] C. O. R. Sarrico, "New solutions for the one-dimensional nonconservative inviscid Burgers equation," *Journal of Mathematical Analysis and Applications*, vol. 317, no. 2, pp. 496–509, 2006.

[22] C. O. R. Sarrico, "A distributional product approach to δ-shock wave solutions for a generalized pressureless gas dynamics system," *International Journal of Mathematics*, vol. 25, no. 1, Article ID 1450007, 12 pages, 2014.

[23] V. G. Danilov and D. Mitrovic, "Delta shock wave formation in the case of triangular hyperbolic system of conservation laws," *Journal of Differential Equations*, vol. 245, no. 12, pp. 3704–3734, 2008.

[24] D. Mitrovic, V. Bojkovic, and V. G. Danilov, "Linearization of the Riemann problem for a triangular system of conservation laws and delta shock wave formation process," *Mathematical Methods in the Applied Sciences*, vol. 33, no. 7, pp. 904–921, 2010.

[25] C. O. R. Sarrico, "The Brio system with initial conditions involving Dirac masses: a result afforded by a distributional product," Chinese Annals of Mathematics.

[26] C. O. R. Sarrico, "About a family of distributional products important in the applications," *Portugaliae Mathematica*, vol. 45, no. 3, pp. 295–316, 1988.

[27] L. Schwartz, *Théorie des Distributions*, Hermann, Paris, France, 1966.

[28] C. O. R. Sarrico, "The multiplication of distributions and the Tsodyks model of synapses dynamics," *International Journal of Mathematical Analysis*, vol. 6, no. 21-24, pp. 999–1014, 2012.