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Research Article

Three Weak Solutions for Nonlocal Fractional Laplacian Equations

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The existence of three weak solutions for the following nonlocal fractional equation $(-\Delta)^s u - \lambda u = \mu f(x,u)$ in Ω, $u = 0$ in $\mathbb{R}^n \setminus \Omega$, is investigated, where $s \in (0,1)$ is fixed, $(-\Delta)^s$ is the fractional Laplace operator, λ and μ are real parameters, Ω is an open bounded subset of $\mathbb{R}^n$, $n > 2s$, and the function f satisfies some regularity and natural growth conditions. The approach is based on a three-critical-point theorem for differential functionals.

1. Introduction

In this work we investigate the existence of three weak solutions to the nonlocal counterpart of perturbed semilinear elliptic partial differential equations of the type

$$\begin{aligned} -\Delta u - \lambda u &= \mu f(x,u) \quad \text{in } \Omega, \\ u &= 0 \quad \text{in } \mathbb{R}^n \setminus \Omega, \end{aligned} \tag{1}$$

namely,

$$\begin{aligned} (-\Delta)^s u - \lambda u &= \mu f(x,u) \quad \text{in } \Omega, \\ u &= 0 \quad \text{in } \mathbb{R}^n \setminus \Omega, \end{aligned} \tag{2}$$

where $s \in (0,1)$ is fixed, Ω is a nonempty bounded open subset of $\mathbb{R}^n$, $n > 2s$, λ and μ are positive real parameters, $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a function satisfying suitable regularity and growth conditions, and $(-\Delta)^s$ is the fractional Laplace operator defined as

$$\begin{aligned} &-(-\Delta)^s u(x) \\ &\quad := \text{P.V.} \int_{\mathbb{R}^n} \frac{u(x+y) + u(x-y) - 2u(x)}{|y|^{n+2s}} dy, \\ &\qquad x \in \mathbb{R}^n. \end{aligned} \tag{3}$$

Fractional Laplace operators have been proved to be valuable tools in the modeling of many phenomena in various fields, such as minimal surfaces, quasi-geostrophic flows, conservation laws, optimization, multiple scattering, anomalous diffusion, ultrarelativistic limits of quantum mechanics, finance, phase transitions, stratified materials, crystal dislocation, semipermeable membranes, flame propagation, soft thin films, and materials science. Recently, there has been significant development in fractional Laplace operators; for examples, see [1–13] and the references therein.

Motivated and inspired by the papers [13–15], in this paper, a variational approach is provided to investigate the existence of three weak solutions to a perturbed nonlocal fractional Laplacian equation (2), by using a three-critical-point theorem obtained by Bonanno and Marano in [14].

2. Preliminaries

Let $s \in (0,1)$ such that $2s < n$, $\Omega \subset \mathbb{R}^n$. The classical fractional Sobolev space $H^s(\mathbb{R}^n)$ is defined by

$$H^s(\mathbb{R}^n) = \left\{ v \in L^2(\mathbb{R}^n) : \frac{|v(x) - v(y)|}{|x-y|^{(n+2s)/2}} \in L^2(\mathbb{R}^n \times \mathbb{R}^n) \right\}, \tag{4}$$

endowed with the norm (the so-called Gagliardo norm)

$$\|v\|_{H^s(\mathbb{R}^n)} = \|v\|_{L^2(\mathbb{R}^n)} + \left(\iint_{\mathbb{R}^n\times\mathbb{R}^n} \frac{|v(x)-v(y)|^2}{|x-y|^{n+2s}} dx\,dy \right)^{1/2}. \tag{5}$$

Let

$$X_0 = \{g \in H^s(\mathbb{R}^n) : g = 0 \text{ a.e. in } \mathbb{R}^n \setminus \Omega\}. \tag{6}$$

By [6] in the sequel we can take the function

$$X_0 \ni v \longmapsto \|v\|_{X_0} = \left(\iint_{(\mathbb{R}^n\times\mathbb{R}^n)\setminus\mathcal{O}} \frac{|v(x)-v(y)|^2}{|x-y|^{n+2s}} dx\,dy \right)^{1/2} \tag{7}$$

as norm on X_0, where $\mathcal{O} = (\mathcal{C}\Omega)\times(\mathcal{C}\Omega) \subset \mathbb{R}^n\times\mathbb{R}^n$. It is easily seen that $(X_0, \|\cdot\|_{X_0})$ is a Hilbert space, with scalar product

$$\langle u, v\rangle_{X_0} = \iint_{(\mathbb{R}^n\times\mathbb{R}^n)\setminus\mathcal{O}} \frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{n+2s}} dx\,dy. \tag{8}$$

Since $v \in X_0$, we have that the integral in (7) (and in the related scalar product) can be extended to all $\mathbb{R}^n \times \mathbb{R}^n$.

By a weak solution u of (2) we mean a function $u \in X_0$ such that

$$\begin{aligned} &\iint_{\mathbb{R}^n\times\mathbb{R}^n} \frac{(u(x)-u(y))(\varphi(x)-\varphi(y))}{|x-y|^{n+2s}} dx\,dy \\ &\quad - \lambda \int_\Omega u(x)\varphi(x)\,dx \\ &\quad = \mu \int_\Omega f(x,u(x))\varphi(x)\,dx \end{aligned} \tag{9}$$

for all $\varphi \in X_0$.

Denote by $\lambda_1 > 0$ the first eigenvalue of the operator $(-\Delta)^s$ with homogeneous Dirichlet boundary data

$$\begin{aligned} (-\Delta)^s u &= \lambda u \quad \text{in } \Omega, \\ u &= 0 \quad \text{in } \mathbb{R}^n \setminus \Omega. \end{aligned} \tag{10}$$

For the existence and the basic properties of λ_1 we may refer to [7]. From [7, 16], we know that if $\lambda < \lambda_1$ then we can take a norm on X_0 as follows:

$$\begin{aligned} \|v\|_{X_0,\lambda} = \Bigg(&\iint_{(\mathbb{R}^n\times\mathbb{R}^n)\setminus\mathcal{O}} \frac{|v(x)-v(y)|^2}{|x-y|^{n+2s}} dx\,dy \\ &- \lambda \int_\Omega |v(x)|^2 dx \Bigg)^{1/2}. \end{aligned} \tag{11}$$

Moreover, we have

$$m_\lambda \|v\|_{X_0} \le \|v\|_{X_0,\lambda} \le M_\lambda \|v\|_{X_0}, \tag{12}$$

where

$$\begin{aligned} m_\lambda &:= \min\left\{ \sqrt{\frac{\lambda_1-\lambda}{\lambda_1}}, 1 \right\}, \\ M_\lambda &:= \max\left\{ \sqrt{\frac{\lambda_1-\lambda}{\lambda_1}}, 1 \right\}. \end{aligned} \tag{13}$$

Remark 1. If $0 < \lambda < \lambda_1$, then

$$m_\lambda = \sqrt{\frac{\lambda_1-\lambda}{\lambda_1}}, \qquad M_\lambda = 1. \tag{14}$$

Taking into account Lemma 8 in [6], we know that the embedding $j : X_0 \hookrightarrow L^\nu(\mathbb{R}^n)$ is continuous for any $\nu \in [1, 2^*]$, while it is compact whenever $\nu \in [1, 2^*)$. Thus, form any $\nu \in [1, 2^*)$ there exists a positive constant c_ν such that

$$\|v\|_{L^\nu(\mathbb{R}^n)} \le c_\nu \|v\|_{X_0} \le c_\nu m_\lambda^{-1} \|v\|_{X_0,\lambda} \tag{15}$$

for any $v \in X_0$.

Let $R := \sup_{x\in\Omega} \operatorname{dist}(x, \partial\Omega)$; simple calculations show that there is $x_0 \in \Omega$ such that $B(x_0, R) \subset \Omega$.

Set

$$u_\delta(x) := \begin{cases} \delta & \text{if } x \in B\left(x_0, \frac{R}{2}\right), \\ \frac{2\delta}{R}(R - |x-x_0|) & \text{if } x \in B(x_0,R) \setminus B\left(x_0, \frac{R}{2}\right), \\ 0 & \text{if } x \in \mathbb{R}^n \setminus B(x_0,R). \end{cases} \tag{16}$$

Lemma 2. *Let $\delta, R > 0$, $s \in (0,1)$, $0 < \lambda < \lambda_1$, and let u_δ be defined by* (16). *Then $u_\delta \in X_0$, and there exist $C_* = C_*(n,s,R) > 0$ and $C^* = C^*(n,s,R)$ such that*

$$C_* m_\lambda^2 \delta^2 \le \|u_\delta\|_{X_0,\lambda}^2 \le C^* \delta^2, \tag{17}$$

where m_λ is as in (14).

Proof. By Proposition 3.4 in [5], we have

$$\|u_\delta\|_{X_0}^2 = 2C(n,s)^{-1} \int_{\mathbb{R}^n} |\xi|^{2s} |\mathcal{F}u_\delta(\xi)|^2 d\xi, \tag{18}$$

where

$$C(n,s) = \left(\int_{\mathbb{R}^n} \frac{1-\cos(\zeta_1)}{|\zeta|^{n+2s}} d\zeta \right)^{-1}. \tag{19}$$

Here $\zeta = (\zeta_1, \ldots, \zeta_n)$. From the trivial inequality $|\xi|^{2s} \le 1 + |\xi|^2$, $s \in (0,1]$, and (18), we obtain

$$\begin{aligned} \|u_\delta\|_{X_0}^2 &\le 2C(n,s)^{-1} \int_{\mathbb{R}^n} \left(1 + |\xi|^2\right) |\mathcal{F}u_\delta(\xi)|^2 d\xi \\ &= 2C(n,s)^{-1} \|u_\delta\|_{H^1(\mathbb{R}^n)}^2. \end{aligned} \tag{20}$$

Moreover, according to the definition of norm for $H^1(\mathbb{R}^n)$, we get

$$\begin{aligned}\|u_\delta\|^2_{H^1(\mathbb{R}^n)} &= \int_{\mathbb{R}^n} |\nabla u_\delta(x)|^2 dx \\ &= \int_{B(x_0,R)\setminus B(x_0,R/2)} \frac{(2\delta)^2}{R^2} dx \\ &= \frac{4\delta^2}{R^2}\left[\operatorname{meas}(B(x_0,R)) - \operatorname{meas}\left(B\left(x_0,\frac{R}{2}\right)\right)\right] \\ &= \frac{4\delta^2}{R^2}\frac{\pi^{n/2}}{\Gamma(1+n/2)}\left(R^n - \left(\frac{R}{2}\right)^n\right) \\ &= \frac{4R^{n-2}\pi^{n/2}(1-1/2^n)}{\Gamma(1+n/2)}\delta^2.\end{aligned} \tag{21}$$

By [5], we have

$$C(n,s)^{-1} = \int_{\mathbb{R}^n} \frac{1-\cos(\zeta_1)}{|\zeta|^{n+2s}} d\zeta = A(n,s)\,D(s), \tag{22}$$

where

$$\begin{aligned}A(n,s) &= \int_{\mathbb{R}^{n-1}} \frac{1}{\left(1+|\eta|^2\right)^{(n+2s)/2}} d\eta, \\ D(s) &= \int_{\mathbb{R}} \frac{1-\cos t}{|t|^{1+2s}} dt.\end{aligned} \tag{23}$$

By polar coordinates, for any $s \in (0,1)$, we obtain

$$\begin{aligned}A(n,s) &= \omega_{n-2}\int_0^{+\infty} \frac{\rho^{n-2}}{(1+\rho^2)^{(n+2s)/2}} d\rho \\ &= \omega_{n-2}\left[\int_0^1 \frac{\rho^{n-2}}{(1+\rho^2)^{(n+2s)/2}} d\rho + \int_1^{+\infty} \frac{\rho^{n-2}}{(1+\rho^2)^{(n+2s)/2}} d\rho\right] \\ &< \omega_{n-2}\left[\int_0^1 \frac{(1+\rho^2)^{(n-2)/2}}{(1+\rho^2)^{(n+2s)/2}} d\rho + \int_1^{+\infty} \frac{\rho^{n-2}}{(\rho^2)^{(n+2s)/2}} d\rho\right] \\ &< \omega_{n-2}\left[\int_0^1 \frac{d\rho}{1+\rho^2} + \int_1^{+\infty} \frac{d\rho}{\rho^{2(1+s)}}\right] \\ &= \omega_{n-2}\left(\frac{\pi}{4} + \frac{1}{1+2s}\right) := C_1,\end{aligned} \tag{24}$$

where ω_{n-2} is the Lebesgue measure of the unit sphere in $\mathbb{R}^{n-1}$. Furthermore, we have

$$\begin{aligned}D(s) &= 2\int_0^{+\infty} \frac{1-\cos t}{t^{1+2s}} dt \\ &= 2\left[\int_0^1 \frac{1-\cos t}{t^{1+2s}} dt + \int_1^{+\infty} \frac{1-\cos t}{t^{1+2s}} dt\right] \\ &< 2\left[\int_0^1 \frac{t^2/2}{t^{1+2s}} dt + 2\int_1^{+\infty} \frac{1}{t^{1+2s}} dt\right] \\ &= \frac{1}{2(1-s)} + \frac{2}{s} := C_2.\end{aligned} \tag{25}$$

Thanks to (20)–(25), we conclude that

$$\|u_\delta\|^2_{X_0} < \frac{8R^{n-2}\pi^{n/2}(1-1/2^n)}{\Gamma(1+n/2)} C_1 C_2 \delta^2 < \infty, \tag{26}$$

which implies that $u_\delta \in X_0$. By (12), (14), and (26), we obtain

$$\|u_\delta\|^2_{X_0,\lambda} \le \|u_\delta\|^2_{X_0} \le C^* \delta^2, \tag{27}$$

where

$$C^* := \frac{8\pi^{n/2}(1-1/2^n)}{\Gamma(1+n/2)} C_1 C_2 R^{n-2}. \tag{28}$$

Hence, the conclusion of right-hand side of (17) holds.

On the other hand, we have

$$\begin{aligned}&\|u_\delta\|^2_{X_0} \\ &= \iint_{\mathbb{R}^n\times\mathbb{R}^n} \frac{|u_\delta(x)-u_\delta(y)|^2}{|x-y|^{n+2s}} dx\,dy \\ &= \iint_{B(x_0,R)\times B(x_0,R)} \frac{|u_\delta(x)-u_\delta(y)|^2}{|x-y|^{n+2s}} dx\,dy \\ &\quad + 2\iint_{B(x_0,R)\times(\mathbb{R}^n\setminus B(x_0,R))} \frac{|u_\delta(x)-u_\delta(y)|^2}{|x-y|^{n+2s}} dx\,dy \\ &= 2\iint_{B(x_0,R/2)\times(B(x_0,R)\setminus B(x_0,R/2))} \frac{|u_\delta(x)-u_\delta(y)|^2}{|x-y|^{n+2s}} dx\,dy \\ &\quad + \iint_{(B(x_0,R)\setminus B(x_0,R/2))\times(B(x_0,R)\setminus B(x_0,R/2))} \times \frac{|u_\delta(x)-u_\delta(y)|^2}{|x-y|^{n+2s}} dx\,dy \\ &\quad + 2\iint_{B(x_0,R)\times(\mathbb{R}^n\setminus B(x_0,R))} \frac{|u_\delta(x)-u_\delta(y)|^2}{|x-y|^{n+2s}} dx\,dy\end{aligned}$$

$$
\begin{aligned}
&= \frac{8\delta^2}{R^2} \\
&\quad \times \int_{B(x_0,R/2)} \left(\int_{B(x_0,R)\setminus B(x_0,R/2)} \frac{\left||x-x_0|-R/2\right|^2}{|x-y|^{n+2s}} dx \right) dy \\
&\quad + \frac{4\delta^2}{R^2} \\
&\quad \times \iint_{(B(x_0,R)\setminus B(x_0,R/2))\times(B(x_0,R)\setminus B(x_0,R/2))} \\
&\qquad \times \frac{\left||x-x_0|-|y-x_0|\right|^2}{|x-y|^{n+2s}} dx\, dy \\
&\quad + 2\int_{B(x_0,R/2)} \left(\int_{\mathbb{R}^n\setminus B(x_0,R)} \frac{\delta^2}{|x-y|^{n+2s}} dx \right) dy \\
&\quad + \frac{8\delta^2}{R^2} \\
&\quad \times \int_{(B(x_0,R)\setminus B(x_0,R/2)} \left(\int_{\mathbb{R}^n\setminus B(x_0,R)} \frac{\left||y-x_0|\,|-R|\right|^2}{|x-y|^{n+2s}} dx \right) dy \\
&\geq \frac{8\delta^2}{R^2} \\
&\quad \times \int_{B(x_0,R/2)} \left(\int_{B(x_0,R)\setminus B(x_0,R/2)} \frac{\left||x-x_0|-R/2\right|^2}{|x-y|^{n+2s}} dx \right) dy \\
&\quad + 2\int_{B(x_0,R/2)} \left(\int_{\mathbb{R}^n\setminus B(x_0,R)} \frac{\delta^2}{|x-y|^{n+2s}} dx \right) dy.
\end{aligned} \tag{29}
$$

For $x \in \mathbb{R}^n \setminus B(x_0, R)$ and $y \in B(x_0, R/2)$, we have

$$
|x-y| \leq |x-x_0| + |y-x_0| \leq |x-x_0| + \frac{R}{2} \leq \frac{3}{2}|x-x_0|. \tag{30}
$$

Thus,

$$
\begin{aligned}
&\int_{B(x_0,R/2)} \left(\int_{\mathbb{R}^n\setminus B(x_0,R)} \frac{1}{|x-y|^{n+2s}} dx \right) dy \\
&\quad \geq \left(\frac{2}{3}\right)^{n+2s} \int_{B(x_0,R/2)} \left(\int_{\mathbb{R}^n\setminus B(x_0,R)} \frac{1}{|x-x_0|^{n+2s}} dx \right) dy \\
&\quad \geq \left(\frac{2}{3}\right)^{n+2s} \cdot \frac{\omega_{n-1}}{n}\left(\frac{R}{2}\right)^n \cdot \omega_{n-1} \int_R^\infty \frac{\rho^{n-1}}{\rho^{n+2s}} d\rho \\
&\quad = \left(\frac{2}{3}\right)^{n+2s} \frac{\omega_{n-1}^2}{n}\left(\frac{R}{2}\right)^n \cdot \frac{1}{2s} R^{-2s} = C_3 R^{n-2s},
\end{aligned} \tag{31}
$$

where $C_3 := (2^{2s-1}/3^{n+2s} ns)\omega_{n-1}^2$. Moreover, we obtain

$$
\begin{aligned}
&\int_{B(x_0,R/2)} \left(\int_{B(x_0,R)\setminus B(x_0,R/2)} \frac{\left||x-x_0|-R/2\right|^2}{|x-y|^{n+2s}} dx \right) dy \\
&\quad \geq \frac{1}{((3/2)R)^{n+2s}} \cdot \frac{\omega_{n-1}}{n}\left(\frac{R}{2}\right)^n \\
&\qquad \cdot \int_{B(x_0,R)\setminus B(x_0,R/2)} \left||x-x_0| - \frac{R}{2}\right|^2 dx \\
&\quad = \frac{2^{2s}\omega_{n-1}}{n 3^{n+2s} R^{2s}} \\
&\qquad \times \left[\int_{B(x_0,R)} \left(|x-x_0|^2 - R|x-x_0| + \frac{R^2}{4} \right) dx \right. \\
&\qquad\quad \left. - \int_{B(x_0,R/2)} \left(|x-x_0|^2 - R|x-x_0| + \frac{R^2}{4} \right) dx \right] \\
&\quad = \frac{2^{2s}\omega_{n-1}}{n 3^{n+2s} R^{2s}} \cdot \omega_{n-1} R^{n+2} \\
&\qquad \times \left[\frac{n^2-n+2}{4n(n+1)(n+2)} + \frac{1}{2^{n+1} n(n+1)(n+2)} \right] \\
&\quad = C_4 R^{n+2-2s},
\end{aligned} \tag{32}
$$

where

$$
C_4 := \frac{2^{2s}\omega_{n-1}^2}{n^2(n+1)(n+2)3^{n+2s}} \left[\frac{n^2-n+2}{4} + \frac{1}{2^{n+1}} \right]. \tag{33}
$$

Substitute (31) and (32) into (29), we get

$$
\iint_{\mathbb{R}^n\times\mathbb{R}^n} \frac{|u_\delta(x) - u_\delta(y)|^2}{|x-y|^{n+2s}} dx\, dy \geq 2(C_3 + 4C_4) R^{n-2s}\delta^2. \tag{34}
$$

From (34) and (12), we obtain

$$
\|u_\delta\|_{X_0,\lambda}^2 \geq m_\lambda^2 \|u_\delta\|_{X_0}^2 \geq C_* m_\lambda^2 \delta^2, \tag{35}
$$

where

$$
C_* := 2(C_3 + 4C_4) R^{n-2s}. \tag{36}
$$

Thus, the conclusion of left-hand side of (17) holds. □

In this paper our main tool is a three-critical-point theorem of [14] which is recalled below.

Theorem 3 (see [14]). *Let X be a reflexive real Banach space; let $\Phi : X \to \mathbf{R}$ be a coercive, continuously Gâteaux differentiable, and sequentially weakly lower semicontinuous functional whose Gâteaux derivative admits a continuous inverse on X^*, and let $\Psi : X \to \mathbf{R}$ be a continuously Gâteaux differentiable functional whose Gâteaux derivative is compact such that*

$$
\Phi(0) = \Psi(0) = 0. \tag{37}
$$

Assume that there exist $r > 0$ and $\overline{x} \in X$, with $r < \Phi(\overline{x})$, such that

(i) $\sup_{\Phi(x)\le r}\Psi(x)/r < \Psi(\overline{x})/\Phi(\overline{x})$;

(ii) *for each* $\mu \in \Lambda_r :=]\Phi(\overline{x})/\Psi(\overline{x}), r/\sup_{\Phi(x)\le r}\Psi(x)[$ *the functional* $\Phi - \mu\Psi$ *is coercive.*

Then, for each $\mu \in \Lambda_r$, the functional $\Phi - \mu\Psi$ has at least three distinct critical points in X.

3. Main Result

Let $f : \Omega \times \mathbb{R} \to \mathbb{R}$ be a Carathéodory function such that

(H1) there exist $a_1, a_2 \ge 0$ and $q \in (1, 2_s^*)$, $2_s^* := 2n/(n-2s)$, such that

$$\left|f(x,v)\right| \le a_1 + a_2|v|^{q-1}, \quad \forall (x,v) \in \Omega \times \mathbb{R}. \tag{38}$$

Theorem 4. *Let function f satisfy condition (H1). Assume that*

(H2) $F(x,v) := \int_0^v f(x,\tau)d\tau \ge 0$ *for all* $(x,v) \in \Omega \times \mathbb{R}^+$;

(H3) *there exist two positive constants b and $p < 2$ such that*

$$F(x,v) \le b\left(1 + |v|^p\right), \tag{39}$$

for almost every $x \in \Omega$ and for every $v \in \mathbb{R}$;

(H4) *let $0 < \lambda < \lambda_1$ such that there exist two positive constants γ and δ, with $\delta > \sqrt{2/C_*}(\gamma/m_\lambda)$ such that*

$$\frac{\inf_{x\in\Omega}F(x,\delta)}{\delta^2} > \frac{b_1}{\gamma} + b_2\gamma^{q-2}, \tag{40}$$

where

$$b_1 = \frac{n2^{n-1/2}a_1c_1C^*}{\omega_{n-1}m_\lambda R^n}, \qquad b_2 = \frac{n2^{n+(q-2)/2}a_2c_q^qC^*}{q\omega_{n-1}m_\lambda^q R^n}, \tag{41}$$

and two positive constants C^* and C_* are as in (28) *and* (36), *respectively. Then, for every μ belonging to*

$$\Lambda := \left]\frac{n2^{n-1}C^*}{\omega_{n-1}R^n}\frac{\delta^2}{\inf_{x\in\Omega}F(x,\delta)}, \frac{n2^{n-1}C^*}{\omega_{n-1}R^n}\frac{1}{b_1/\gamma + b_2\gamma^{q-2}}\right[, \tag{42}$$

problem (2) *possesses at least three weak solutions in X_0.*

Proof. Let us apply Theorem 3 with X_0 and

$$\mathcal{J}_{\lambda,\mu}(u) := \Phi_\lambda(u) - \mu\Psi(u), \quad u \in X_0, \tag{43}$$

where

$$\Phi_\lambda(u) := \frac{1}{2}\iint_{\mathbb{R}^n\times\mathbb{R}^n}\frac{\left|u(x)-u(y)\right|^2}{\left|x-y\right|^{n+2s}}dx\,dy - \frac{\lambda}{2}\int_\Omega |u(x)|^2dx,$$
$$\Psi(u) := \int_\Omega F(x,u(x))\,dx. \tag{44}$$

For each $u, v \in X_0$, one has

$$\Phi_\lambda'(u)(v) = \iint_{\mathbb{R}^n\times\mathbb{R}^n}\frac{(u(x)-u(y))(v(x)-v(y))}{\left|x-y\right|^{n+2s}}dx\,dy - \lambda\int_\Omega u(x)v(x)\,dx,$$
$$\Psi'(u)(v) = \int_\Omega f(x,u(x))v(x)\,dx. \tag{45}$$

From the proof of Theorem 1 in [16], we obtain that Φ_λ is coercive, continuously Gâteaux differentiable, and sequentially weakly lower semicontinuous functional. Moreover, similar to the proof of proposition in [17], we get by (12) that

$$\begin{aligned}\left(\Phi_\lambda'(u) - \Phi_\lambda'(v)\right)(u-v) &= \iint_{\mathbb{R}^n\times\mathbb{R}^n}\frac{\left[(u-v)(x)-(u-v)(y)\right]^2}{\left|x-y\right|^{n+2s}}dx\,dy \\ &\quad - \lambda\int_\Omega (u(x)-v(x))^2dx \\ &= \|u-v\|_{X_0,\lambda}^2 \ge m_\lambda^2\|u-v\|_{X_0}^2,\end{aligned} \tag{46}$$

for every u and v belonging to X_0. This actually means that Φ_λ' is a uniformly monotone operator in X_0. In addition, standard arguments ensure that Φ_λ' also turns out to be coercive and hemicontinuos in X_0. Therefore, Φ_λ' admits that a continuous inverse in X_0^* follows immediately by applying Theorem 26. A. of [18]. Furthermore, the functional Ψ is well defined, continuously Gâteaux differentiable with compact derivative and $\Phi_\lambda(0) = \Psi(0) = 0$.

By [16] we know that being u a weak solution of problem (2) is equivalent to being a critical point of the functional $\mathcal{J}_{\lambda,\mu}$. Since $0 < \lambda < \lambda_1$, from Lemma 2, one has

$$\Phi_\lambda(u_\delta) = \frac{1}{2}\|u_\delta\|_{X_0,\lambda}^2 \ge \frac{1}{2}C_*m_\lambda^2\delta^2. \tag{47}$$

Bearing in mind that $\delta > \sqrt{2/C_*}(\gamma/m_\lambda)$ (H4), it follows that $\Phi_\lambda(u_\delta) > \gamma^2$. By (H2), we obtain

$$\begin{aligned}\Psi(u_\delta) &= \int_\Omega F(x,u_\delta(x))\,dx \ge \int_{B(x_0,R/2)}F(x,\delta)\,dx \\ &\ge \inf_{x\in\Omega}F(x,\delta)\cdot\frac{\omega_{n-1}}{n}\left(\frac{R}{2}\right)^n.\end{aligned} \tag{48}$$

By Lemma 2, we have

$$\Phi_\lambda(u_\delta) = \frac{1}{2}\|u_\delta\|_{X_0,\lambda}^2 \le \frac{1}{2}C^*\delta^2. \tag{49}$$

So, by (48) and (49), one has

$$\frac{\Psi(u_\delta)}{\Phi_\lambda(u_\delta)} \ge \frac{\omega_{n-1}R^n}{n2^{n-1}C^*}\frac{\inf_{x\in\Omega}F(x,\delta)}{\delta^2}. \tag{50}$$

Thanks to (H1), one has

$$F(x,u) \le a_1 |u| + \frac{a_2}{q}|u|^q, \quad (x,u) \in \Omega \times \mathbb{R}. \tag{51}$$

Thus, by (15) and (51), for every $u \in X_0 : \Phi_\lambda(u) \le r$, we obtain

$$\begin{aligned}\Psi(u) &= \int_\Omega F(x,u(x))\,dx \le a_1 \|u\|_{L^1(\Omega)} + \frac{a_2}{q}\|u\|^q_{L^q(\Omega)} \\ &\le \frac{\sqrt{2}a_1c_1}{m_\lambda}\sqrt{r} + \frac{2^{q/2}c_q^q a_2}{qm_\lambda^q} r^{q/2}.\end{aligned} \tag{52}$$

Therefore

$$\sup_{u\in\Phi_\lambda^{-1}(]-\infty,r])} \Psi(u) \le \frac{\sqrt{2}a_1c_1}{m_\lambda}\sqrt{r} + \frac{2^{q/2}c_q^q a_2}{qm_\lambda^q} r^{q/2}. \tag{53}$$

Denote the function

$$\chi(r) := \frac{\sup_{u\in\Phi_\lambda^{-1}(]-\infty,r])}\Psi(u)}{r}, \quad r > 0. \tag{54}$$

By (53), we have

$$\chi(r) \le \frac{\sqrt{2}a_1c_1}{m_\lambda} r^{-1/2} + \frac{2^{q/2}c_q^q a_2}{qm_\lambda^q} r^{q/2-1}, \quad r > 0. \tag{55}$$

Owing to (50), (55), and (H4), we have

$$\begin{aligned}\chi(\gamma^2) &\le \frac{\sqrt{2}a_1c_1}{m_\lambda\gamma} + \frac{2^{q/2}c_q^q a_2}{qm_\lambda^q}\gamma^{q-2} = \frac{\omega_{n-1}R^n}{n2^{n-1}C^*}\left(\frac{b_1}{\gamma} + b_2\gamma^{q-2}\right) \\ &< \frac{\omega_{n-1}R^n}{n2^{n-1}C^*}\frac{\inf_{x\in\Omega}F(x,\delta)}{\delta^2} \le \frac{\Psi(u_\delta)}{\Phi_\lambda(u_\delta)}.\end{aligned} \tag{56}$$

Hence, the assumption (i) of Theorem 3 is satisfied.

Furthermore, if $p < 2$, for each $u \in X_0$, $|u|^p \in L^{2/p}(\Omega)$, Hölder's inequality and (15) give

$$\begin{aligned}\int_\Omega |u(x)|^p dx &\le \|u\|^p_{L^2(\Omega)} \operatorname{meas}(\Omega)^{(2-p)/2} \\ &\le \frac{c_2^p}{m_\lambda^p}\|u\|^p_{X_0,\lambda} \operatorname{meas}(\Omega)^{(2-p)/2}, \quad \forall u \in X_0.\end{aligned} \tag{57}$$

Due to (H3) and (57), we deduce that

$$\begin{aligned}\mathcal{J}_{\lambda,\mu}(u) &\ge \frac{1}{2}\|u\|^2_{X_0,\lambda} - \frac{\mu b c_2^p}{m_\lambda^p}\operatorname{meas}(\Omega)^{(2-p)/2}\|u\|^p_{X_0,\lambda} \\ &\quad - \mu b \cdot \operatorname{meas}(\Omega), \quad \forall u \in X_0.\end{aligned} \tag{58}$$

Hence, $\mathcal{J}_{\lambda,\mu}(u)$ is a coercive functional for every positive parameter μ, in particular, for each $\mu \in \Lambda \subset]\Phi_\lambda(u_\delta)/\Psi(u_\delta), \gamma^2/\sup_{\Phi_\lambda(u)\le\gamma^2}\Psi(u)[$. So also condition (ii) holds. So all the assumptions of Theorem 3 are satisfied. Thus, for each $0 < \lambda < \lambda_1$ there exists $\mu > 0$, depending on λ, such that, for any $\mu \in \Lambda$, the functional $\mathcal{J}_{\lambda,\mu}(u)$ has at least three distinct critical points that are weak solutions to problem (2). □

Remark 5. Similar to Example 3.1 in [15], we can give a concrete example of function satisfying hypotheses (H1)–(H4). Set $q \in (2, 2_s^*)$, $s \in (0,1)$, and $0 < p < 2$ and let

$$h := \max\left\{1, \frac{1}{m_\lambda}\sqrt{\frac{2}{C_*}}, (A_1 + A_2)^{1/(q-2)} q^{1/(q-2)}\right\}, \tag{59}$$

where

$$A_1 = \frac{n2^{n-1/2}c_1C^*}{\omega_{n-1}m_\lambda R^n}, \qquad A_2 = \frac{n2^{n+(q-2)/2}c_q^q C^*}{q\omega_{n-1}m_\lambda^q R^n}. \tag{60}$$

From (H4) we know that $b_1 = a_1A_1$ and $b_2 = a_2A_2$. Let r be a positive constant such that $r > h$ and consider the following continuous and positive function $f : \Omega \times \mathbb{R} \to \mathbb{R}$:

$$f(x,v) := \begin{cases} 1 + |v|^{q-1} & \text{if } v \le r, \\ 1 + r^{q-p}v^{p-1} & \text{if } v > r. \end{cases} \tag{61}$$

Obviously, $f(x,v) \le 1 + |v|^{q-1}$ for each $(x,v) \in \Omega \times \mathbb{R}$, and (H1) holds. Furthermore, for every $\eta \in \mathbb{R}$, we have

$$F(x,\eta) \le \left(r + \frac{r^q}{p}\right)\left(1 + |\eta|^{\max\{1,p\}}\right). \tag{62}$$

Thus the conditions (H2) and (H3) are satisfied. Moreover, $r > h \ge (1/m_\lambda)\sqrt{2/C_*}$ and

$$\frac{\int_0^r f(x,t)\,dt}{r^2} = \frac{r^{q-2}}{q} + \frac{1}{r} > A_1 + A_2, \tag{63}$$

which implies that (H4) holds.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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References

[1] L. A. Caffarelli, S. Salsa, and L. Silvestre, "Regularity estimates for the solution and the free boundary of the obstacle problem for the fractional Laplacian," *Inventiones Mathematicae*, vol. 171, no. 2, pp. 425–461, 2008.

[2] L. Silvestre, "Regularity of the obstacle problem for a fractional power of the Laplace operator," *Communications on Pure and Applied Mathematics*, vol. 60, no. 1, pp. 67–112, 2007.

[3] X. Cabré and J. Tan, "Positive solutions of nonlinear problems involving the square root of the Laplacian," *Advances in Mathematics*, vol. 224, no. 5, pp. 2052–2093, 2010.

[4] X. Cabré and Y. Sire, "Nonlinear equations for fractional Laplacians I: regularity, maximum principles, and Hamiltonian estimates," *Annales de l'Institut Henri Poincare C*, 2013.

[5] E. Di Nezza, G. Palatucci, and E. Valdinoci, "Hitchhiker's guide to the fractional Sobolev spaces," *Bulletin des Sciences Mathématiques*, vol. 136, no. 5, pp. 521–573, 2012.

[6] R. Servadei and E. Valdinoci, "Mountain pass solutions for nonlocal elliptic operators," *Journal of Mathematical Analysis and Applications*, vol. 389, no. 2, pp. 887–898, 2012.

[7] R. Servadei and E. Valdinoci, "Variational methods for nonlocal operators of elliptic type," *Discrete and Continuous Dynamical Systems A*, vol. 33, no. 5, pp. 2105–2137, 2013.

[8] R. Servadei and E. Valdinoci, "Lewy-Stampacchia type estimates for variational inequalities driven by (non)local operators," *Revista Matemática Iberoamericana*, vol. 29, no. 3, pp. 1091–1126, 2013.

[9] M. M. Fall and T. Weth, "Nonexistence results for a class of fractional elliptic boundary value problems," *Journal of Functional Analysis*, vol. 263, no. 8, pp. 2205–2227, 2012.

[10] F. Ferrari and I. E. Verbitsky, "Radial fractional Laplace operators and Hessian inequalities," *Journal of Differential Equations*, vol. 253, no. 1, pp. 244–272, 2012.

[11] R. Servadei and E. Valdinoci, "The Brezis-Nirenberg result for the fractional Laplacian," *Transactions of the American Mathematical Society*. In press.

[12] C. Bai, "Existence results for non-local operators of elliptic type," *Nonlinear Analysis A*, vol. 83, pp. 82–90, 2013.

[13] S. Dipierro, G. Palatucci, and E. Valdinoci, "Existence and symmetry results for a Schrödinger type problem involving the fractional Laplacian," *Le Matematiche*, vol. 68, no. 1, pp. 201–216, 2013.

[14] G. Bonanno and S. A. Marano, "On the structure of the critical set of non-differentiable functions with a weak compactness condition," *Applicable Analysis*, vol. 89, no. 1, pp. 1–10, 2010.

[15] G. Bonanno and G. Molica Bisci, "Three weak solutions for elliptic Dirichlet problems," *Journal of Mathematical Analysis and Applications*, vol. 382, no. 1, pp. 1–8, 2011.

[16] G. Molica Bisci and R. Servadei, "A bifurcation result for nonlocal fractional equations," *Journal of Applied Analysis*. In press.

[17] G. Bonanno and P. Candito, "Three solutions to a Neumann problem for elliptic equations involving the p-Laplacian," *Archiv der Mathematik*, vol. 80, no. 4, pp. 424–429, 2003.

[18] E. Zeidler, *Nonlinear Functional Analysis and Its Applications*, vol. 2, Springer, New York, NY, USA, 1985.