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Research Article

Nonuniform Dependence on Initial Data of a Periodic Camassa-Holm System

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This paper is concerned with some properties of a periodic two-component Camassa-Holm system. By constructing two sequences of solutions of the two-component Camassa-Holm system, we prove that the solution map of the Cauchy problem of the two-component Camassa-Holm system is not uniformly continuous in $H^s(\mathbb{S})$, $s > 5/2$.

1. Introduction

In this paper, we consider the Cauchy problem of the following two-component periodic Camassa-Holm system:

$$\begin{gathered} m_t + um_x + 2mu_x = -\rho\rho_x, \quad t > 0,\ x \in \mathbb{R}, \\ \rho_t + (\rho u)_x = 0, \quad t > 0,\ x \in \mathbb{R}, \\ u(0,x) = u_0(x), \qquad \rho(0,x) = \rho_0(x), \quad x \in \mathbb{R}, \\ u(t,x+1) = u(t,x), \qquad \rho(t,x+1) = \rho(t,x), \\ t \geq 0,\ x \in \mathbb{R}, \end{gathered} \tag{1}$$

where $m = u - u_{xx}$. The Camassa-Holm equation can be obtained via the obvious reduction $\rho \equiv 0$.

The Camassa-Holm (CH) equation has been extended to a two-component integrable system (CH2) by combining its integrability property with compressibility, or free-surface elevation dynamics in its shallow-water interpretation [1, 2]; that is,

$$\begin{gathered} m_t + um_x + 2mu_x + \sigma\rho\rho_x = 0, \quad t > 0,\ x \in \mathbb{R}, \\ \rho_t + (\rho u)_x = 0, \quad t > 0,\ x \in \mathbb{R}, \end{gathered} \tag{2}$$

where $m = u - u_{xx}$ and $\sigma = \pm 1$. Local well-posedness of system (2) with $\sigma = 1$ was obtained by [1, 3]. The precise blow-up scenarios and blow-up phenomena of strong solution for system (2) was established by [1, 3–6]. Just recently, Gui and Liu [7] studied system (1) with $\sigma = 1$ in Besov space and they obtained the local well-posedness. In this paper, we consider the Cauchy problem of system (1) and study some properties of it.

If $\rho \equiv 0$, then system (2) becomes the well-known Camassa-Holm equation [8]. In the past decade, the Camassa-Holm equation has attracted much attention because of its integrability and the existence of multipeakon solution; see [4, 8–22] for the details. The Cauchy problem and initial boundary value problem of the Camassa-Holm equation have been studied extensively [10, 23]. It has been shown that the Camassa-Holm equation is locally well-posedness [10] for initial data $u_0 \in H^s(\mathbb{S})$, $s > 3/2$. Moreover, it has global strong solutions [10, 18] and finite time blow-up solutions [10]. On the other hand, it has global weak solution in $H^1(\mathbb{S})$ [8, 9, 13, 19]. The advantage of the Camassa-Holm equation in comparison with the KdV equation lies in the fact that the Camassa-Holm equation has peaked solutions and models wave breaking (i.e., the solution remains bounded while its slope becomes unbounded in finite time [8, 10, 24]).

Recently, some properties of solutions to the Camassa-Holm equation have been studied by many authors. Himonas et al. [15] studied the persistence properties and unique continuation of solutions of the Camassa-Holm equation. They showed that a strong solution of the Camassa-Holm equation, initially decaying exponentially together with its spatial derivative, must be identically equal to zero if it also

decays exponentially at a later time; see [11, 22] for the same properties of solutions to other shallow water equations. Just recently, Himonas and Kenig [16] and Himonas et al. [14, 17] considered the nonuniform dependence on initial data for the Camassa-Holm equation on the line and on the circle, respectively. Lv et al. [25] obtained the nonuniform dependence on initial data for μ-b equation. Lv and Wang [26] considered the system (1) with $\rho = \gamma - \gamma_{xx}$ and obtained the nonuniform dependence on initial data.

In this paper, we will consider the nonuniform dependence on initial data to system (1). We remark that there is significant difference between system (1) and system (1) with $\rho = \gamma - \gamma_{xx}$. It is easy to see that when $\rho = \gamma - \gamma_{xx}$, there are some similar properties between the two equations in system (1). Thus the proof of nonuniform dependence on initial data to system (1) with $\rho = \gamma - \gamma_{xx}$ is similar to the single equation, for example, Camassa-Holm equation. But in system (1), ρ and u have different properties; see Theorem 1. This needs constructing different asymptotic solution; see Section 3.

This paper is organized as follows. In Section 2, we recall the well-posedness result of Hu and Yin [27] and use it to prove the basic energy estimate from which we derive a lower bound for the lifespan of the solution as well as an estimate of the $H^s(\mathbb{S}) \times H^{s-1}(\mathbb{S})$ norm of the solution $(u(t,x), \rho(t,x))$ in terms of $H^s(\mathbb{S}) \times H^{s-1}(\mathbb{S})$ norm of the initial data (u_0, ρ_0). In Section 3, we construct approximate solutions, compute the error, and estimate the H^1-norm of this error. In Section 4, we estimate the difference between approximate and actual solutions, where the exact solution is a solution to system (1) with initial data given by the approximate solutions evaluated at time zero. The nonuniform dependence on initial data for system (1) is established in Section 5 by constructing two sequences of solutions to system (1) in a bounded subset of the Sobolev space $H^s(\mathbb{S})$, whose distance at the initial time is converging to zero while at any later time it is bounded below by a positive constant. During preparing our paper, we find another paper [28] where the same problem has been considered, but our method is different from theirs.

Notation. In the following, we denote by $*$ the spatial convolution. Given a Banach space Z, we denote its norm by $\|\cdot\|_Z$. Since all space of functions are over $\mathbb{S}$, for simplicity, we drop $\mathbb{S}$ in our notations of function spaces if there is no ambiguity. Let $[A,B] = AB - BA$ denote the commutator of linear operator A and B; see [29, 30] for the details. Set $\|z\|^2_{H^s\times H^{s-1}} = \|u\|^2_{H^s} + \|\rho\|^2_{H^{s-1}}$, where $z = (u,\rho)$.

2. Local Well-Posedness

In this section we first recall the known results of Hu and Yin [27] and give a new estimate of the solution to (1).

Let $\Lambda = (1-\partial_x^2)^{1/2}$. Then the operator Λ^{-2} acting on $L^2(\mathbb{S})$ can be expressed by its associated Green's function $G(x) = \cosh(x-[x]-(1/2))/2\sinh(1/2)$, where $[x]$ stands for the integer part of x, as

$$\begin{aligned}\Lambda^{-2} f(x) &= (G * f)(x)\\ &= \frac{1}{2}\int_{\mathbb{S}} \frac{\cosh(x-y-[x-y]-(1/2))}{\sinh(1/2)} f(y)\,dy,\\ &\qquad f \in L^2(\mathbb{S}).\end{aligned} \tag{3}$$

Hence (1) is equivalent to the following system:

$$\begin{aligned} &u_t + uu_x = -\partial_x\Lambda^{-2}\left(u^2 + \frac{1}{2}u_x^2 + \frac{1}{2}\rho^2\right), \quad t>0,\ x\in\mathbb{R},\\ &\rho_t + u\rho_x = -u_x\rho, \quad t>0,\ x\in\mathbb{R},\\ &u(0,x) = u_0(x), \quad \rho(0,x) = \rho_0(x),\ x\in\mathbb{R},\\ &u(t,x+1) = u(t,x), \quad \rho(t,x+1) = \rho(t,x), \quad t\ge 0,\ x\in\mathbb{R}.\end{aligned} \tag{4}$$

In the rest of this paper, we will consider the following system:

$$\begin{aligned} &u_t + uu_x = -\partial_x\Lambda^{-2}\left(u^2 + \frac{1}{2}u_x^2 + \frac{1}{2}\rho^2\right), \quad t>0,\ x\in\mathbb{S},\\ &\rho_t + u\rho_x = -u_x\rho, \quad t>0,\ x\in\mathbb{S},\\ &u(0,x) = u_0(x), \quad \rho(0,x) = \rho_0(x), \quad x\in\mathbb{S}.\end{aligned} \tag{5}$$

The following result is obtained by Hu and Yin [27].

Theorem 1 (see [27]). *Given $z_0 = (u_0,\rho_0) \in H^s \times H^{s-1}$, $s \ge 2$. Then there exists a maximal existence time $T = T(\|z_0\|_{H^s\times H^{s-1}}) > 0$ and a unique solution $z = (u,\rho)$ to system (5) such that*

$$\begin{aligned} z &= z(\cdot, z_0)\\ &\in C\left([0,T); H^s\times H^{s-1}\right) \cap C^1\left([0,T); H^{s-1}\times H^{s-2}\right).\end{aligned} \tag{6}$$

Moreover, the solution depends continuously on the initial data; that is, the mapping

$$\begin{aligned} z_0 &\longrightarrow z(\cdot, z_0) : H^s\times H^{s-1}\\ &\longrightarrow C\left([0,T); H^s\times H^{s-1}\right)\\ &\quad \cap C^1\left([0,T); H^{s-1}\times H^{s-2}\right)\end{aligned} \tag{7}$$

is continuous.

Next, we will give an explicit estimate for the maximal existence time T. Also, we will show that at any time t in

the time interval $[0, T_0]$ the H^s-norm of the solution $z(t, x)$ is dominated by the H^s-norm of the initial data $z_0(x)$. In order to do this, we need the following lemmas.

Lemma 2 (see [29]). *If $r > 0$, then $H^r \cap L^\infty$ is an algebra. Moreover,*

$$\|fg\|_{H^r} \le C\left(\|f\|_\infty \|g\|_{H^r} + \|f\|_{H^r}\|g\|_\infty\right), \tag{8}$$

where C is a positive constant depending only on r.

Lemma 3 (see [29]). *If $r > 0$, then*

$$\|[\Lambda^r, f]\, g\|_2 \le C\left(\|f_x\|_\infty \left\|\Lambda^{r-1} g\right\|_2 + \|\Lambda^r f\|_2 \|g\|_\infty\right), \tag{9}$$

where C is a positive constant depending only on r.

Theorem 4. *Let $s > 5/2$. If $z = (u, \rho)$ is a solution of system (5) with initial data z_0 described in Theorem 1, then the maximal existence time T satisfies*

$$T \ge T_0 := \frac{1}{2C_s \|z_0\|_{H^s \times H^{s-1}}}, \tag{10}$$

where C_s is a constant depending only on s. Also, we have

$$\|z(t)\|_{H^s \times H^{s-1}} \le 2\|z_0\|_{H^s \times H^{s-1}}, \quad 0 \le t \le T_0. \tag{11}$$

Proof. The derivation of the lower bound for the maximal existence time (10) and the solution size estimate (11) is based on the following differential inequality for the solution z:

$$\frac{1}{2}\frac{d}{dt}\|z(t)\|^2_{H^s \times H^{s-1}} \le C_s \|z(t)\|^3_{H^s \times H^{s-1}}, \quad 0 \le t < T. \tag{12}$$

Suppose that (12) holds. Then, integrating (12) from 0 to t, we have

$$\|z(t)\|_{H^s \times H^{s-1}} \le \frac{\|z_0\|_{H^s \times H^{s-1}}}{1 - C_s \|z_0\|_{H^s \times H^{s-1}} t}. \tag{13}$$

It follows from the above inequality that $\|z(t)\|_{H^s \times H^{s-1}}$ is finite if $C_s \|z_0\|_{H^s \times H^{s-1}} t < 1$. Let $T_0 = 1/2C_s \|z_0\|_{H^s \times H^{s-1}}$, then, for $0 \le t \le T_0$, we have

$$\|z(t)\|_{H^s \times H^{s-1}} \le \frac{\|z_0\|_{H^s \times H^{s-1}}}{1 - C_s \|z_0\|_{H^s \times H^{s-1}} T_0} = 2\|z_0\|_{H^s \times H^{s-1}}. \tag{14}$$

Now we prove inequality (12). Note that the products uu_x and $u\rho_x$ are only in H^{s-1} if $u, \rho \in H^s$. To deal with this problem, we will consider the following modified system:

$$\begin{aligned} &(J_\varepsilon u)_t + J_\varepsilon (uu_x) \\ &\quad = -\partial_x \Lambda^{-2}\left(J_\varepsilon u^2 + \frac{1}{2} J_\varepsilon u_x^2 + \frac{1}{2} J_\varepsilon \rho^2\right), \\ &\qquad\qquad t > 0,\ x \in \mathbb{S}, \\ &(J_\varepsilon \rho)_t + J_\varepsilon (u\rho_x) = -J_\varepsilon (u_x \rho), \quad t > 0,\ x \in \mathbb{S}, \end{aligned} \tag{15}$$

where for each $\varepsilon \in (0, 1]$ the operator J_ε is the Friedrichs mollifier defined by

$$J_\varepsilon f(x) = J_\varepsilon (f)(x) = j_\varepsilon * f. \tag{16}$$

Here $j_\varepsilon(x) = (1/\varepsilon) j(x/\varepsilon)$, and $j(x)$ is a C^∞ function supported in the interval $[-1, 1]$ such that $j(x) \ge 0$, $\int_{\mathbb{S}} j(x)dx = 1$. Applying the operator Λ^s and Λ^{s-1} to the first and second equations of (15), respectively, then multiplying the resulting equations by $\Lambda^s J_\varepsilon u$ and $\Lambda^{s-1} J_\varepsilon \rho$, respectively, and integrating them with respect to $x \in \mathbb{S}$, we obtain

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\|J_\varepsilon u\|^2_{H^s} = &-\int_{\mathbb{S}} \Lambda^s J_\varepsilon (uu_x) \Lambda^s J_\varepsilon u\, dx \\ &-\int_{\mathbb{S}} \partial_x \Lambda^{s-2} \partial_x \Lambda^{-2}\left(J_\varepsilon u^2 + \frac{1}{2} J_\varepsilon u_x^2 \right. \\ &\qquad \left. + \frac{1}{2} J_\varepsilon \rho^2\right) \Lambda^s J_\varepsilon u\, dx, \end{aligned} \tag{17}$$

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\|J_\varepsilon \rho\|^2_{H^{s-1}} = &-\int_{\mathbb{S}} \Lambda^{s-1} J_\varepsilon (u\rho_x) \Lambda^{s-1} J_\varepsilon \rho\, dx \\ &-\int_{\mathbb{S}} \Lambda^{s-1} J_\varepsilon (u_x \rho) \Lambda^{s-1} J_\varepsilon \rho\, dx. \end{aligned} \tag{18}$$

We estimate the right-hand sides of (17) and (18), and we will use the fact that Λ^s and J_ε are commutative and

$$(J_\varepsilon f, g)_0 = (f, J_\varepsilon g)_0, \qquad \|J_\varepsilon u\|_{H^s} \le \|u\|_{H^s}. \tag{19}$$

To estimate the first integrals in the right-hand sides of (17) and (18) we write them as follows:

$$\begin{aligned} \int_{\mathbb{S}} \Lambda^s J_\varepsilon (uu_x) \Lambda^s J_\varepsilon u\, dx &= \int_{\mathbb{S}} \Lambda^s (uu_x) J_\varepsilon \Lambda^s J_\varepsilon u\, dx \\ &= ([\Lambda^s, u]\, u_x, J_\varepsilon \Lambda^s J_\varepsilon u)_0 \\ &\quad + (u\Lambda^s u_x, J_\varepsilon \Lambda^s J_\varepsilon u)_0, \\ \int_{\mathbb{S}} \Lambda^{s-1} J_\varepsilon (u\rho_x) \Lambda^{s-1} J_\varepsilon \rho\, dx &= \int_{\mathbb{S}} \Lambda^{s-1} (u\rho_x) J_\varepsilon \Lambda^{s-1} J_\varepsilon \rho\, dx \\ &= \left(\left[\Lambda^{s-1}, u\right] \rho_x, J_\varepsilon \Lambda^{s-1} J_\varepsilon \rho\right)_0 \\ &\quad + \left(u\Lambda^{s-1} \rho_x, J_\varepsilon \Lambda^{s-1} J_\varepsilon \rho\right)_0. \end{aligned} \tag{20}$$

Using Lemma 3 and (19), we can estimate the first part in the right-hand sides of (20)

$$\begin{aligned} \left|([\Lambda^s, u]\, u_x, J_\varepsilon \Lambda^s J_\varepsilon u)_0\right| &\le \|[\Lambda^s, u]\, u_x\|_2 \|J_\varepsilon \Lambda^s J_\varepsilon u\|_2 \\ &\le C_s \|u_x\|_\infty \|u\|^2_{H^s}, \\ \left|\left(\left[\Lambda^{s-1}, u\right] \rho_x, J_\varepsilon \Lambda^{s-1} J_\varepsilon \rho\right)_0\right| &\le \|u_x\|_\infty \|\rho\|^2_{H^{s-1}} \\ &\quad + \|\rho_x\|_\infty \|u\|_{H^s} \|\rho\|_{H^{s-1}}, \end{aligned} \tag{21}$$

where we use the fact that $\|u\|_{H^s} = \|\Lambda^s u\|_2$. Noting that

$$\|[J_\varepsilon, u]\, f_x\|_2 \le C\|u_x\|_\infty \|f\|_2, \tag{22}$$

which is obtained by Himonas and Kenig (see [16, Lemma 2]), and integrating by parts, we obtain

$$\begin{aligned}
\left|(u\Lambda^s u_x, J_\varepsilon \Lambda^s J_\varepsilon u)_0\right| &= \left|(J_\varepsilon u \Lambda^s u_x, \Lambda^s J_\varepsilon u)_0\right| \\
&= \left|([J_\varepsilon, u]\, \partial_x \Lambda^s u, \Lambda^s J_\varepsilon u)_0\right. \\
&\quad \left. + (u J_\varepsilon \partial_x \Lambda^s u, \Lambda^s J_\varepsilon u)_0\right| \\
&\le \|[J_\varepsilon, u]\, \partial_x \Lambda^s u\|_2 \|u\|_{H^s} \\
&\quad + \frac{1}{2}\left|(u_x \Lambda^s J_\varepsilon u, \Lambda^s J_\varepsilon u)_0\right| \\
&\le C_s \,\|\, u_x\|_\infty \,\|\, u\|_{H^s}^2,
\end{aligned} \tag{23}$$

$$\begin{aligned}
\left|\left(u\Lambda^{s-1}\rho_x, J_\varepsilon \Lambda^{s-1} J_\varepsilon \rho\right)_0\right| &= \left|\left([J_\varepsilon, u]\, \partial_x \Lambda^{s-1}\rho, \Lambda^{s-1} J_\varepsilon \rho\right)_0\right| \\
&\quad + \frac{1}{2}\left|\left(u_x \Lambda^{s-1} J_\varepsilon \rho, \Lambda^{s-1} J_\varepsilon \rho\right)_0\right| \\
&\le C_s \|u_x\|_\infty \|\rho\|_{H^{s-1}}^2.
\end{aligned} \tag{24}$$

Combining (21)–(24), we have

$$\left|\int_{\mathbb{S}} \Lambda^s J_\varepsilon (u u_x)\, \Lambda^s J_\varepsilon u \, dx\right| \le C_s \|u_x\|_\infty \|u\|_{H^s}^2, \tag{25}$$

$$\begin{aligned}
&\left|\int_{\mathbb{S}} \Lambda^{s-1} J_\varepsilon (u\rho_x)\, \Lambda^{s-1} J_\varepsilon \rho \, dx\right| \\
&\quad \le C_s \left(\|u_x\|_\infty + \|\rho_x\|_\infty\right)\left(\|u\|_{H^s}^2 + \|\rho\|_{H^{s-1}}^2\right).
\end{aligned} \tag{26}$$

For the second integral in the right-hand side of (17), we have

$$\begin{aligned}
&\left|\int_{\mathbb{S}} \partial_x \Lambda^{s-2}\left(J_\varepsilon u^2 + \frac{1}{2} J_\varepsilon u_x^2 + \frac{1}{2} J_\varepsilon \rho^2\right) \Lambda^s J_\varepsilon u \, dx\right| \\
&\quad \le \left\|\partial_x \Lambda^{s-2}\left(J_\varepsilon u^2 + \frac{1}{2} J_\varepsilon u_x^2 + \frac{1}{2} J_\varepsilon \rho^2\right)\right\|_2 \|u\|_{H^s} \\
&\quad \le \left\|J_\varepsilon u^2 + \frac{1}{2} J_\varepsilon u_x^2 + \frac{1}{2} J_\varepsilon \rho^2\right\|_{H^{s-1}} \|u\|_{H^s} \\
&\quad \le C_s \left(\|u\|_\infty + \|\rho\|_\infty + \|u_x\|_\infty\right)\left(\|u\|_{H^s}^2 + \|\rho\|_{H^{s-1}}^2\right),
\end{aligned} \tag{27}$$

where we have used Lemma 2 with $r = s-1$. Similarly, for the second and third integrals in the right-hand side of (18), we get

$$\begin{aligned}
\left|\int_{\mathbb{S}} \Lambda^{s-1} J_\varepsilon (u_x \rho)\, \Lambda^{s-1} J_\varepsilon \rho \, dx\right| &\le \|\rho u_x\|_{H^{s-1}} \|\rho\|_{H^{s-1}} \\
&\le C_s \|u_x\|_\infty \|\rho\|_{H^{s-1}}^2.
\end{aligned} \tag{28}$$

Submitting (25), (27), and (26), (28) into (17) and (18), respectively, we obtain

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|J_\varepsilon u\|_{H^s}^2 &\le C_s \left(\|u\|_\infty + \|\rho\|_\infty + \|u_x\|_\infty \|\rho_x\|_\infty\right) \\
&\quad \times \left(\|u\|_{H^s}^2 + \|\rho\|_{H^{s-1}}^2\right), \\
\frac{1}{2}\frac{d}{dt}\|J_\varepsilon \rho\|_{H^{s-1}}^2 &\le C_s \left(\|u\|_\infty + \|\rho\|_\infty + \|u_x\|_\infty + \|\rho_x\|_\infty\right) \\
&\quad \times \left(\|u\|_{H^s}^2 + \|\rho\|_{H^{s-1}}^2\right).
\end{aligned} \tag{29}$$

Consequently,

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left(\|J_\varepsilon u\|_{H^s}^2 + \|J_\varepsilon \rho\|_{H^{s-1}}^2\right) \\
&\quad \le C_s \left(\|u\|_\infty + \|\rho\|_\infty + \|u_x\|_\infty + \|\rho_x\|_\infty\right) \\
&\quad \times \left(\|u\|_{H^s}^2 + \|\rho\|_{H^{s-1}}^2\right).
\end{aligned} \tag{30}$$

Then, letting ε go to 0, we have

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left(\|u\|_{H^s}^2 + \|\rho\|_{H^{s-1}}^2\right) \\
&\quad \le C_s \left(\|u\|_\infty + \|\rho\|_\infty + \|u_x\|_\infty + \|\rho_x\|_\infty\right) \\
&\quad \times \left(\|u\|_{H^s}^2 + \|\rho\|_{H^{s-1}}^2\right),
\end{aligned} \tag{31}$$

or

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|z(t)\|_{H^s\times H^{s-1}}^2 &\le C_s \left(\|u(t)\|_{C^1} + \|\rho\|_{C^1}\right) \\
&\quad \times \|z(t)\|_{H^s\times H^{s-1}}^2.
\end{aligned} \tag{32}$$

Since $s > 5/2$, using Sobolev's inequality we have that

$$\|u(t)\|_{C^1} \le C_s \|u(t)\|_{H^s}, \qquad \|\rho(t)\|_{C^1} \le C_s \|\rho(t)\|_{H^{s-1}}. \tag{33}$$

From (32) we obtain the desired inequality (12). This completes the proof of Theorem 4. □

Recall that $\|z(t)\|_{H^s\times H^{s-1}}^2 = \|u(t)\|_{H^s}^2 + \|\rho(t)\|_{H^{s-1}}^2$, where $z(t) = (u(t), \rho(t))$. It follows from Theorem 4 that

$$\begin{aligned}
\|u(t)\|_{H^s}, \|\rho(t)\|_{H^{s-1}} &\le \|z(t)\|_{H^s\times H^{s-1}} \\
&\le 2\|z_0\|_{H^s\times H^{s-1}}, \qquad 0 \le t \le T_0.
\end{aligned} \tag{34}$$

3. Approximate Solutions

In this section we first construct a two-parameter family of approximate solutions by using a similar method to [17] and then compute the error and last estimate the $H^1 \times L^2$-norm of the error.

Following [17], our approximate solutions $u^{\omega,\lambda} = u^{\omega,\lambda}(t, x)$ and $\rho^{\omega,\lambda} = \rho^{\omega,\lambda}(t, x)$ to (5) will consist of a low frequency and a high frequency part, that is,

$$\begin{aligned}
u^{\omega,\lambda} &= \omega\lambda^{-1} + \lambda^{-s}\cos(\lambda x - \omega t), \\
\rho^{\omega,\lambda} &= \omega\lambda^{-1} + \lambda^{-s+1}\cos(\lambda x - \omega t),
\end{aligned} \tag{35}$$

where ω is in a bounded set of $\mathbb{S}$ and λ is in the set of positive integers $\mathbb{Z}_+$.

Now we compute the error. Substituting the approximate solution $(u^{\omega,\lambda}, \rho^{\omega,\lambda})$ into the first and second equation of (5), we get the following error:

$$\begin{aligned} E &= u_t^{\omega,\lambda} + u^{\omega,\lambda} u_x^{\omega,\lambda} \\ &\quad + \partial_x \Lambda^{-2}\left(\left(u^{\omega,\lambda}\right)^2 + \frac{1}{2}\left(u_x^{\omega,\lambda}\right)^2 + \frac{1}{2}\left(\rho^{\omega,\lambda}\right)^2\right), \\ F &= \rho_t^{\omega,\lambda} + u^{\omega,\lambda}\rho_x^{\omega,\lambda} + u_x^{\omega,\lambda}\rho^{\omega,\lambda}. \end{aligned} \tag{36}$$

Direct calculation shows that

$$\begin{aligned} & u_t^{\omega,\lambda} + u^{\omega,\lambda} u_x^{\omega,\lambda} \\ &\quad = -\frac{1}{2}\lambda^{-2s+1}\sin(2\lambda x - 2\omega t) := E_1, \\ & \partial_x \Lambda^{-2}\left(\left(u^{\omega,\lambda}\right)^2 + \frac{1}{2}\left(u_x^{\omega,\lambda}\right)^2 + \frac{1}{2}\left(\rho^{\omega,\lambda}\right)^2\right) \\ &\quad = -\frac{3}{2}\lambda^{-2s+1}\Lambda^{-2}\sin(2\lambda x - 2\omega t) \\ &\qquad - 3\omega\lambda^{-s}\Lambda^{-2}\sin(\lambda x - \omega t) \\ &\qquad + \frac{1}{2}\lambda^{-2s+3}\Lambda^{-2}\sin(2\lambda x - 2\omega t) \\ &\qquad - \omega\lambda^{-s+1}\Lambda^{-2}\sin(\lambda x - \omega t) \\ &\quad := E_2 + E_3 + E_4 + E_5. \end{aligned} \tag{37}$$

Similarly, we have

$$\begin{aligned} & \rho_t^{\omega,\lambda} + u^{\omega,\lambda}\rho_x^{\omega,\lambda} + +u_x^{\omega,\lambda}\rho^{\omega,\lambda} \\ &\quad = -\lambda^{-s}\sin(\lambda x - \omega t) \\ &\qquad - \lambda^{-2s+2}\sin(2\lambda x - 2\omega t) \\ &\quad := F_1 + F_2. \end{aligned} \tag{38}$$

Let C be a generic positive constant. For any positive quantities P and Q, we write $P \lesssim Q$ $(P \gtrsim Q)$ meaning that $P \le CQ$ $(P \ge CQ)$ in the following.

Next, we estimate the error. We remark that the error of the periodic Camassa-Holm equation contains $E_i (i = 1, 2, 3)$ and the estimate of E_i was contained in paper [17]. In [17], they obtained that

$$\begin{aligned} \|E_1\|_{H^1}, \|E_2\|_{H^1}, \|E_4\|_{H^1} &\lesssim \lambda^{-2s+2}, \\ \|E_3\|_{H^1} &\lesssim \lambda^{-s-1}. \end{aligned} \tag{39}$$

Now, we estimate E_5 and F_i $(i = 1, 2)$. We need the following lemma.

Lemma 5 (see [17]). *Let $\sigma \in \mathbb{R}$. If $\lambda \in \mathbb{Z}_+$ and $\lambda \gg 1$ then*

$$\|\cos(\lambda x - \alpha)\|_{H^\sigma(\mathbb{S})} \approx \lambda^\sigma, \quad \alpha \in \mathbb{R}. \tag{40}$$

The above relation also holds if $\cos(\lambda x - \alpha)$ is replaced with $\sin(\lambda x - \alpha)$.

Estimating the H^1-Norm of E_5. By using the definition of Λ, we have

$$\begin{aligned} \|E_5\|_{H^1} &= \left\|\omega\lambda^{-s+1}\Lambda^{-2}\sin(\lambda x - \omega t)\right\|_{H^1} \\ &= \left|\omega\lambda^{-s+1}\right| \|\sin(\lambda x - \omega t)\|_{H^{-1}} \\ &\lesssim \lambda^{-s}, \end{aligned} \tag{41}$$

where we used Lemma 5.

Estimating the L^2-Norms of F_1 and F_2. Also, we have

$$\begin{aligned} \|F_1\|_{L^2} &= \|\lambda^{-s}\sin(\lambda x - \omega t)\|_{L^2} \lesssim \lambda^{-s}, \\ \|F_2\|_{L^2} &= \left\|\lambda^{-2s+2}\sin(2\lambda x - 2\omega t)\right\|_{L^2} \lesssim \lambda^{-2s+2}. \end{aligned} \tag{42}$$

Collecting all error estimates together, we have the following Theorem.

Theorem 6. *Let $s > 5/2$. If ω is bounded, then for $\lambda \gg 1$ we have*

$$\|E\|_{H^1}, \|F\|_{L^2} \lesssim \lambda^{-s}. \tag{43}$$

4. Difference between Approximate and Actual Solutions

In this section, we will estimate the difference between the approximate and actual solutions.

Let $(u_{\omega,\lambda}(t,x), \rho_{\omega,\lambda}(t,x))$ be the solution to system (5) with initial data of the value of the approximate solution $(u^{\omega,\lambda}(t,x), \rho^{\omega,\lambda}(t,x))$ at time zero; that is, $(u_{\omega,\lambda}(t,x), \rho_{\omega,\lambda}(t,x))$ satisfies

$$\begin{aligned} & \partial_t u_{\omega,\lambda} + u_{\omega,\lambda}\partial_x u_{\omega,\lambda} \\ &\quad + \partial_x \Lambda^{-2}\left(u_{\omega,\lambda}^2 + \frac{1}{2}\left(\partial_x u_{\omega,\lambda}\right)^2 + \frac{1}{2}\rho_{\omega,\lambda}^2\right) = 0, \\ &\qquad t > 0,\ x \in \mathbb{S}, \\ & \partial_t \rho_{\omega,\lambda} + u_{\omega,\lambda}\partial_x \rho_{\omega,\lambda} + u_{\omega,\lambda}\partial\rho_{\omega,\lambda} = 0, \quad t > 0,\ x \in \mathbb{S}, \\ & u_{\omega,\lambda}(0,x) = u^{\omega,\lambda}(0,x) = \omega\lambda^{-1} + \lambda^{-s}\cos(\lambda x), \quad x \in \mathbb{S}, \\ & \rho_{\omega,\lambda}(0,x) = \rho^{\omega,\lambda}(0,x) = \omega\lambda^{-1} + \lambda^{-s+1}\cos(\lambda x), \quad x \in \mathbb{S}. \end{aligned} \tag{44}$$

Note that $(u_{\omega,\lambda}(0,x), \rho_{\omega,\lambda}(0,x)) \in H^s \times H^{s-1}$, $s \ge 0$. Moreover, we have

$$\begin{aligned} \|u_{\omega,\lambda}(0,x)\|_{H^s} &\lesssim |\omega|\lambda^{-1} + 1, \quad \lambda \gg 1, \\ \|\rho_{\omega,\lambda}(0,x)\|_{H^{s-1}} &\lesssim |\omega|\lambda^{-1} + 1, \quad \lambda \gg 1. \end{aligned} \tag{45}$$

Therefore, if $s > 5/2$, by using Theorems 1 and 4, we have that, for any ω in a bounded set and $\lambda \gg 1$, problem (44) has a unique solution $z_{\omega,\lambda} \in C([0,T]; H^s) \times C([0,T]; H^{s-1})$ with

$$T \gtrsim \frac{1}{\|z_{\omega,\lambda}(0)\|_{H^s \times H^{s-1}}} \gtrsim \frac{1}{1 + \lambda^{-1}} \gtrsim 1. \tag{46}$$

To estimate the difference between the approximate and actual solutions, we let

$$v = u^{\omega,\lambda} - u_{\omega,\lambda}, \qquad \sigma = \rho^{\omega,\lambda} - \rho_{\omega,\lambda}. \tag{47}$$

Then (v, σ) satisfies

$$\begin{aligned}
&v_t - vv_x + u^{\omega,\lambda} v_x + vu_x^{\omega,\lambda} \\
&\quad - \partial_x \Lambda^{-2} \Big[v^2 + \frac{1}{2} v_x^2 + \frac{1}{2}\sigma^2 - 2u^{\omega,\lambda} v \\
&\qquad - u_x^{\omega,\lambda} v_x - \rho^{\omega,\lambda}\sigma \Big] = E, \quad t > 0,\ x \in \mathbb{S}, \\
&\sigma_t - v\sigma_x + u^{\omega,\lambda}\sigma_x + v\rho_x^{\omega,\lambda} \\
&\quad - \left(\sigma v_x - u^{\omega,\lambda}\sigma - \rho^{\omega,\lambda} v_x\right) = F, \quad t > 0,\ x \in \mathbb{S}, \\
&\quad v(0,x) = \sigma(0,x) = 0, \quad x \in \mathbb{S},
\end{aligned} \tag{48}$$

where E and F are defined as in Section 3.

Now we prove that the H^1-norm of difference decays.

Theorem 7. *Let $s > 5/2$; then*

$$\|v(t)\|_{H^1} \lesssim \lambda^{-s}, \quad \|\sigma(t)\|_{L^2} \lesssim \lambda^{-s}, \quad 0 \le t \le T,\ \lambda \gg 1. \tag{49}$$

Proof. Note that

$$\frac{1}{2}\frac{d}{dt}\|v(t)\|_{H^1}^2 = \int_{\mathbb{S}} \left(vv_t + v_x v_{xt}\right) dx, \tag{50}$$

$$\frac{1}{2}\frac{d}{dt}\|\sigma(t)\|_{L^2}^2 = \int_{\mathbb{S}} \sigma\sigma_t \, dx. \tag{51}$$

Applying the operator $1 - \partial_x^2 = \Lambda^2$ to both sides of the first equations of system (48), we have

$$\begin{aligned}
v_t &= \Lambda^2 E - \Lambda^2 \left(u^{\omega,\lambda} v_x - vu_x^{\omega,\lambda}\right) \\
&\quad - \left(2u^{\omega,\lambda} v + u_x^{\omega,\lambda} v_x + \rho^{\omega,\lambda}\sigma\right)_x \\
&\quad + \frac{1}{2}\left(\sigma^2\right)_x + 3vv_x - 2v_x v_{xx} - vv_{xxx} + v_{xxt},
\end{aligned} \tag{52}$$

$$\sigma_t = F - \left(u^{\omega,\lambda}\sigma_x + v\rho_x^{\omega,\lambda}\right) - \left(u_x^{\omega,\lambda}\sigma + \rho^{\omega,\lambda} v_x\right) + (v\sigma)_x. \tag{53}$$

Substituting (52) and (53) into (50) and (51), respectively, we obtain

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|v(t)\|_{H^1}^2 &= \int_{\mathbb{S}} v\Lambda^2 \widetilde{E}\, dx - \int_{\mathbb{S}} v\Lambda^2 \left(u^{\omega,\lambda} v_x + vu_x^{\omega,\lambda}\right) dx \\
&\quad - \int_{\mathbb{S}} v\left(2u^{\omega,\lambda} v + u_x^{\omega,\lambda} v_x + \rho^{\omega,\lambda}\sigma\right)_x dx \\
&\quad + \frac{1}{2}\int_{\mathbb{S}} v\left(\sigma^2\right)_x dx \\
&\quad + \int_{\mathbb{S}} \left(v\left(3vv_x - 2v_x v_{xx} - vv_{xxx} + v_{xxt}\right)\right. \\
&\qquad \left. + v_x v_{xt}\right) dx,
\end{aligned} \tag{54}$$

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|\sigma(t)\|_{L^2}^2 &= \int_{\mathbb{S}} \sigma\widetilde{F}\, dx - \int_{\mathbb{S}} \sigma\left(u^{\omega,\lambda}\sigma_x + v\rho_x^{\omega,\lambda}\right) dx \\
&\quad - \int_{\mathbb{S}} \sigma\left(\rho^{\omega,\lambda} v_x + \sigma u_x^{\omega,\lambda}\right) dx + \int_{\mathbb{S}} \sigma(v\sigma)_x dx.
\end{aligned} \tag{55}$$

It is direct to calculate that

$$\begin{aligned}
&\int_{\mathbb{S}} \left(v\left(3vv_x - 2v_x v_{xx} - vv_{xxx} + v_{xxt}\right) + v_x v_{xt}\right) dx \\
&\quad = \int_{\mathbb{S}} \left[\left(v^3\right)_x - \left(v^2 v_{xx}\right)_x + \left(vv_{xt}\right)_x\right] dx = 0.
\end{aligned} \tag{56}$$

Substituting the above equalities into (54) and adding the resulting equations, we get

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left(\|v(t)\|_{H^1}^2 + \|\sigma(t)\|_{L^2}^2\right) \\
&\quad = \int_{\mathbb{S}} v\Lambda^2 \widetilde{E}\, dx + \int_{\mathbb{S}} \sigma\widetilde{F}\, dx \\
&\qquad - \int_{\mathbb{S}} v\Lambda^2 \left(u^{\omega,\lambda} v_x + vu_x^{\omega,\lambda}\right) dx \\
&\qquad - \int_{\mathbb{S}} \sigma\left(u^{\omega,\lambda}\sigma_x + v\rho_x^{\omega,\lambda}\right) dx \\
&\qquad - \int_{\mathbb{S}} v\left(2u^{\omega,\lambda} v + u_x^{\omega,\lambda} v_x + \rho^{\omega,\lambda}\sigma\right)_x dx \\
&\qquad - \int_{\mathbb{S}} \sigma\left(\rho^{\omega,\lambda} v_x + \sigma u_x^{\omega,\lambda}\right) dx \\
&\qquad + \int_{\mathbb{S}} \left[\frac{1}{2} v\left(\sigma^2\right)_x + \sigma(v\sigma)_x\right] dx \\
&\quad := I_1 + I_2 + \cdots + I_7.
\end{aligned} \tag{57}$$

We first look at the last term I_7. Integrating by parts gives

$$I_7 = \int_{\mathbb{S}} \left[\frac{1}{2} v\left(\sigma^2\right)_x + \sigma(v\sigma)_x\right] dx = 0. \tag{58}$$

Estimates of Integrals I_1 and I_2. Integrating by parts and applying the Cauchy-Schwarz inequality, we have

$$\begin{aligned}
&\left|\int_{\mathbb{S}} v\Lambda^2 E\, dx\right| = \left|\int_{\mathbb{S}} \left(vE - v_x E_x\right) dx\right| \le \|E\|_{H^1}\|v(t)\|_{H^1}, \\
&\left|\int_{\mathbb{S}} \sigma F\, dx\right| \le \|F\|_{L^2}\|\sigma(t)\|_{L^2}.
\end{aligned} \tag{59}$$

Estimates of Integrals I_3. Integrating by parts, we get

$$\begin{aligned} -I_3 &= \int_{\mathbb{S}} v\Lambda^2\left(u^{\omega,\lambda}v_x + vu_x^{\omega,\lambda}\right)dx \\ &= \int_{\mathbb{S}} v\left(u^{\omega,\lambda}v_x + vu_x^{\omega,\lambda}\right)dx \\ &\quad + \int_{\mathbb{S}} v_x\left(u^{\omega,\lambda}v_x\right)_x dx + \int_{\mathbb{S}} v_x\left(vu_x^{\omega,\lambda}\right)_x dx, \end{aligned} \tag{60}$$

and estimate its first part by

$$\begin{aligned} &\left|\int_{\mathbb{S}} v\left(u^{\omega,\lambda}v_x + vu_x^{\omega,\lambda}\right)dx\right| \\ &\quad \le \left(\left\|u^{\omega,\lambda}(t)\right\|_\infty + \left\|u_x^{\omega,\lambda}(t)\right\|_\infty\right)\|v(t)\|_{H^1}^2. \end{aligned} \tag{61}$$

Its second part can be estimated by

$$\begin{aligned} \left|\int_{\mathbb{S}} v_x\left(u^{\omega,\lambda}v_x\right)_x dx\right| &= \left|\int_{\mathbb{S}}\left(u_x^{\omega,\lambda}v_x^2 + \frac{1}{2}u^{\omega,\lambda}\left(v_x^2\right)_x\right)dx\right| \\ &= \frac{1}{2}\left|\int_{\mathbb{S}} u_x^{\omega,\lambda}v_x^2 dx\right| \le \left\|u_x^{\omega,\lambda}\right\|_\infty \|v(t)\|_{H^1}^2. \end{aligned} \tag{62}$$

For the last part, integrating by parts, we obtain

$$\begin{aligned} \left|\int_{\mathbb{S}} v_x\left(vu_x^{\omega,\lambda}\right)_x dx\right| &\le \left|\int_{\mathbb{S}} u_x^{\omega,\lambda}v_x^2 dx\right| + \left|\int_{\mathbb{S}} vv_x u_{xx}^{\omega,\lambda} dx\right| \\ &\le \left(\left\|u_x^{\omega,\lambda}(t)\right\|_\infty + \left\|u_{xx}^{\omega,\lambda}(t)\right\|_\infty\right)\|v(t)\|_{H^1}^2. \end{aligned} \tag{63}$$

Estimates of Integrals I_4 and I_6. Integrating by parts, we can deal with the integral I_4:

$$\begin{aligned} |I_4| &= \left|-\int_{\mathbb{S}} \sigma\left(u^{\omega,\lambda}\sigma_x + v\rho_x^{\omega,\lambda}\right)dx\right| \\ &= \left|\int_{\mathbb{S}} \frac{\sigma^2}{2}u_x^{\omega,\lambda} + v\sigma\rho_x^{\omega,\lambda}dx\right| \\ &\le \left(\left\|u_x^{\omega,\lambda}(t)\right\|_{L^\infty} + \left\|\rho_x^{\omega,\lambda}(t)\right\|_{L^\infty}\right) \\ &\quad \times \left(\|\sigma\|_{L^2}^2 + \|v(t)\|_{H^1}^2\right). \end{aligned} \tag{64}$$

Similarly, we can estimate the term I_6,

$$\begin{aligned} |I_6| &= \left|-\int_{\mathbb{S}} \sigma\left(\rho^{\omega,\lambda}v_x + \sigma u_x^{\omega,\lambda}\right)dx\right| \\ &\le \left(\left\|u_x^{\omega,\lambda}(t)\right\|_{L^\infty} + \left\|\rho^{\omega,\lambda}(t)\right\|_{L^\infty}\right) \\ &\quad \times \left(\|\sigma\|_{L^2}^2 + \|v(t)\|_{H^1}^2\right). \end{aligned} \tag{65}$$

Estimate of the Integral I_5. Integrating by parts, we have

$$\begin{aligned} |I_5| &= \left|\int_{\mathbb{S}} v\left(2u^{\omega,\lambda}v + u_x^{\omega,\lambda}v_x + \rho^{\omega,\lambda}\sigma\right)_x dx\right| \\ &= \left|\int_{\mathbb{S}} v_x\left(2u^{\omega,\lambda}v + u_x^{\omega,\lambda}v_x + \rho^{\omega,\lambda}\sigma\right)dx\right| \\ &\le \left(\left\|u^{\omega,\lambda}(t)\right\|_\infty + \left\|u_x^{\omega,\lambda}(t)\right\|_\infty + \left\|\rho^{\omega,\lambda}(t)\right\|_\infty\right) \\ &\quad \times \left(\|v(t)\|_{H^1}^2 + \|\sigma(t)\|_{L^2}^2\right). \end{aligned} \tag{66}$$

Combining the estimations of I_1–I_7, we have that

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\left(\|v(t)\|_{H^1}^2 + \|\sigma(t)\|_{L^2}^2\right) \\ &\quad \le \left(\|E\|_{H^1} + \|F\|_{H^1}\right)\left(\|v(t)\|_{H^1} + \|\sigma(t)\|_{L^2}\right) \\ &\qquad + \left(\left\|u^{\omega,\lambda}(t)\right\|_\infty + \left\|u_x^{\omega,\lambda}(t)\right\|_\infty + \left\|u_{xx}^{\omega,\lambda}(t)\right\|_\infty \right. \\ &\qquad\quad \left. + \left\|\rho^{\omega,\lambda}(t)\right\|_\infty + \left\|\rho_x^{\omega,\lambda}(t)\right\|_\infty\right) \\ &\qquad \times \left(\|v(t)\|_{H^1}^2 + \|\sigma(t)\|_{H^1}^2\right). \end{aligned} \tag{67}$$

Note that

$$\begin{aligned} u^{\omega,\lambda} &= \omega\lambda^{-1} + \lambda^{-s}\cos(\lambda x - \omega t), \\ \rho^{\omega,\lambda} &= \omega\lambda^{-1} + \lambda^{-s}\cos(\lambda x - \omega t); \end{aligned} \tag{68}$$

we have $\partial_x u^{\omega,\lambda} = -\lambda^{-s+1}\sin(\lambda x - \omega t)$. Furthermore, we have

$$\begin{aligned} &\left\|u^{\omega,\lambda}\right\|_{L^\infty}, \lesssim \lambda^{-1} + \lambda^{-s}, \\ &\left\|\partial_x u^{\omega,\lambda}\right\|_{L^\infty}, \left\|\partial\rho^{\omega,\lambda}\right\|_{L^\infty} \lesssim \lambda^{-s+1}, \\ &\left\|u_{xx}^{\omega,\lambda}\right\|_{L^\infty} \lesssim \lambda^{-s+2}. \end{aligned} \tag{69}$$

Submitting (69) into (67) and using Theorem 6, we obtain

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\|z(t)\|_{H^1\times L^2}^2 \\ &\quad \lesssim \lambda^{-s}\|z(t)\|_{H^1\times L^2} + \lambda^{-s+2}\|z(t)\|_{H^1\times L^2}^2, \end{aligned} \tag{70}$$

which implies that

$$\|z(t)\|_{H^1\times L^2} \lesssim \lambda^{-s}. \tag{71}$$

Noting that

$$\|v(t)\|_{H^1}, \left\|\varphi(t)\right\|_{L^2} \le \|z(t)\|_{H^1\times L^2}, \tag{72}$$

combining the above inequality, we complete the proof of Theorem 7. □

5. Nonuniform Dependence

In this section, we will prove nonuniform dependence for system (5) by taking advantage of the information provided by Theorems 1–4, 6, and 7. Our main result is the following.

Theorem 8. *If $s > 5/2$, then the data-to-solution $z(0) \to z(t)$ for system (5) is not uniformly continuous from any bounded subset of $H^s \times H^{s-1}$ into $C([-T,T];H^s) \times C([-T,T];H^{s-1})$, where $z(0) = (u_0(x), \rho_0(x))$ and $z(t) = (u(t,x), \rho(t,x))$. More precisely, there exist two sequences of solutions $(u_\lambda(t), \rho_\lambda(t))$ and $(\tilde{u}_\lambda(t), \tilde{\rho}_\lambda(t))$ to the differential equations of (5) in $C([-T,T];H^s) \times C([-T,T];H^{s-1})$ such that*

$$\|u_\lambda(t)\|_{H^s} + \|\tilde{u}_\lambda(t)\|_{H^s} + \|\rho_\lambda(t)\|_{H^{s-1}} + \|\tilde{\rho}_\lambda(t)\|_{H^{s-1}} \lesssim 1,$$
$$\lim_{\lambda\to\infty} \|u_\lambda(0) - \tilde{u}_\lambda(0)\|_{H^s} = \lim_{\lambda\to\infty} \|\rho_\lambda(0) - \tilde{\rho}_\lambda(0)\|_{H^{s-1}} = 0, \tag{73}$$

$$\liminf_{\lambda\to\infty} \left(\|u_\lambda(t) - \tilde{u}_\lambda(t)\|_{H^s} + \|\rho_\lambda(t) - \tilde{\rho}_\lambda(t)\|_{H^{s-1}}\right) \gtrsim \sin t, \quad |t| < T \le 1. \tag{74}$$

Proof. Let $(u_\lambda(t), \rho_\lambda(t)) = (u_{1,\lambda}(t,x), \rho_{1,\lambda}(t,x))$ and $(\tilde{u}_\lambda(t), \tilde{\rho}_\lambda(t)) = (u_{-1,\lambda}(t,x), \rho_{-1,\lambda}(t,x))$, where $(u_{1,\lambda}(t,x), \rho_{1,\lambda}(t,x))$ and $(u_{-1,\lambda}(t,x), \rho_{-1,\lambda}(t,x))$ are the unique solution to Cauchy problem (44) with initial data $(u^{1,\lambda}(0,x), \rho^{1,\lambda}(0,x))$ and $(u^{-1,\lambda}(0,x), \rho^{-1,\lambda}(0,x))$, respectively.

It follows from Theorem 1 that these solutions belong in $C([0,T];H^s) \times C([0,T];H^{s-1})$. By (48) and the assumptions after Theorem 1, we see that T is independent of $\lambda \gg 1$. Letting $k = [s] + 2$ and using estimate (34), we have

$$\|u_{\pm1,\lambda}(t)\|_{H^k}, \|\rho_{\pm1,\lambda}(t)\|_{H^{k-1}} \lesssim \|z^{\pm1,\lambda}(0)\|_{H^k\times H^{k-1}}, \tag{75}$$

where $z^{\pm1,\lambda}(0) = (u^{\pm1,\lambda}(0), \rho^{\pm1,\lambda}(0))$ and $\|z^{\pm1,\lambda}(0)\|^2_{H^k\times H^{k-1}} = \|u^{\pm1,\lambda}(0)\|^2_{H^k} + \|\rho^{\pm1,\lambda}(0)\|^2_{H^{k-1}}$. If λ is large enough, then from Lemma 5 we have

$$\begin{aligned}\|u^{\pm1,\lambda}(t)\|_{H^k} &\le \|u_{\pm1,\lambda}(t)\|_{H^k} \\ &\quad + \lambda^{-(1/2)\delta-s}\left\|\phi\left(\frac{x}{\lambda^\delta}\right)\cos(\lambda x - \omega t)\right\|_{H^k} \\ &\lesssim \lambda^{-1+(1/2)\delta} + \lambda^{k-s}\|\phi\|_2,\end{aligned} \tag{76}$$

which gives

$$\|u^{\pm1,\lambda}(t)\|_{H^k} \lesssim \lambda^{k-s}. \tag{77}$$

Combining (75) with (77), we obtain

$$\|u_{\pm1,\lambda}(t)\|_{H^k} \lesssim \lambda^{k-s}, \quad \lambda \gg 1. \tag{78}$$

Estimate (77) together with (78) yields

$$\|u^{\pm1,\lambda}(t) - u_{\pm1,\lambda}(t)\|_{H^k} \lesssim \lambda^{k-s}, \quad \lambda \gg 1. \tag{79}$$

Theorem 7 implies that

$$\|u^{\pm1,\lambda}(t) - u_{\pm1,\lambda}(t)\|_{H^1} \lesssim \lambda^{-s}, \quad \lambda \gg 1. \tag{80}$$

Now, applying the interpolation inequality

$$\|\varphi\|_{H^s} \le \|\varphi\|_{H^{s_1}}^{(s_2-s)/(s_2-s_1)} \|\varphi\|_{H^{s_2}}^{(s-s_1)/(s_2-s_1)} \tag{81}$$

with $s_1 = 1$ and $s_2 = [s] + 2 = k$ and using estimates (79) and (80), we get

$$\begin{aligned}\|u^{\pm1,\lambda}(t) - u_{\pm1,\lambda}(t)\|_{H^s} &\le \|u^{\pm1,\lambda}(t) - u_{\pm1,\lambda}(t)\|_{H^1}^{(k-s)/(k-1)} \\ &\quad \times \|u^{\pm1,\lambda}(t) - u_{\pm1,\lambda}(t)\|_{H^k}^{(s-1)/(k-1)} \\ &\lesssim \lambda^{-s(k-s)/(k-1)}\lambda^{(k-s)(s-1)/(k-1)} \\ &\lesssim \lambda^{-(k-s)/(k-1)}, \quad \lambda \gg 1.\end{aligned} \tag{82}$$

Hence

$$\|u^{\pm1,\lambda}(t) - u_{\pm1,\lambda}(t)\|_{H^s} \lesssim \lambda^{-\varepsilon_s}, \quad \lambda \gg 1, \tag{83}$$

where $\varepsilon_s = 1/(s+2)$.

Next, we prove (73) and (74). From (44), we have

$$\begin{aligned}\|u_{1,\lambda}(0) - u_{-1,\lambda}(0)\|_{H^s} &= \|2\lambda^{-1}\|_{H^s} \\ &\cong \lambda^{-1} \longrightarrow 0 \quad \text{as } \lambda \longrightarrow \infty, \\ \|\rho_{1,\lambda}(0) - \rho_{-1,\lambda}(0)\|_{H^{s-1}} &= \|2\lambda^{-1}\|_{H^{s-1}} \\ &\cong \lambda^{-1} \longrightarrow 0 \quad \text{as } \lambda \longrightarrow \infty,\end{aligned} \tag{84}$$

which implies that (73) holds.

Now, we prove (74). Obviously, we have

$$\begin{aligned}&\liminf_{\lambda\to\infty} \left(\|u_\lambda(t) - \tilde{u}_\lambda(t)\|_{H^s} + \|\rho_\lambda(t) - \tilde{\rho}_\lambda(t)\|_{H^{s-1}}\right) \\ &\quad \ge \liminf_{\lambda\to\infty} \|u_\lambda(t) - \tilde{u}_\lambda(t)\|_{H^s}.\end{aligned} \tag{85}$$

Thus we only prove that

$$\liminf_{\lambda\to\infty} \|u_\lambda(t) - \tilde{u}_\lambda(t)\|_{H^s} \gtrsim \sin t, \quad |t| < T \le 1. \tag{86}$$

It is easy to see that

$$\begin{aligned}\|u_{1,\lambda}(t) - u_{-1,\lambda}(t)\|_{H^s} &\ge \|u^{1,\lambda}(t) - u^{-1,\lambda}(t)\|_{H^s} \\ &\quad - \|u^{1,\lambda}(t) - u_{1,\lambda}(t)\|_{H^s} \\ &\quad - \|u^{-1,\lambda}(t) - u_{-1,\lambda}(t)\|_{H^s}.\end{aligned} \tag{87}$$

It follows from (77) that

$$\|u_{1,\lambda}(t) - u_{-1,\lambda}(t)\|_{H^s} \ge \|u^{1,\lambda}(t) - u^{-1,\lambda}(t)\|_{H^s} - c\lambda^{-\varepsilon_s}, \quad \lambda \gg 1, \tag{88}$$

which implies that

$$\begin{aligned}&\liminf_{\lambda\to\infty} \|u_{1,\lambda}(t) - u_{-1,\lambda}(t)\|_{H^s} \\ &\quad \ge \liminf_{\lambda\to\infty} \|u^{1,\lambda}(t) - u^{-1,\lambda}(t)\|_{H^s}.\end{aligned} \tag{89}$$

Using the identity

$$\cos\alpha - \cos\beta = -2\sin\frac{\alpha+\beta}{2}\sin\frac{\alpha-\beta}{2} \tag{90}$$

gives

$$u^{1,\lambda}(t) - u^{-1,\lambda}(t) = 2\lambda^{-1} + 2\lambda^{-s}\sin(\lambda x)\sin t. \tag{91}$$

Thus,

$$\begin{aligned}\left\|u^{1,\lambda}(t) - u^{-1,\lambda}(t)\right\|_{H^s} &\geq 2\lambda^{-s}\|\sin\lambda x\|_{H^s}|\sin t| - 2\lambda^{-1}\|1\|_{H^s} \\ &\gtrsim \lambda^{-s}\|\sin\lambda x\|_{H^s}|\sin t| - \lambda^{-1}, \quad \lambda \gg 1.\end{aligned} \tag{92}$$

Letting $\lambda \to \infty$ in the above inequality, we have

$$\lim_{\lambda\to\infty}\inf\left\|u^{1,\lambda}(t) - u^{-1,\lambda}(t)\right\|_{H^s} \gtrsim |\sin t|. \tag{93}$$

Summing inequalities (89) and (93) up, it yields inequality (74). This completes the proof of this Theorem. □

Conflict of Interests

The author declares that there is no conflict of interests regarding the publication of this paper.

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