Research Article

Three Weak Solutions for Nonlocal Fractional Laplacian Equations

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The existence of three weak solutions for the following nonlocal fractional equation

\[-(\Delta)^s u - \lambda u = \mu f(x, u) \text{ in } \Omega, \quad u = 0 \text{ in } \mathbb{R}^n \setminus \Omega,\]

is investigated, where \( s \in (0, 1) \) is fixed, \( (\Delta)^s \) is the fractional Laplace operator, \( \lambda \) and \( \mu \) are real parameters, \( \Omega \) is an open bounded subset of \( \mathbb{R}^n \), \( n > 2s \), and the function \( f \) satisfies some regularity and natural growth conditions. The approach is based on a three-critical-point theorem for differential functionals.

1. Introduction

In this work we investigate the existence of three weak solutions to the nonlocal counterpart of perturbed semilinear elliptic partial differential equations of the type

\[-\Delta u - \lambda u = \mu f(x, u) \text{ in } \Omega, \quad u = 0 \text{ in } \mathbb{R}^n \setminus \Omega,\]

or

\[-(\Delta)^s u - \lambda u = \mu f(x, u) \text{ in } \Omega, \quad u = 0 \text{ in } \mathbb{R}^n \setminus \Omega,\]

where \( s \in (0, 1) \) is fixed, \( \Omega \) is a nonempty bounded open subset of \( \mathbb{R}^n \), \( n > 2s \), and \( \lambda \) and \( \mu \) are positive real parameters, \( f : \Omega \times \mathbb{R} \to \mathbb{R} \) is a function satisfying suitable regularity and growth conditions, and \( (\Delta)^s \) is the fractional Laplace operator defined as

\[-(\Delta)^s u(x) := \text{P.V.} \int_{\mathbb{R}^n} \frac{u(x + y) + u(x - y) - 2u(x)}{|y|^{n+2s}} dy, \quad x \in \mathbb{R}^n.\]

Fractional Laplace operators have been proved to be valuable tools in the modeling of many phenomena in various fields, such as minimal surfaces, quasi-geostrophic flows, conservation laws, optimization, multiple scattering, anomalous diffusion, ultrarelativistic limits of quantum mechanics, finance, phase transitions, stratified materials, crystal dislocation, semipermeable membranes, flame propagation, soft thin films, and materials science. Recently, there has been significant development in fractional Laplace operators; for examples, see [1–13] and the references therein.

Motivated and inspired by the papers [13–15], in this paper, a variational approach is provided to investigate the existence of three weak solutions to a perturbed nonlocal fractional Laplacian equation (2), by using a three-critical-point theorem obtained by Bonanno and Marano in [14].

2. Preliminaries

Let \( s \in (0, 1) \) such that \( 2s < n, \Omega \subset \mathbb{R}^n \). The classical fractional Sobolev space \( H^s(\mathbb{R}^n) \) is defined by

\[H^s(\mathbb{R}^n) = \left\{ v \in L^2(\mathbb{R}^n) : \frac{|v(x) - v(y)|}{|x - y|^{(n+2s)/2}} \in L^2(\mathbb{R}^n \times \mathbb{R}^n) \right\},\]

equipped with the norm (the so-called Gagliardo norm)
\[ \|v\|_{H^s(R^n)} = \left( \int_{R^n} \frac{|v(x) - v(y)|^2}{|x-y|^{n+2s}} dx dy \right)^{1/2}. \]  

Let
\[ X_0 = \{ g \in H^s(R^n) : g = 0 \text{ a.e. in } R^n \setminus \Omega \}. \]

By [6] in the sequel we can take the function
\[ X_0 \ni v \mapsto \|x\|_{X_0} = \left( \int_{(R^n \times R^n)\setminus\Gamma} \frac{|v(x) - v(y)|^2}{|x-y|^{n+2s}} dx dy \right)^{1/2} \tag{7} \]
as norm on \( X_0 \), where \( \Gamma = \{ (\psi \Omega) \times (\psi \Omega) : \psi \in R^n \times R^n \}. \) It is easily seen that \((X_0, \| \cdot \|_{X_0})\) is a Hilbert space, with scalar product
\[ \langle u, v \rangle_{X_0} = \int_{(R^n \times R^n)\setminus\Gamma} \frac{(u(x) - u(y))(v(x) - v(y))}{|x-y|^{n+2s}} dx dy. \tag{8} \]

Since \( v \in X_0 \), we have that the integral in (7) (and in the related scalar product) can be extended to all \( R^n \times R^n \).

By a weak solution of (2) we mean a function \( u \in X_0 \) such that
\[ \int_{R^n \times R^n} \frac{(u(x) - u(y))(\varphi(x) - \varphi(y))}{|x-y|^{n+2s}} dx dy - \lambda \int_{\Omega} u(x) \varphi(x) dx = \mu \int_{\Omega} f(x, u(x)) \varphi(x) dx \tag{9} \]
for all \( \varphi \in X_0 \).

Denote by \( \lambda_1 > 0 \) the first eigenvalue of the operator \((-\Delta)^s\) with homogeneous Dirichlet boundary data
\[ (-\Delta)^s u = \lambda u \quad \text{in } \Omega, \quad u = 0 \quad \text{in } R^n \setminus \Omega. \tag{10} \]

For the existence and the basic properties of \( \lambda_1 \) we may refer to [7]. From [7, 16], we know that if \( \lambda < \lambda_1 \) then we can take a norm on \( X_0 \) as follows:
\[ \|v\|_{X_0,\lambda} = \left( \int_{(R^n \times R^n)\setminus\Gamma} \frac{|v(x) - v(y)|^2}{|x-y|^{n+2s}} dx dy \right)^{1/2} - \lambda \int_{\Omega} |v(x)|^2 dx \right)^{1/2}. \tag{11} \]

Moreover, we have
\[ m_\lambda \|v\|_{X_0} \leq \|v\|_{X_0,\lambda} \leq M_\lambda \|v\|_{X_0}, \tag{12} \]

where
\[ m_\lambda := \min \left\{ \frac{\lambda_1 - \lambda}{\lambda_1}, 1 \right\}, \tag{13} \]
\[ M_\lambda := \max \left\{ \frac{\lambda_1 - \lambda}{\lambda_1}, 1 \right\}. \tag{14} \]

**Remark 1.** If \( 0 < \lambda < \lambda_1 \), then
\[ m_\lambda = \sqrt{\frac{\lambda_1 - \lambda}{\lambda_1}}, \quad M_\lambda = 1. \tag{15} \]

Taking into account Lemma 8 in [6], we know that the embedding \( j : X_0 \hookrightarrow L^p(R^n) \) is continuous for any \( p \in [1, 2^*] \), while it is compact whenever \( p \in [1, 2^*). \) Thus, form any \( \varphi \in [1, 2^*] \) there exists a positive constant \( c_\varphi \) such that
\[ \|v\|_{L^p(R^n)} \leq c_\varphi \|v\|_{X_0} \leq c_\varphi m_\lambda^{-1/2} \|v\|_{X_0,\lambda} \tag{16} \]
for any \( v \in X_0 \).

Let \( R := \sup_{x \in \Omega} \text{dist}(x, \partial \Omega) \); simple calculations show that there is \( x_0 \in \Omega \) such that \( B(x_0, R) \subset \Omega \).

Set
\[ u_\delta(x) := \begin{cases} \delta & \text{if } x \in B\left(x_0, \frac{R}{2}\right), \\ \frac{2\delta}{R} (R - |x - x_0|) & \text{if } x \in B\left(x_0, R \setminus B\left(x_0, \frac{R}{2}\right), \\ 0 & \text{if } x \in R^n \setminus B(x_0, R). \end{cases} \tag{17} \]

**Lemma 2.** Let \( \delta, R > 0, s \in (0, 1), 0 < \lambda < \lambda_1, \) and let \( u_\delta \) be defined by (17). Then \( u_\delta \in X_0, \) and there exist \( C_* = C_*(n, s, R) > 0 \) and \( C^* = C^*(n, s, R) \) such that
\[ C_* m_\lambda^2 \delta^2 \leq \|u_\delta\|_{X_0,\lambda}^2 \leq C^* \delta^2, \tag{18} \]
where \( m_\lambda \) is as in (14).

**Proof.** By Proposition 3.4 in [5], we have
\[ \|u_\delta\|_{X_0}^2 = 2C(n, s)^{-1} \int_{R^n} |\xi|^2 |\mathcal{F}u_\delta(\xi)|^2 d\xi, \tag{19} \]

where
\[ C(n, s) = \left( \int_{R^n} \frac{1 - \cos(\zeta_1) \cdots \zeta_n)}{|\xi|^2 - 1} \right)^{-1}. \tag{20} \]

Here \( \zeta = (\zeta_1, \ldots, \zeta_n) \). From the trivial inequality \( |\xi|^2 \leq 1 + \Delta |\xi|^2, s \in (0, 1], \) and (18), we obtain
\[ \|u_\delta\|_{X_0,\lambda}^2 \leq 2C(n, s)^{-1} \int_{R^n} \left( 1 + |\xi|^2 \right) |\mathcal{F}u_\delta(\xi)|^2 d\xi \]
\[ = 2C(n, s)^{-1} \|u_\delta\|_{H^s(R^n)}^2 \tag{21} \]
Moreover, according to the definition of norm for $H^1(\mathbb{R}^n)$, we get
\[
\left\| u_\delta \right\|_{H^1(\mathbb{R}^n)}^2 = \int_{\mathbb{R}^n} \left| \nabla u_\delta (x) \right|^2 \, dx
\]
\[
= \int_{B(x_0, R) \times B(x_0, R/2)} \frac{(2\delta)^2}{R^2} \, dx
\]
\[
= \frac{4\delta^2}{R^2} \int \left[ \text{meas} (B(x_0, R)) - \text{meas} (B(x_0, R/2)) \right]
\]
\[
= \frac{4\delta^2}{R^2} \frac{\pi^{n/2}}{\Gamma(1+n/2)} \left( R^n - \left( \frac{R}{2} \right)^n \right)
\]
\[
= \frac{4R^{n-2}\pi^{n/2} (1-1/2^n)}{\Gamma(1+n/2)} \delta^2.
\]
where $\omega_{n-2}$ is the Lebesgue measure of the unit sphere in $\mathbb{R}^{n-1}$. Furthermore, we have
\[
D(s) = 2 \int_0^\infty \frac{1 - \cos t}{t^{1+2s}} \, dt
\]
\[
= 2 \left[ \int_0^1 \frac{1 - \cos t}{t^{1+2s}} \, dt + \int_1^{\infty} \frac{1 - \cos t}{t^{1+2s}} \, dt \right]
\]
\[
< 2 \left[ \frac{1}{t^{1+2s}} \int_0^1 \frac{t^2}{2} \, dt + \frac{1}{t^{1+2s}} \int_1^{\infty} \, dt \right]
\]
\[
= \frac{1}{2} \left( 1 - s \right) + \frac{2}{s} := C_2.
\]

Thanks to (20)–(25), we conclude that
\[
\left\| u_\delta \right\|_{X_0}^2 < \frac{8R^{n-2}\pi^{n/2} (1-1/2^n)}{\Gamma(1+n/2)} C_1 C_2 \delta^2 < \infty,
\]
which implies that $u_\delta \in X_0$. By (12), (14), and (26), we obtain
\[
\left\| u_\delta \right\|_{X_0}^2 \leq \left\| u_0 \right\|_{X_0}^2 \leq C^* \delta^2,
\]
where
\[
C^* := \frac{8\pi^{n/2} (1-1/2^n)}{\Gamma(1+n/2)} C_1 C_2 R^{n-2}.
\]

Hence, the conclusion of right-hand side of (17) holds.

On the other hand, we have
\[
\left\| u_\delta \right\|_{X_1}^2
\]
\[
= \iint_{\mathbb{R}^n \times \mathbb{R}^n} \frac{ \left| u_\delta (x) - u_\delta (y) \right|^2 }{ \left| x - y \right|^{\frac{n+2s}{2}} } \, dx \, dy
\]
\[
= \iint_{B(x_0, R) \times B(x_0, R)} \frac{ \left| u_\delta (x) - u_\delta (y) \right|^2 }{ \left| x - y \right|^{\frac{n+2s}{2}} } \, dx \, dy
\]
\[
+ 2 \iint_{B(x_0, R) \times (\mathbb{R}^n \setminus B(x_0, R))} \frac{ \left| u_\delta (x) - u_\delta (y) \right|^2 }{ \left| x - y \right|^{\frac{n+2s}{2}} } \, dx \, dy
\]
\[
= 2 \iint_{B(x_0, R/2) \times \left( B(x_0, R) \setminus B(x_0, R/2) \right)} \frac{ \left| u_\delta (x) - u_\delta (y) \right|^2 }{ \left| x - y \right|^{\frac{n+2s}{2}} } \, dx \, dy
\]
\[
+ \iint_{B(x_0, R/2) \times \left( B(x_0, R) \setminus B(x_0, R/2) \right)} \frac{ \left| u_\delta (x) - u_\delta (y) \right|^2 }{ \left| x - y \right|^{\frac{n+2s}{2}} } \, dx \, dy
\]
\[
= 2 \int_{B(x_0, R/2) \times \left( B(x_0, R) \setminus B(x_0, R/2) \right)} \frac{ \left| u_\delta (x) - u_\delta (y) \right|^2 }{ \left| x - y \right|^{\frac{n+2s}{2}} } \, dx \, dy
\]
\[
+ 2 \int_{R^2 \setminus B(x_0, R/2)} \frac{ \left| u_\delta (x) - u_\delta (y) \right|^2 }{ \left| x - y \right|^{\frac{n+2s}{2}} } \, dx \, dy
\]
\[ \int_{\mathbb{R}^n} \frac{|x - x_0| - R/2|^2}{|x - y|^{n+2s}} \, dx \, dy \]
\[ \geq \frac{8\delta^2}{R^2} \cdot \int_{B(x_0,R/2)} \int_{B(x_0,R/2)} \frac{|x - x_0| - R/2|^2}{|x - y|^{n+2s}} \, dx \, dy \]
\[ + \frac{4\delta^2}{R^2} \cdot \int_{B(x_0,R/2)} \int_{B(x_0,R/2)} \frac{|y - x_0|^2}{|x - y|^{n+2s}} \, dx \, dy \]
\[ + \frac{8\delta^2}{R^2} \cdot \int_{B(x_0,R/2)} \int_{B(x_0,R/2)} \frac{|y - x_0| - R|^2}{|x - y|^{n+2s}} \, dx \, dy \]
\[ \geq \frac{8\delta^2}{R^2} \cdot \int_{B(x_0,R/2)} \int_{B(x_0,R/2)} \frac{|x - x_0| - R/2|^2}{|x - y|^{n+2s}} \, dx \, dy \]
\[ + \frac{2\delta^2}{R^2} \cdot \int_{B(x_0,R/2)} \int_{B(x_0,R/2)} \frac{|y - x_0|^2}{|x - y|^{n+2s}} \, dx \, dy \]
\[ + \frac{8\delta^2}{R^2} \cdot \int_{B(x_0,R/2)} \int_{B(x_0,R/2)} \frac{|y - x_0| - R|^2}{|x - y|^{n+2s}} \, dx \, dy. \]

For \( x \in \mathbb{R}^n \setminus B(x_0, R) \) and \( y \in B(x_0, R/2) \), we have
\[ |x - y| \leq |x - x_0| + |y - x_0| \leq |x - x_0| + R/2 \leq \frac{3}{2} |x - x_0|. \]

Thus,
\[ \int_{B(x_0,R/2)} \int_{B(x_0,R/2)} \frac{1}{|x - y|^{n+2s}} \, dx \, dy \]
\[ \geq \left( \frac{2}{3} \right)^{n+2s} \cdot \frac{\omega_{n-1}}{n} \cdot \frac{R^n}{2^n} \cdot \frac{\omega_{n-1}}{2} \int_{\mathbb{R}^n} \frac{R^{n-1}}{\rho^{n+2s}} \, d\rho \]
\[ = \left( \frac{2}{3} \right)^{n+2s} \frac{\omega_{n-1}}{n} \cdot \frac{1}{2} \cdot 2^{n-2s} = C_3 R^{n-2s}, \]

where \( C_3 := \frac{2^{2s-1}}{3^{n+2s} m_2} \omega_{n-1}^2 \). Moreover, we obtain
\[ \int_{B(x_0,R/2)} \int_{B(x_0,R/2)} \frac{|x - x_0| - R/2|^2}{|x - y|^{n+2s}} \, dx \, dy \]
\[ \geq \frac{1}{((3/2)R)^{n+2s}} \cdot \frac{\omega_{n-1}}{n} \left( \frac{R}{2} \right)^n \int_{B(x_0,R/2)} \int_{B(x_0,R/2)} \frac{|x - x_0| - R/2|^2}{|x - y|^{n+2s}} \, dx \, dy \]
\[ = \frac{2^{2s} \omega_{n-1}}{n^{n+2s} 2^{n+2}} \cdot \omega_{n-1} R^{n+2} \]
\[ \times \left[ \frac{n^2 - n + 2}{4n(n + 1)(n + 2)} + \frac{1}{2^{n+1}(n + 1)(n + 2)} \right] \]
\[ = C_4 R^{n+2-2s}, \]

where
\[ C_4 := \frac{2^{2s} \omega_{n-1}}{n^2(n + 1)(n + 2) 3^{n+2s}} \left[ \frac{n^2 - n + 2}{4n + 1} \right]. \]

Substitute (31) and (32) into (29), we get
\[ \int_{\mathbb{R}^n} \frac{|u_0(x) - u_0(y)|^2}{|x - y|^{n+2s}} \, dx \, dy \geq 2 (C_3 + 4C_4) R^{n-2s} \delta^2. \]

From (34) and (12), we obtain
\[ \|u_0\|^2_{X,4} \geq m_2^2 \|u_0\|^2_{X,0} \geq C_* \delta^2, \]

where
\[ C_* := 2 (C_3 + 4C_4) R^{n-2s}. \]

Thus, the conclusion of left-hand side of (17) holds. \( \square \)

In this paper our main tool is a three-critical-point theorem of [14] which is recalled below.

**Theorem 3** (see [14]). Let \( X \) be a reflexive real Banach space; let \( \Phi : X \to \mathbb{R} \) be a coercive, continuously Gâteaux differentiable, and sequentially weakly lower semicontinuous functional whose Gâteaux derivative admits a continuous inverse on \( X^* \), and let \( \Psi : X \to \mathbb{R} \) be a continuously Gâteaux differentiable functional whose Gâteaux derivative is compact such that
\[ \Phi(0) = \Psi(0) = 0. \]
Assume that there exist \( r > 0 \) and \( x \in X \), with \( r < \Phi(x) \), such that

(i) \( \sup_{\Phi(x) > r} \Psi(x) / r < \Psi(\Phi(x)) / \Phi(x) \);

(ii) for each \( \mu \in \Lambda \), \( | \Phi(\Phi(x)) / \Psi(x) | \) is coercive.

Then, for each \( \mu \in \Lambda \), the functional \( \Phi - \mu \Psi \) has at least three distinct critical points in \( X \).

3. Main Result

Let \( f : \Omega \times \mathbb{R} \rightarrow \mathbb{R} \) be a Carathéodory function such that

(H1) there exist \( a_1, a_2 \geq 0 \) and \( q \in (1, 2_*^*) \), \( 2_*^* := 2n/(n-2s) \), such that

\[ |f(x, v)| \leq a_1 + a_2 |v|^{q-1}, \quad \forall (x, v) \in \Omega \times \mathbb{R}. \]  (38)

Theorem 4. Let function \( f \) satisfy condition (H1). Assume that

(H2) \( F(x, v) := \int_0^v f(x, r) \, dr \geq 0 \) for all \( (x, v) \in \Omega \times \mathbb{R}^+ \);

(H3) there exist two positive constants \( b \) and \( p < 2 \) such that

\[ F(x, v) \leq b (1 + |v|^p), \]  (39)

for almost every \( x \in \Omega \) and for every \( v \in \mathbb{R} \);

(H4) let \( 0 < \lambda < \Lambda \) such that there exist two positive constants \( \gamma \) and \( \delta \), with \( \delta > \sqrt{2/C_\ast (\gamma/m_1)} \) such that

\[ \inf_{x \in \Omega} F(x, \delta) \delta^-2 > b_1 \gamma^{-2}, \]  (40)

where

\[ b_1 = \frac{n2^{n-1}a_1c_\ast}{\omega_{n-1}m_1^{2n}}, \quad b_2 = \frac{n2^{n+(q-2)/2}a_2c_\ast \gamma}{\omega_{n-1}m_1^{2n}}, \]  (41)

and two positive constants \( C_\ast \) and \( C_* \) are as in (28) and (36), respectively. Then, for every \( \mu \) belonging to

\[ \Lambda := \left\{ \frac{n2^{n-1}C_\ast}{\omega_{n-1}R^n} \delta^{-2}, \frac{n2^{n+(q-2)/2}a_2c_\ast \gamma}{\omega_{n-1}m_1^{2n}} \delta^{-2} \right\}, \]  (42)

problem (2) possesses at least three weak solutions in \( X_\mu \).

Proof. Let us apply Theorem 3 with \( X_\mu \) and

\[ J_{\lambda, \mu}(u) := \Phi_{\lambda}(u) - \mu \Psi(u), \quad u \in X_\mu, \]  (43)

where

\[ \Phi_{\lambda}(u) := \frac{1}{2} \int_{\Omega} \int_{\mathbb{R}^+} \frac{|u(x) - u(y)|^2}{|x - y|^{n+2s}} \, dx \, dy - \frac{\lambda}{2} \int_\Omega |u(x)|^2 \, dx, \]

\[ \Psi(u) := \int_\Omega F(x, u(x)) \, dx. \]  (44)

For each \( u, v \in X_\mu \), one has

\[ \Phi_{\lambda}'(u)(v) = \int_{\Omega} \int_{\mathbb{R}^+} \frac{(u(x) - u(y))(v(x) - v(y))}{|x - y|^{n+2s}} \, dx \, dy \]

\[ - \lambda \int_\Omega v(x) \, dx, \]

\[ \Psi'(u)(v) = \int_\Omega f(x, u(x)) \, v(x) \, dx. \]  (45)

From the proof of Theorem 1 in [16], we obtain that \( \Phi_{\lambda} \) is coercive, continuously Gâteaux differentiable, and sequentially weakly lower semicontinuous functional. Moreover, similar to the proof of proposition in [17], we get by (12) that

\[ \left( \Phi_{\lambda}'(u) - \Phi_{\lambda}'(v) \right)(u - v) \]

\[ = \int_{\Omega} \int_{\mathbb{R}^+} \frac{(u - v)(x) - (u - v)(y))^2}{|x - y|^{n+2s}} \, dx \, dy \]

\[ - \lambda \int_\Omega (u - v(x))^2 \, dx \]

\[ = \|u - v\|_{X_\mu}^2 \geq m_1^2 \|u - v\|_{X_\mu}^2 \]

for every \( u \) and \( v \) belonging to \( X_\mu \). This actually means that \( \Phi_{\lambda} \) is a uniformly monotone operator in \( X_\mu \). In addition, standard arguments ensure that \( \Phi_{\lambda} \) also turns out to be coercive and hemicontinuous in \( X_\mu \). Therefore, \( \Phi_{\lambda} \) admits that a continuous inverse in \( X_\mu^* \) follows immediately by applying Theorem 26. A. of [18]. Furthermore, the functional \( \Psi \) is well defined, continuously Gâteaux differentiable with compact derivative and \( \Phi_\lambda(0) - \Psi(0) = 0 \).

By [16] we know that being \( u \) a weak solution of problem (2) is equivalent to being a critical point of the functional \( J_{\lambda, \mu} \). Since \( 0 < \lambda < \Lambda \), from Lemma 2, one has

\[ \Phi_{\lambda}(u_{\delta}) = \frac{1}{2} \|u_{\delta}\|_{X_{\mu}}^2 \geq \frac{1}{2} C_\ast m_1^2 \delta^2. \]  (47)

Bearing in mind that \( \delta > \sqrt{2/C_\ast (\gamma/m_1)} \) (H4), it follows that \( \Phi_{\lambda}(u_{\delta}) > \gamma^2 \). By (H2), we obtain

\[ \Psi(u_{\delta}) = \int_\Omega F(x, u_{\delta}(x)) \, dx \geq \int_{B(\Omega, R/2)} F(x, \delta) \, dx \]

\[ \geq \inf_{x \in \Omega} F(x, \delta) \cdot \frac{\omega_{n-1} R^n}{n} \left( \frac{R}{2} \right)^n. \]  (48)

By Lemma 2, we have

\[ \Phi_{\lambda}(u_{\delta}) = \frac{1}{2} \|u_{\delta}\|_{X_{\mu}}^2 \leq \frac{1}{2} C_* \delta^2. \]  (49)

So, by (48) and (49), one has

\[ \frac{\Psi(u_{\delta})}{\Phi_{\lambda}(u_{\delta})} \geq \frac{\omega_{n-1} R^n \inf_{x \in \Omega} F(x, \delta)}{n2^{n-1}C_* \delta^2}. \]  (50)
Thanks to (H1), one has
\[ F(x, u) \leq a_1 |u| + \frac{a_2}{q} |u|^q, \quad (x, u) \in \Omega \times \mathbb{R}. \]  
(51)

Thus, by (15) and (51), for every \( u \in X_0 : \Phi_\lambda(u) \leq r \), we obtain
\[ \Psi(u) = \int_\Omega F(x, u(x)) \, dx \leq a_1 \|u\|_{L^q(\Omega)} + \frac{a_2}{q} \|u\|_{L^q(\Omega)} \]  
\[ \leq \frac{\sqrt{2}a_1c_1}{m_\lambda} \sqrt{r} + \frac{2^{q/2}c_2^2}{q^2 m^2_\lambda} r^{q/2}. \]  
(52)

Therefore
\[ \sup_{u \in \Phi_\lambda^{-1}([0, r])} \Psi(u) \leq \frac{\sqrt{2}a_1c_1}{m_\lambda} \sqrt{r} + \frac{2^{q/2}c_2^2}{q^2 m^2_\lambda} r^{q/2}. \]  
(53)

Denote the function
\[ \chi(r) := \sup_{u \in \Phi_\lambda^{-1}([0, r])} \Psi(u), \quad r > 0. \]  
(54)

By (53), we have
\[ \chi(r) \leq \frac{\sqrt{2}a_1c_1}{m_\lambda} r^{-1/2} + \frac{2^{q/2}c_2^2}{q^2 m^2_\lambda} r^{q/2-1}, \quad r > 0. \]  
(55)

Owing to (50), (55), and (H4), we have
\[ \chi(y^3) \leq \frac{\sqrt{2}a_1c_1}{m_\lambda y} + \frac{2^{q/2}c_2^2}{q^2 m^2_\lambda} y^{q/2} = \omega_{n-1} R^n \frac{1}{n^{2(q-1)/2}} \left( \frac{b_1}{y} + b_2 y^{q-2} \right) \]  
\[ < \frac{\omega_{n-1} R^n}{2^{n-1} C^*} \inf_{F(x, \delta)} F(x, \delta) \leq \Psi(\delta), \quad \Phi_\lambda(\delta) \]  
\[ \leq \Psi(\delta), \quad \Phi_\lambda(\delta). \]  
(56)

Hence, the assumption (i) of Theorem 3 is satisfied.

Furthermore, if \( p < 2 \), for each \( u \in X_0 \), \( |u|^p \in L^{2/p}(\Omega) \), Hölder’s inequality and (15) give
\[ \int |u(x)|^p \, dx \leq \|u\|_{L^q(\Omega)}^{p/q} \meas(\Omega)^{(2-p)/2} \]  
\[ \leq \frac{c_p}{m_\lambda} \|u\|_{X_{\lambda, \lambda}}^{p} \meas(\Omega)^{(2-p)/2}, \quad \forall u \in X_0. \]  
(57)

Due to (H3) and (57), we deduce that
\[ J_\lambda \mu(u) \geq \frac{1}{2} \|u\|_{X_{\lambda, \lambda}}^{2} - \mu b \cdot \meas(\Omega)^{(2-p)/2} \|u\|_{X_{\lambda, \lambda}}^{p} - \mu b \cdot \meas(\Omega), \quad \forall u \in X_0. \]  
(58)

Hence, \( J_\lambda \mu(u) \) is a coercive functional for every positive parameter \( \mu \), in particular, for each \( \mu \in \Lambda < \Phi \theta(u_\lambda)^{1/p}(n_\theta(u_\lambda)), \) \( y^2/\sup_{|v| < \theta} \phi V(\theta) \). So also condition (ii) holds. So all the assumptions of Theorem 3 are satisfied. Thus, for each \( 0 < \lambda < \lambda_1 \) there exists \( \mu > 0 \), depending on \( \lambda \), such that, for any \( \mu \in \Lambda \), the functional \( J_\lambda \mu(u) \) has at least three distinct critical points that are weak solutions to problem (2).

**Remark 5.** Similar to Example 3.1 in [15], we can give a concrete example of function satisfying hypotheses (H1)–(H4). Set \( q \in (2, 2^*_s), s \in (0, 1), \) and let
\[ h := \max \left\{ 1, \frac{2}{m_\lambda} \left( A_1 + A_2 \right)^{1/(q-2)} q^{1/(q-2)} \right\}, \]  
(59)

where
\[ A_1 = \frac{n^{2(q-1)/2} c^*_s}{\omega_{n-1} m_{\lambda} R^n}, \quad A_2 = \frac{n^{2(q-2)/2} c^*_q c^*_s}{q \omega_{n-1} m_{\lambda}^2 R^n}. \]  
(60)

From (H4) we know that \( b_1 = a_1 A_1 \) and \( b_2 = a_2 A_2 \). Let \( r \) be a positive constant such that \( r > h \) and consider the following continuous and positive function \( f : \Omega \times \mathbb{R} \rightarrow \mathbb{R} \):
\[ f(x, v) := \begin{cases} 1 + |v|^{q-1} & \text{if } v \leq r, \\ 1 + r^{q-1} v^{q-1} & \text{if } v > r. \end{cases} \]  
(61)

Obviously, \( f(x, v) \leq 1 + |v|^{q-1} \) for each \( (x, v) \in \Omega \times \mathbb{R} \), and (H1) holds. Furthermore, for every \( \eta \in \mathbb{R} \), we have
\[ F(x, \eta) \leq \left( r + \frac{r^q}{p} \right) \left( 1 + |\eta|^{\max(1, p)} \right). \]  
(62)

Thus the conditions (H2) and (H3) are satisfied. Moreover, \( r > h \geq (1/m_{\lambda}) \sqrt{2/C^*} \), and
\[ \int_0^r f(x, t) \, dt = \frac{r^{q-2}}{q} + \frac{1}{r} > A_1 + A_2, \]  
(63)

which implies that (H4) holds.

**Conflict of Interests**

The authors declare that there is no conflict of interests regarding the publication of this paper.

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