Research Article

Iterates of Bernstein Type Operators on a Triangle with All Curved Sides

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We consider some Bernstein-type operators as well as their product and Boolean sum for a function defined on a triangle with all curved sides. Using the weakly Picard operators technique and the contraction principle, we study the convergence of the iterates of these operators.

1. Bernstein Type Operators

In this paper, using the weakly Picard operators technique and the contraction principle, we study the convergence of the iterates of some operators introduced in [1]. Similar operators with the ones from [1] were studied in [2, 3] and [4], where the authors construct interpolation and Bernstein-type operators on triangles and squares with one and all curved sides. They studied the operators, their product and Boolean sum, as well as their interpolation properties, the order of accuracy, and the remainder of the corresponding approximation formulas.

We recall some results regarding Bernstein-type operators on a triangle with all curved sides from [1].

Let $\tilde{T}_h$ be the triangle with all curved sides, which has the vertices $V_1 = (0, h)$, $V_2 = (h, 0)$, $V_3 = (0, 0)$, and the three curved sides $\gamma_1$, $\gamma_2$ (along the coordinate axis), and $\gamma_3$ (opposite to the vertex $V_3$); $h \in \mathbb{R}_+$. We have that $\gamma_1$ is defined by $(x, f_1(x))$, with $f_1(0) = f_1(h) = 0$, $f_1(x) \leq 0$, for $x \in [0, h]$; $\gamma_2$ is defined by $(g_2(y), y)$, with $g_2(0) = g_2(h) = 0$, $g_2(y) \leq 0$, for $y \in [0, h]$ and $\gamma_3$ is defined by the one-to-one functions $f_3$ and $g_3$, where $g_3$ is the inverse of the function $f_3$; that is, $y = f_3(x)$ and $x = g_3(y)$, with $x, y \in [0, h]$ and $f_3(0) = g_3(0) = h$ (see Figure 1). In the sequel we denote by $e_{ij}(x, y) = x^i y^j$, for $i, j \in \mathbb{N}$.

Let $F$ be a real-valued function defined on $\tilde{T}_h$ and $(g_2(y), y), (g_3(y), y)$, respectively, and let $(x, f_1(x)), (x, f_3(x))$ be the points in which the parallel lines to the coordinate axes, passing through the point $(x, y) \in \tilde{T}_h$, intersecting the sides $\gamma_1$, $\gamma_2$, and $\gamma_3$. We consider the uniform partitions of the intervals $[g_2(y), g_3(y)]$ and $[f_1(x), f_3(x)]$, $x, y \in [0, h]$:

$$\Delta_m^x = \left\{ g_2(y) + i \frac{g_3(y) - g_2(y)}{m} \right\}_{i = 0, m},$$

$$\Delta_n^y = \left\{ f_1(x) + j \frac{f_3(x) - f_1(x)}{n} \right\}_{j = 0, n},$$

and the Bernstein-type operators $B_m^x$ and $B_n^y$ defined by

$$B_m^x F(x, y) = \sum_{i=0}^{m} p_{m,i}(x, y) F\left( g_2(y) + i \frac{g_3(y) - g_2(y)}{m}, y \right),$$

with

$$p_{m,i}(x, y) = \binom{m}{i} \left( \frac{x - g_2(y)}{g_3(y) - g_2(y)} \right)^i \left( 1 - \frac{x - g_2(y)}{g_3(y) - g_2(y)} \right)^{m-i}.$$
We have
\[ (P_{mn}F)(x, y) = \sum_{i=0}^{m} \sum_{j=0}^{n} P_{mj}(x, y) q_{n,j}(x, y) \]
with $x_i = g_2(y) + i((g_3(y) - g_4(y))/m)$, respectively,
\[ (Q_{mn}F)(x, y) = \sum_{i=0}^{m} \sum_{j=0}^{n} P_{mj}(x, y) q_{n,j}(x, y) \]
with $y_j = f_1(x) + j((f_3(x) - f_4(x))/n)$.

**Theorem 3.** If $F$ is a real-valued function defined on $\tilde{T}$, then
(i) $(P_{mn}F)(V_\gamma) = F(V_\gamma)$, $P_{mn}F = F$, on $\gamma_2$ and
(ii) $(Q_{mn}F)(V_\gamma) = F(V_\gamma), Q_{mn}F = F$, on $\gamma_3$.

We consider the Boolean sums of the operators $B^x_m$ and $B^y_n$; that is,
\[ S_{mn} := B^x_m \oplus B^y_n = B^x_m + B^y_n - B^x_mB^y_n, \]
respectively,
\[ T_{mn} := B^y_n \oplus B^x_m = B^y_n + B^x_m - B^x_mB^y_n. \]

**Theorem 4.** If $F$ is a real-valued function defined on $\tilde{T}$, then
\[ S_{mn}F|_{I_{\tilde{T}^\gamma}} = F|_{I_{\tilde{T}^\gamma}}, \]
\[ T_{mn}F|_{I_{\tilde{T}^\gamma}} = F|_{I_{\tilde{T}^\gamma}}. \]

**2. Weakly Picard Operators**

We recall some results regarding weakly Picard operators that will be used in the sequel (see, e.g., [5]).

Let $(X, d)$ be a metric space and $A : X \rightarrow X$ an operator. We denote by $F_A := \{ x \in X \mid A(x) = x \}$, the fixed points set of $A$; $I(A) := \{ Y \subset X \mid A(Y) \subset Y, Y \neq \emptyset \}$, the family of the nonempty invariant subsets of $A$; $A^0 := 1_X, A^1 := A, \ldots, A^{n+1} := A \circ A^n, n \in \mathbb{N}$.

**Definition 5.** The operator $A : X \rightarrow X$ is a Picard operator if there exists $x^* \in X$ such that
(i) $F_A = \{ x^* \}$;
(ii) the sequence $(A^n(x_0))_{n \in \mathbb{N}}$ converges to $x^*$ for all $x_0 \in X$.  

Definition 6. The operator $A$ is a weakly Picard operator if the sequence $(A^n(x))_{n \in \mathbb{N}}$ converges, for all $x \in X$, and the limit (which may depend on $x$) is a fixed point of $A$.

Definition 7. If $A$ is a weakly Picard operator, then we consider the operator $A^\infty, A^\infty : X \to X$, defined by

$$A^\infty (x) := \lim_{n \to \infty} A^n (x).$$

Theorem 8. An operator $A$ is a weakly Picard operator if and only if there exists a partition of $X, X = \bigcup_{\lambda \in \Lambda} X_\lambda$, such that

(a) $X_\lambda \in I(A), \forall \lambda \in \Lambda$;

(b) $A|_{X_\lambda} : X_\lambda \to X_\lambda$ is a Picard operator, $\forall \lambda \in \Lambda$.

3. Iterates of Bernstein Type Operators

Let $F$ be a real-valued function defined on $\tilde{T}_h, h \in \mathbb{R}_+$. Using the weakly Picard operators technique and the contraction principle, we study the convergence of the iterates of the Bernstein-type operators (3) and (5) and of their product and Boolean sum operators (7), (8), (9) and (10). The same approach for some other linear and positive operators leads to similar results in [6–12].

The limit behavior for the iterates of some classes of positive linear operators was also studied, for example, in [13–23]. In the papers [19–21] new methods were introduced (e.g., Korovkin type technique) for the study of the asymptotic behavior of the iterates of positive linear operators, positive linear operators preserving the affine functions and defined on the space of bounded real-valued functions on $[0, 1]$. This techniques enlarge the class of operators for which the limit of the iterates can be computed. In [13, 14] some methods were proposed to determine the degree of convergence for the iterates of certain positive linear operators towards the first Bernstein operator. Using the spectrum of the operators involved in [15], convergence results were proved for overiterates of certain (generalized) Bernstein-Stancu operators (see, e.g., [24–26]). In [16, 17] new techniques were introduced (infinite products, rates of convergence), based on the results from [18], in order to prove that infinite products of certain positive linear operators weakly converge to the first Bernstein operator.

Now we study the convergence of the iterates of the Bernstein-type operators (3) and (5).

Theorem 9. The operators $B^*_m$ and $B^*_n$ are weakly Picard operators and

$$B^{x,\infty}_m (x, y) = \frac{F(g_3(y), y) - F(g_2(y), y)}{g_3(y) - g_2(y)} x + \frac{g_3(y) F(g_2(y), y) - g_2(y) F(g_3(y), y)}{g_3(y) - g_2(y)},$$

(13)

$$B^{y,\infty}_n (x, y) = \frac{F(x, f_3(x)) - F(x, f_1(x))}{f_3(x) - f_1(x)} y + \frac{f_3(x) F(x, f_1(x)) - f_1(x) F(x, f_3(x))}{f_3(x) - f_1(x)}.$$  

(14)

Proof. Taking into account the interpolation properties of $B^*_m$ and $B^*_n$ (from Theorem 2), let us consider

$$X_{\mu_3, \varphi}^{(y)} = \{F \in C(T_h) \mid F(g_3(y), y) = \varphi \}_{\mu_3},$$

$$F(g_3(y), y) = \varphi_{\mu_3}^y, \quad \text{for } y \in [0, h],$$

where $\varphi_{\mu_3}^y$ is the $\mu_3$th convolution derivative of $\varphi$ at $y$.
and denote
\[
F^{(1)}_{\psi|y_1, \varphi|y_3}(x, y) := \frac{\varphi|y_1 - \varphi|y_3}{g_3(y) - g_2(y)} x + \frac{g_3(y) \varphi|y_3 - g_2(y) \varphi|y_1}{g_3(y) - g_2(y)},
\]
\[
F^{(2)}_{\psi|y_1, \varphi|y_3}(x, y) := \frac{\psi|y_1 - \psi|y_3}{f_3(x) - f_1(x)} x + \frac{f_3(x) \psi|y_3 - f_1(x) \psi|y_1}{f_3(x) - f_1(x)},
\]
with \( \varphi, \psi \in C(\overline{I}_h) \).

We have the following properties:

(i) \( X^{(1)}_{\varphi|y_1, \psi|y_3} \) and \( X^{(2)}_{\varphi|y_1, \psi|y_3} \) are closed subsets of \( C(\overline{I}_h) \);

(ii) \( X^{(1)}_{\varphi|y_1, \psi|y_3} \) is an invariant subset of \( B^*_m \) and \( X^{(2)}_{\varphi|y_1, \psi|y_3} \) is an invariant subset of \( B^*_n \) for \( \varphi, \psi \in C(\overline{I}_h) \) and \( m, n \in \mathbb{N}^* \);

(iii) \( C(\overline{I}_h) = \bigcup_{\varphi \in C(\overline{I}_h)} X^{(1)}_{\varphi|y_1, \psi|y_3} \) and \( C(\overline{I}_h) = \bigcup_{\varphi \in C(\overline{I}_h)} X^{(2)}_{\varphi|y_1, \psi|y_3} \) are partitions of \( C(\overline{I}_h) \);

(iv) \( F^{(1)}_{\varphi|y_1, \psi|y_3} \in X^{(1)}_{\varphi|y_1, \psi|y_3} \cap F_{B_m} \) and \( F^{(2)}_{\varphi|y_1, \psi|y_3} \in X^{(2)}_{\varphi|y_1, \psi|y_3} \cap F_{B_n} \), where \( F_{B_m} \) and \( F_{B_n} \) denote the fixed points sets of \( B^*_m \) and \( B^*_n \).

The statements (i) and (iii) are obvious.

(ii), by linearity of Bernstein operators and Theorem 2, it follows that \( \forall F^{(1)}_{\varphi|y_1, \psi|y_3} \in X^{(1)}_{\varphi|y_1, \psi|y_3} \) and \( \forall F^{(2)}_{\varphi|y_1, \psi|y_3} \in X^{(2)}_{\varphi|y_1, \psi|y_3} \); we have
\[
B^*_m F^{(1)}_{\varphi|y_1, \psi|y_3}(x, y) = F^{(1)}_{\varphi|y_1, \psi|y_3}(x, y),
\]
\[
B^*_n F^{(2)}_{\varphi|y_1, \psi|y_3}(x, y) = F^{(2)}_{\varphi|y_1, \psi|y_3}(x, y).
\]
So, \( X^{(1)}_{\varphi|y_1, \psi|y_3} \) and \( X^{(2)}_{\varphi|y_1, \psi|y_3} \) are invariant subsets of \( B^*_m \) and, respectively, of \( B^*_n \), for \( \varphi, \psi \in C(\overline{I}_h) \) and \( m, n \in \mathbb{N}^* \).

(iv), we prove that
\[
B^*_m X^{(1)}_{\varphi|y_1, \psi|y_3} \rightarrow X^{(1)}_{\varphi|y_1, \psi|y_3},
\]
\[
B^*_n X^{(2)}_{\varphi|y_1, \psi|y_3} \rightarrow X^{(2)}_{\varphi|y_1, \psi|y_3},
\]
are contractions for \( \varphi, \psi \in C(\overline{I}_h) \) and \( m, n \in \mathbb{N}^* \).

Let \( F, G \in X^{(1)}_{\varphi|y_1, \psi|y_3} \). From (3) we have
\[
|B^*_m(F(x, y) - B^*_m(G)(x, y))|
\]
\[
= |B^*_m(F - G)(x, y)|
\]
\[
\leq \left| 1 - \left( 1 - \frac{x - g_2(y)}{g_3(y) - g_2(y)} \right)^m \right| - \left( \frac{x - g_2(y)}{g_3(y) - g_2(y)} \right)^m
\]
\[
\|F - G\|_{\infty} \leq \left( 1 - \frac{1}{2^{m-1}} \right) \|F - G\|_{\infty},
\]
where \( \| \cdot \|_{\infty} \) denotes the Chebyshev norm. So,
\[
\|B^*_m(F(x, y) - B^*_m(G)(x, y))\|_{\infty}
\]
\[
\leq \left( 1 - \frac{1}{2^{m-1}} \right) \|F - G\|_{\infty}, \quad \forall F, G \in X^{(1)}_{\varphi|y_1, \psi|y_3},
\]
that is, \( B^*_m X^{(1)}_{\varphi|y_1, \psi|y_3} \) is a contraction for \( \varphi \in C(\overline{I}_h) \).

Analogously we have
\[
|B^*_n(F(x, y) - B^*_n(G)(x, y))|
\]
\[
= |B^*_n(F - G)(x, y)|
\]
\[
\leq \left| 1 - \left( 1 - \frac{y - f_1(x)}{f_3(x) - f_1(x)} \right)^n \right| - \left( \frac{y - f_1(x)}{f_3(x) - f_1(x)} \right)^n
\]
\[
\|F - G\|_{\infty} \leq \left( 1 - \frac{1}{2^{n-1}} \right) \|F - G\|_{\infty},
\]
whence
\[
\|B^*_n(F(x, y) - B^*_n(G)(x, y))\|_{\infty}
\]
\[
\leq \left( 1 - \frac{1}{2^{n-1}} \right) \|F - G\|_{\infty}, \quad \forall F, G \in X^{(2)}_{\varphi|y_1, \psi|y_3},
\]
that is, \( B^*_n X^{(2)}_{\varphi|y_1, \psi|y_3} \) is a contraction for \( \psi \in C(\overline{I}_h) \).

On the other hand, \( ((\varphi|y_1 - \varphi|y_3)/(g_3(y) - g_2(y)))\) · \( (g_3(y)\varphi|y_2 - g_2(y)\varphi|y_1)/(g_3(y) - g_2(y)) \) \( \in X^{(1)}_{\varphi|y_1, \psi|y_3} \), \( ((\psi|y_1 - \psi|y_3)/(f_3(x) - f_1(x)))\) · \( (f_3(x)\psi|y_3 - f_1(x)\psi|y_1)/(f_3(x) - f_1(x)) \) \( \in X^{(2)}_{\varphi|y_1, \psi|y_3} \) are fixed points of \( B^*_m \) and \( B^*_n \); that is,
\[
B^*_m \left( \frac{\varphi|y_1 - \varphi|y_3}{g_3(y) - g_2(y)} \right) = \frac{g_3(y) \varphi|y_3 - g_2(y) \varphi|y_1}{g_3(y) - g_2(y)},
\]
\[
B^*_n \left( \frac{\psi|y_1 - \psi|y_3}{f_3(x) - f_1(x)} \right) = \frac{f_3(x) \psi|y_3 - f_1(x) \psi|y_1}{f_3(x) - f_1(x)}.
\]
From the contraction principle, $F^{(1)}_{\psi_{1},\psi_{3}}(x, y) := ((\psi_{1}|_{\gamma_{1}} - \psi_{3}|_{\gamma_{3}})/(g_{1}(y) - g_{2}(y)))x + (g_{1}(y)\psi_{3}|_{\gamma_{1}} - g_{2}(y)\psi_{3}|_{\gamma_{3}})/(g_{2}(y) - g_{3}(y)))$ is the unique fixed point of $B^{x}_{m}$ in $X^{(1)}_{\psi_{1},\psi_{3}}$ and $B^{x}_{m}X^{(1)}_{\psi_{1},\psi_{3}}$ is a Picard operator, with

$$
(B^{x}_{m}F)(x, y) = \frac{F(g_{1}(y), y) - F(g_{2}(y), y)}{g_{2}(y) - g_{3}(y)}x + g_{3}(y)F(g_{2}(y), y) - g_{2}(y)F(g_{3}(y), y),
$$

and, similarly, $F^{(2)}_{\psi_{1},\psi_{3}}(x, y) := ((\psi_{1}|_{\gamma_{1}} - \psi_{3}|_{\gamma_{3}})/(f_{3}(x) - f_{1}(x)))y + ((f_{1}(x)\psi_{3}|_{\gamma_{1}} - f_{3}(x)\psi_{3}|_{\gamma_{3}})/(f_{3}(x) - f_{1}(x)))$ is the unique fixed point of $B^{y}_{n}$ in $X^{(2)}_{\psi_{1},\psi_{3}}$ and $B^{y}_{n}X^{(2)}_{\psi_{1},\psi_{3}}$ is a Picard operator, with

$$
(B^{y}_{n}F)(x, y) = \frac{F(x, f_{1}(x)) - F(x, f_{2}(x))}{f_{2}(x) - f_{1}(x)}y + \frac{f_{2}(x)F(x, f_{1}(x)) - f_{1}(x)F(x, f_{2}(x))}{f_{2}(x) - f_{1}(x)}.
$$

Consequently, taking into account (ii), by Theorem 8 it follows that the operators $B^{x}_{m}$ and $B^{y}_{n}$ are weakly Picard operators.

Now we study the convergence of the product and Boolean sum operators (7) and (9).

**Theorem 10.** The operator $P_{m+n}$ is a weakly Picard operator and

$$
(P_{m+n}^{x}F)(x, y) = \frac{1}{\|g_{2}(y) - g_{3}(y)\|} \times \left[ g_{3}(y)F(x_{0}, f_{1}(x_{0})) + g_{2}(y)F(x_{1}, f_{1}(x_{1})) \times \frac{\alpha + g_{2}(y)F(x_{1}, f_{1}(x_{1})))}{\beta} \right]
$$

with $\alpha, \beta, \gamma, \delta \in \mathbb{R}$.

We remark that

(i) $X_{\alpha,\beta,\gamma,\delta}$ is a closed subset of $C(\bar{T}_{h})$;

(ii) $X_{\alpha,\beta,\gamma,\delta}$ is an invariant subset of $P_{m+n}$ for $\alpha, \beta, \gamma, \delta \in \mathbb{R}$ and $m, n \in \mathbb{N}$;

(iii) $C(\bar{T}_{h}) = \bigcup_{\alpha,\beta,\gamma,\delta} X_{\alpha,\beta,\gamma,\delta}$ is a partition of $C(\bar{T}_{h})$;

(iv) $F_{\alpha,\beta,\gamma,\delta} \in X_{\alpha,\beta,\gamma,\delta} \cap F_{m+n}$, where $F_{m+n}$ denote the fixed points sets of $P_{m+n}$.

The statements (i) and (iii) are obvious.

(ii), similarly with the proof of Theorem 9, by linearity of Bernstein operators and Theorem 3, it follows that $X_{\alpha,\beta,\gamma,\delta}$ is an invariant subset of $P_{m+n}$, for $\alpha, \beta, \gamma, \delta \in \mathbb{R}$ and $m, n \in \mathbb{N}$.

(iv), we prove that

$$
P_{m+n}\mid X_{\alpha,\beta,\gamma,\delta} : X_{\alpha,\beta,\gamma,\delta} \rightarrow X_{\alpha,\beta,\gamma,\delta}
$$

is a contraction for $\alpha, \beta, \gamma, \delta \in \mathbb{R}$ and $m, n \in \mathbb{N}$. Let $F, G \in X_{\alpha,\beta,\gamma,\delta}$. From [7, Lemma 8] it follows that

$$
|P_{m+n}(F)(x, y) - P_{m+n}(G)(x, y)| \leq (1 - \frac{1}{2^{m+n-2}})\|F - G\|_{\infty}.
$$

\textbf{Proof.} Let $X_{\alpha,\beta,\gamma,\delta} = \{F \in C(\bar{T}_{h}) \mid F(x_{0}, f_{1}(x_{1})) = \alpha, F(x_{1}, f_{3}(x_{1})) = \beta, F(x_{1}, f_{5}(x_{1})) = \gamma, F(x_{0}, f_{3}(x_{0})) = \delta\}$ and denote

$$
F_{\alpha,\beta,\gamma,\delta}(x, y) := (g_{3}(y) f_{5}(x_{0}) + g_{2}(y) f_{1}(x_{1}))
$$

$$
- (g_{3}(y) f_{1}(x_{1}) - g_{2}(y) f_{3}(x_{0} \beta)) \times \left[ (g_{3}(y) - g_{2}(y)) \left[ f_{3}(x) - f_{1}(x) \right]^{-1} \right]
$$

$$
+ \frac{f_{1}(x_{1}) \delta + f_{3}(x_{0}) \beta - f_{3}(x_{0}) \alpha - f_{1}(x_{1}) \gamma}{g_{3}(y) - g_{2}(y)} \left[ f_{3}(x) - f_{1}(x) \right]^{y}
$$

$$
+ \frac{g_{3}(y) \delta + g_{2}(y) \beta - g_{2}(y) \alpha - g_{2}(y) \gamma}{g_{3}(y) - g_{2}(y)} \left[ f_{3}(x) - f_{1}(x) \right]^{y}
$$

$$
+ \alpha \beta - \delta \times \left[ g_{3}(y) - g_{2}(y) \right][f_{3}(x) - f_{1}(x)]^{xy}
$$

\text{with} \alpha, \beta, \gamma, \delta \in \mathbb{R}.
So,
\[ \| P_{mn}(F)(x, y) - P_{mn}(G)(x, y) \|_\infty \leq \left( 1 - \frac{1}{2^{m+n-2}} \right) \| F - G \|_\infty, \forall F, G \in X_{\alpha, \beta, \gamma, \delta}, \]
that is, \( P_{mn} \mid X_{\alpha, \beta, \gamma, \delta} \) is a contraction for \( \alpha, \beta, \gamma, \delta \in \mathbb{R} \).

From the contraction principle we have that \( P_{mn} \mid X_{\alpha, \beta, \gamma, \delta} \) is the unique fixed point of \( P_{mn} \) in \( X_{\alpha, \beta, \gamma, \delta} \) and \( P_{mn} \mid X_{\alpha, \beta, \gamma, \delta} \) is a Picard operator, so (26) holds. Consequently, taking into account (ii), by Theorem 8 it follows that the operator \( P_{mn} \) is a weakly Picard operator.

Remark 11. We have a similar result for the operator \( Q_{mn} \).

**Theorem 12.** The operator \( S_{mn} \) is a weakly Picard operator and
\[
(S_{mn}^n F)(x, y) = \frac{F(g_3(y), y) - F(g_1(y), y)}{g_3(y) - g_1(y)} \times [g_3(y) F(g_2(y), y) - g_1(y) F(g_3(y), y) \\
+ \frac{F(x, f_3(x)) - F(x, f_1(x))}{f_3(x) - f_1(x)} + \frac{f_3(x) F(x, f_1(x)) - f_1(x) F(x, f_3(x))}{f_3(x) - f_1(x)} \times [g_3(y) - g_2(y)] [f_3(x) - f_1(x)] \\
\times [f_1(x)] F(x, f_3(x)) + f_3(x) F(x, f_1(x)) - f_3(x) F(x, f_1(x)) - f_1(x) F(x, f_3(x)) \\
- \frac{y}{[g_3(y) - g_2(y)] [f_3(x) - f_1(x)]} \times [g_3(y) F(x_0, f_3(x_0)) + g_2(y) F(x_1, f_1(x_1)) \\
- g_3(y) F(x_1, f_1(x_1)) - g_2(y) F(x_1, f_3(x_1)) \times \frac{xy}{[g_3(y) - g_2(y)] [f_3(x) - f_1(x)]} \times [F(x_0, f_3(x_0)) + F(x_1, f_3(x_1)) \\
- \frac{[F(x_0, f_3(x_0)) - F(x_1, f_1(x_1))]}], \\
with x_0 = g_2(y), x_1 = g_3(y). \tag{31} \]

**Proof.** The proof follows the same steps as in the previous theorems but using the following inequality:
\[
\| S_{mn}(F)(x, y) - S_{mn}(G)(x, y) \|_\infty \leq \left[ 1 - \left( \frac{1}{2^{m-1}} + \frac{1}{2^{m-1}} - \frac{1}{2^{m+n-2}} \right) \right] \| F - G \|_\infty, \tag{32} \]
in order to prove that \( S_{mn} \) is a contraction. \( \square \)

**Conflict of Interests**

The authors declare that there is no conflict of interests regarding the publication of this paper.

**References**


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