Research Article
On Growth of Meromorphic Solutions of Complex Functional Difference Equations

Jing Li, 1,2 Jianjun Zhang, 3 and Liangwen Liao 1

1 Department of Mathematics, Nanjing University, Nanjing 210093, China
2 Nankai University Binhai College, Tianjin 300270, China
3 Mathematics and Information Technology School, Jiangsu Second Normal University, Nanjing 210013, China

Correspondence should be addressed to Jianjun Zhang; zhangjianjun1982@163.com

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The main purpose of this paper is to investigate the growth order of the meromorphic solutions of complex functional difference equations by utilizing Nevanlinna theory. They obtained the following result.

Theorem A. Suppose that $f$ is a transcendental meromorphic solution of the equation

$$\sum_{j \in I} \alpha_j (z) \left( \prod_{j \in J} f(z + c_j) \right) = f(p(z)),$$

(1)

where $\{J\}$ is a collection of all subsets of $\{1, 2, \ldots, n\}$, $c_j$'s are distinct complex constants, and $p(z)$ is a polynomial of degree $k \geq 2$. Moreover, we assume that the coefficients $\alpha_j(z)$ are small functions relative to $f$ and that $n \geq k$. Then

$$T(r, f) = O\left( (\log r)^{\alpha + \varepsilon} \right),$$

(2)

where $\alpha = \log n / \log k$.


Theorem B. Suppose that $f$ is a transcendental meromorphic function. Let $Q(z, f)$, $R(z, f)$ be rational functions in $f$ with small meromorphic coefficients relative to $f$ such that $0 < q := \deg_q Q \leq d := \deg_f R$ and $p(z) = p_k z^k + p_{k-1} z^{k-1} +$
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\[ R(z, f(z)) = Q(z, f(p(z))), \]

then \( qk \leq d \), and for any \( \epsilon, 0 < \epsilon < 1 \), there exist positive real constants \( K_1 \) and \( K_2 \) such that

\[ K_1 (\log r)^{\alpha - \epsilon} \leq T(r, f) \leq K_2 (\log r)^{\alpha - \epsilon}, \quad \alpha = \frac{\log d - \log q}{\log k}, \]

when \( r \) is large enough.

Rieppo [6] also considered the growth order of meromorphic solutions of functional equation (3) when \( k = 1 \) and got the following.

**Theorem C.** Suppose that \( f \) is a transcendental meromorphic solution of (3), where \( p(z) = az + b, a, b \in \mathbb{C}, a \neq 0 \) and \( |a| \neq 1 \). Then

\[ \mu(f) = \rho(f) = \frac{\log d - \log q}{\log |a|}, \]

Two years later, Zheng et al. [7] extended Theorem A to more general type and obtained a similar result of Theorem C. In fact, they got the following two results.

**Theorem D.** Suppose that \( f \) is a transcendental meromorphic solution of the equation

\[ \sum \alpha_j(z) \left( \prod_{j \in J} f(z + c_j) \right) = Q(z, f(p(z))), \]

where \( J \) is a collection of all nonempty subsets of \{1, 2, \ldots, n\}, \( c_j (j = 1, \ldots, n) \) are distinct complex constants, \( p(z) = \alpha z + \beta, \alpha, \beta \in \mathbb{C} \), and \( Q(z, w) \) is a rational function in \( w \) of \( \deg Q = q > 0 \). Also suppose that all the coefficients of (6) are small functions relative to \( f \). Then \( qk \leq n \), and

\[ T(r, f) = O\left( (\log r)^{\alpha + \epsilon} \right), \]

where \( \alpha = (\log n - \log q)/\log k \).

**Theorem E.** Suppose that \( f \) is a transcendental meromorphic solution of (6), where \( J \) is a collection of all nonempty subsets of \{1, 2, \ldots, n\}, \( c_j (j = 1, \ldots, n) \) are distinct complex constants, \( p(z) = \alpha z + \beta, \alpha, \beta \in \mathbb{C} \), and \( Q(z, w) \) is a rational function in \( w \) of \( \deg Q = q > 0 \). Also suppose that all the coefficients of (6) are small functions relative to \( f \).

(i) If \( 0 < |a| < 1 \), then we have

\[ \mu(f) \geq \frac{\log q - \log n}{-\log |a|}. \]

(ii) If \( |a| > 1 \), then we have \( q \leq n \) and

\[ \rho(f) \leq \frac{\log n - \log q}{\log |a|}. \]

(iii) If \( |a| = 1, q > n \), then we have \( \rho(f) = \mu(f) = \infty \).

In this paper, we will consider a more general class of complex functional difference equations. We prove the following results, which generalize the above related results.

**Theorem 1.** Suppose that \( f(z) \) is a transcendental meromorphic solution of the functional difference equation

\[ \sum_{k=1}^{n} \alpha_k(z) \left( \prod_{r=1}^{n} f(z + c_r) \right)^{\lambda_r} = Q(z, f(p(z))), \]

where \( c_r (r = 1, \ldots, n) \) are distinct complex constants, \( \alpha_r \) are two finite index sets, \( p(z) = \alpha z + \beta, \alpha, \beta \in \mathbb{C} \), and \( Q(z, u) \) is a rational function in \( u \) of \( \deg Q = q > 0 \). Also suppose that all the coefficients of (10) are small functions relative to \( f \). Denoting

\[ \sigma_v = \max_{\lambda, \mu} \left\{ l_\lambda, m_{\mu, v} \right\} \quad (v = 1, 2, \ldots, n), \quad \sigma = \sum_{v=1}^{n} \sigma_v \]

Then \( qk \leq \sigma \), and

\[ T(r, f) = O\left( (\log r)^{\alpha + \epsilon} \right), \]

where \( \alpha = (\log \sigma - \log q)/\log k \).

**Theorem 2.** Suppose that \( f(z) \) is a transcendental meromorphic solution of the equation

\[ \sum_{k=1}^{n} \alpha_k(z) \left( \prod_{r=1}^{n} f(z + c_r) \right)^{\lambda_r} = Q(z, f(p(z))), \]

where \( c_r (r = 1, \ldots, n) \) are distinct complex constants, \( \alpha_r \) are two finite index sets, \( p(z) = \alpha z + \beta, \alpha, \beta \in \mathbb{C} \), and \( Q(z, u) \) is a rational function in \( u \) of \( \deg Q = q > 0 \). Also suppose that all the coefficients of (10) are small functions relative to \( f \).

Denoting

\[ \sigma_v = \max_{\lambda, \mu} \left\{ l_\lambda, m_{\mu, v} \right\} \quad (v = 1, 2, \ldots, n), \quad \sigma = \sum_{v=1}^{n} \sigma_v \]

(i) If \( 0 < |a| < 1 \), then we have

\[ \mu(f) \geq \frac{\log q - \log \sigma}{-\log |a|}. \]

(ii) If \( |a| > 1 \), then we have \( q \leq \sigma \) and

\[ \rho(f) \leq \frac{\log \sigma - \log q}{\log |a|}. \]

(iii) If \( |a| = 1, q > \sigma \), then we have \( \mu(f) = \rho(f) = \infty \).
Next we will give some examples to show that our results are best in some extent.

**Example 3.** Let $c_1 = \arctan 2$, $c_2 = -\pi/4$. Then it is easy to check that $f(z) = \tan z$ solves the following equation:

$$
\frac{f(z + c_1)^2 f(z + c_2)}{f(z + c_1) + f(z + c_2)^2} = \frac{-4f(z^2) + 8z^7 + 28f(z^2)^6 - 56f(z^2)^5}{\left(-4f(z^2) + 8z^7 + 28f(z^2)^6 - 56f(z^2)^5\right)}
$$

Next, we give an example to show that case (iii) in Theorem 2 may hold.

**Example 6.** Meromorphic function $f(z) = \tan z$ solves the following equation:

$$
\frac{f(z + \pi/4)^2}{f(z + \pi/4) + f(z - \pi/4)^2} = \frac{(f(z) + 1)^3}{4f(z)(1 - f(z))},
$$

where $a = 1$ and $q = 3$, but $\rho(f) = \mu(f) = 1$.

Next, we give an example to show that case (iii) in Theorem 2 may hold.

**Example 7.** Function $f(z) = z e^z$ satisfies the following equation:

$$
\frac{(z + \log 6)(z + \log 2)^5 [f(z + \log 4)^4 + f(z + \log 4)]}{(z + \log 4)f(z + \log 6)} = \frac{(z + \log 4)^5 f(z + \log 2)^6 + (z + \log 2)^6}{f(z + \log 2)},
$$

where $a = 1$ and $q = 6 > 5 = \sigma$. Obviously, $\rho(f) = \mu(f) = \infty$.

### 2. Main Lemmas

In order to prove our results, we need the following lemmas.

**Lemma 1** (see [4, 8]). Let $f(z)$ be a meromorphic function. Then for all irreducible rational functions in $f$,

$$
\frac{R(z, f)}{Q(z, f)} = \sum_{i=0}^{p} a_i (z) f^i \sum_{j=0}^{q} b_j (z) f^j,
$$

such that the meromorphic coefficients $a_i(z), b_j(z)$ satisfy

$$
T(r, a_i) = S(r, f), \quad i = 0, 1, \ldots, p, \quad T(r, b_j) = S(r, f), \quad j = 0, 1, \ldots, q;
$$

then one has

$$
T(r, R(z, f)) = \max \{p, q\} \cdot T(r, f) + S(r, f).
$$

From the proof of Theorem 1 in [9], we have the following estimate for the Nevanlinna characteristic.

**Lemma 2.** Let $f_1, f_2, \ldots, f_n$ be distinct meromorphic functions and

$$
F(z) = \frac{P(z)}{Q(z)} = \sum_{i=1}^{\infty} a_i(z) f_1^{i_{1}} f_2^{i_{2}} \cdots f_n^{i_{n}}.
$$

(27)
Then
\[
T(r, F(z)) \leq \sum_{\nu=1}^{n} \sigma_{\nu} T(r, f_{\nu}) + S(r, f),
\]
(28)

where \(I = \{ \lambda = (l_{1,\lambda}, l_{2,\lambda}, \ldots, l_{n,\lambda}) | l_{\nu,\lambda} \in \mathbb{N} \setminus \{0\}, \nu = 1, 2, \ldots, n \} \) and \(J = \{ \mu = (m_{1,\mu}, m_{2,\mu}, \ldots, m_{n,\mu}) | m_{\mu,\nu} \in \mathbb{N} \setminus \{0\}, \nu = 1, 2, \ldots, n \} \) are two finite index sets, \(\sigma_{\nu} = \max_{\lambda \in I} \{ l_{\nu,\lambda} \} \) (\(\nu = 1, 2, \ldots, n\)), \(\alpha_{\nu}(\lambda) = o(T(r, f_{\nu})(\lambda \in I))\) and \(\beta_{\nu}(z) = o(T(r, f_{\nu})(\mu \in J))\) hold for all \(\nu \in \{1, 2, \ldots, n\}\) and satisfy \(T(r, \alpha_{\nu}) = S(r, f)\) (\(\lambda \in I\)) and \(T(r, \beta_{\nu}) = S(r, f)\) (\(\mu \in J\)).

Lemma 3 (see [7]). Let \(c\) be a complex constant. Given \(\varepsilon > 0\) and a meromorphic function \(f\), one has
\[
T(r, f(z \pm c)) \leq (1 + \varepsilon) T(r + |c|, f),
\]
(29)

for all \(r > r_{0}\), where \(r_{0}\) is some positive constant.

Lemma 4 (see [4]). Let \(g : (0, +\infty) \to \mathbb{R}, h : (0, +\infty) \to \mathbb{R}\) be monotone increasing functions such that \(g(r) \leq h(r)\) outside of an exceptional set \(E\) of finite linear measure. Then, for any \(\alpha > 1\), there exists \(r_{0} > 0\) such that \(g(r) \leq h(ar)\) for all \(r > r_{0}\).

Lemma 5 (see [10]). Let \(f\) be a transcendental meromorphic function, and \(p(z) = a_{k} e^{z} + a_{k-1} e^{z-1} + \cdots + a_{0} e^{z} + a_{0} \neq 0\), be a nonconstant polynomial of degree \(k\). Given \(0 < \delta < |a_{k}|\), denote \(\lambda = |a_{k}| + \delta\) and \(\mu = |a_{k}| - \delta\). Then given \(\varepsilon > 0\) and \(a \in \mathbb{C} \setminus \{0\}\), one has
\[
kn(\mu^{k}, a, f) \leq n(r, a, f(p(z))) \leq kn(\lambda^{k}, a, f)
\]
\[
N(\mu^{k}, a, f) + O(\log r) \leq N(r, a, f(p(z)))
\]
\[
\leq N(\lambda^{k}, a, f) + O(\log r)
\]
\[
(1 + \varepsilon) T(\mu^{k}, f) \leq T(r, f(p(z))) \leq (1 + \varepsilon) T(\lambda^{k}, f),
\]
(30)

for all \(r\) large enough.

Lemma 6 (see [11]). Let \(\phi : [r_{0}, +\infty) \to (0, +\infty)\) be positive and bounded in every finite interval, and suppose that \(\phi(\mu^{m}) \leq A\phi(r) + B\) holds for all \(r\) large enough, where \(\mu > 0\), \(m > 1\), \(A > 1\) and \(B\) are real constants. Then
\[
\phi(r) = O\left( (\log r)^{\alpha} \right),
\]
(31)

where \(\alpha = \log A / \log m\).

Lemma 7 (see [6]). Let \(\phi : (r_{0}, \infty) \to (1, \infty), \) where \(r_{0} \geq 1\), be a monotone increasing function. If for some real constant \(\alpha > 1\), there exists a real number \(K > 1\) such that \(\phi(ar) \geq K\phi(r)\), then
\[
\lim_{r \to \infty} \frac{\log \phi(r)}{\log r} \geq \frac{\log K}{\log \alpha}
\]
(32)

Lemma 8 (see [12]). Let \(\phi : (1, \infty) \to (0, \infty)\) be a monotone increasing function and let \(f\) be a nonconstant meromorphic function. If, for some real constant \(\alpha \in (0, 1)\), there exist real constants \(K_{1} > 0\) and \(K_{2} \geq 1\) such that
\[
T(r, f) \leq K_{1} \phi(ar) + K_{2} T(ar, f) + S(ar, f),
\]
(33)

then
\[
\rho(f) \leq \frac{\log K_{2}}{-\log \alpha} + \lim_{r \to \infty} \frac{\log \phi(r)}{\log r}.
\]
(34)

3. Proof of Theorems

Proof of Theorem 1. We assume \(f(z)\) is a transcendental meromorphic solution of (10). Denoting \(C = \max{|c_{1}|, |c_{2}|, \ldots, |c_{n}|}\). According to Lemmas 1, 2, and 3 and the last assertion of Lemma 5, we get that for any \(\varepsilon_{1} > 0\),
\[
q \left( 1 - \varepsilon_{1} \right) T\left( \mu^{k}, f \right) + S(r, f)
\]
\[
\leq qT(r, f(p(z))) + S(r, f)
\]
\[
= T(r, Q(z, f(p(z)))
\]
\[
= T\left( r, \left( \sum_{\lambda \in I} \alpha_{\lambda}(z) \left( \prod_{\nu=1}^{n} f(z + \varepsilon_{\nu})^{l_{\nu,\lambda}} \right) \right) \right)
\]
\[
\leq \sum_{\nu=1}^{n} \sigma_{\nu} T(r, f(z + \varepsilon_{\nu})) + S(r, f)
\]
\[
\leq \sum_{\nu=1}^{n} \sigma_{\nu} \left( 1 + \varepsilon_{1} \right) T(r + C, f(z)) + S(r, f)
\]
\[
= \left( \sum_{\nu=1}^{n} \sigma_{\nu} \right) \left( 1 + \varepsilon_{1} \right) T(r + C, f(z)) + S(r, f)
\]
\[
= \sigma (1 + \varepsilon_{1}) T(r + C, f(z)) + S(r, f),
\]
where \(r\) is large enough and \(\mu = |p_{\nu}| - \delta\) for some \(0 < \delta < |p_{\nu}|\).

Since \(T(r + C, f(z)) \leq T(\beta r, f)\) holds for \(r\) large enough for \(\beta > 1\), we may assume \(r\) to be large enough to satisfy
\[
q \left( 1 - \varepsilon_{1} \right) T\left( \mu^{k}, f \right) \leq \sigma (1 + \varepsilon_{1}) T(\beta r, f)
\]
(36)

outside a possible exceptional set of finite linear measure. By Lemma 4, we know that whenever \(\gamma > 1\),
\[
q \left( 1 - \varepsilon_{1} \right) T\left( \mu^{k}, f \right) \leq \sigma (1 + \varepsilon_{1}) T(\gamma \beta r, f)
\]
(37)

holds for all \(r\) large enough. Denote \(t = \gamma \beta r\); thus the inequality (37) may be written in the form
\[
T\left( \frac{\mu}{(\gamma \beta)^{t}}, f \right) \leq \frac{\sigma (1 + \varepsilon_{1})}{q \left( 1 - \varepsilon_{1} \right)} T(t, f).
\]
(38)

By Lemma 6, we have
\[
T(r, f) = O\left( (\log r)^{\alpha_{1}} \right),
\]
(39)
where
\[
\alpha_1 = \frac{\log(\sigma(1+\epsilon_1)/q(1-\epsilon_1))}{\log k} = \frac{\log \sigma - \log q + \log((1+\epsilon_1)/(1-\epsilon_1))}{\log k}.
\] (40)

Denoting now \(\alpha = (\log \sigma - \log q)/\log k\) and \(\epsilon = \log((1+\epsilon_1)/(1-\epsilon_1))/\log k\); thus we obtain the required form.

Finally, we show that \(qk \leq \sigma\). If \(qk > \sigma\), then we have \(\alpha < 1\). For sufficiently small \(\epsilon > 0\), we have \(\alpha + \epsilon < 1\), which contradicts with the transcendency of \(f\). Thus Theorem 1 is proved.

**Proof of Theorem 2.** Suppose \(f(z)\) is a transcendental meromorphic solution of (13). Denoting \(C = \max\{|c_1|, |c_2|, \ldots, |c_n|\}\).

(i) \(0 < |a| < 1\). We may assume that \(q > \sigma\), since the case \(q \leq \sigma\) is trivial by the fact that \(\mu(f) \geq 0\). By Lemmas 1–3, we have for any \(\epsilon > 0\) and \(\beta > 1\),

\[
qT(r,f(p(z)))+S(r,f) = T(r,Q(z,f(p(z))))
\]

\[
= T\left(r, \frac{\sum_{\lambda \in I} a_{\lambda} (z) \left( \prod_{n=1}^m f(z+c_n) \right)^{\lambda_n}}{\sum_{\mu \in J} b_{\mu} (z) \left( \prod_{n=1}^m f(z+c_n) \right)^{\mu_n}} \right)
\]

\[
\leq \sum_{n=1}^n \sigma_n T(r,f(z+c_n))+S(r,f)
\]

\[
\leq \sum_{n=1}^n \sigma_n (1+\epsilon) T(r+C,f(z))+S(r,f)
\]

\[
= \left( \sum_{n=1}^n \sigma_n (1+\epsilon) \right) T(r+C,f(z))+S(r,f)
\]

\[
= \sigma (1+\epsilon) T(r+C,f(z))+S(r,f)
\]

\[
\leq \sigma (1+\epsilon) T(\beta r,f)+S(r,f),
\] (41)

where \(r\) is large enough.

By the last assertion of Lemma 5 and (41), we obtain that, for \(\mu = |a| - \delta\) \((0 < \delta < |a|, 0 < \mu < 1)\), the following inequality

\[
q (1-\epsilon) T(\mu r,f) \leq \sigma (1+\epsilon) T(\beta r,f)
\] (42)

holds, where \(r\) is large enough outside of a possible set of finite linear measure. By Lemma 4, we get that for any \(\gamma > 1\) and sufficiently large \(r\),

\[
q (1-\epsilon) T(\mu r,f) \leq \sigma (1+\epsilon) T(\gamma \beta r,f).
\] (43)

Therefore,

\[
\frac{q (1-\epsilon)}{\sigma (1+\epsilon)} T(r,f) \leq T\left( \frac{\gamma \beta}{\mu} r,f \right).
\] (44)

Since \(\beta > 1, \gamma > 1, 0 < \mu < 1\) and \(q > \sigma\), we have \(\beta \gamma / \mu > 1\) and \(q(1-\epsilon)/\sigma(1+\epsilon) > 1\) when \(\epsilon\) is small enough. Using Lemma 7, we see that

\[
\mu(f) \geq \frac{\log q - \log \sigma (1+\epsilon)}{\log \gamma \beta - \log \mu}.
\] (45)

Letting \(\epsilon \to 0, \delta \to 0, \beta \to 1\) and \(\gamma \to 1\), we have

\[
\mu(f) \geq \frac{\log q - \log \sigma}{-\log |a|}.
\] (46)

(ii) \(|a| > 1\). By the similar reasoning as is (i), we easily obtain that

\[
q (1-\epsilon) T(\mu r,f) \leq q T(r,f(p(z)))+S(r,f)
\]

\[
\leq \sigma (1+\epsilon) T(r+C,f(z))+S(r,f)
\]

\[
\leq \sigma (1+\epsilon) T(r+C,f(z))+S(r,f)
\]

\[
\leq \sigma (1+\epsilon) T(\gamma r,f(z))
\] (47)

namely,

\[
T(\mu r,f) \leq \frac{\sigma (1+\epsilon)}{q(1-\epsilon)} T(r+C,f(z))+S(r,f)
\]

holds for all sufficiently large \(r\). By Lemma 8, we obtain

\[
\rho(f) \leq \frac{\log \sigma - \log q + \log (1+\epsilon) - \log (1-\epsilon)}{-\log (\gamma / \mu)}.
\] (52)

Letting \(\epsilon \to 0, \delta \to 0\) and \(\gamma \to 1\), we have

\[
\rho(f) \leq \frac{\log \sigma - \log q}{-\log |a|}.
\] (53)

(iii) \(|a| = 1\) and \(q > \sigma\). The proof of this case is completely similar as in the case in (i). In fact, we set \(\mu = |a| - \delta = 1 - \delta (0 < \delta < 1, 0 < \mu < 1)\). Similarly, we can get

\[
\mu(f) \geq \frac{\log q - \log \sigma}{-\log |a|}.
\] (54)

Since \(|a| = 1\), we have \(\mu(f) = \rho(f) = \infty\).
Conflict of Interests
The authors declare that there is no conflict of interests regarding the publication of this paper.

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