Research Article

On Growth of Meromorphic Solutions of Complex Functional Difference Equations

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The main purpose of this paper is to investigate the growth order of the meromorphic solutions of complex functional difference equation of the form $(\sum_{\lambda \in I} \alpha_{\lambda}(z)(\prod_{\nu=1}^n f(z + c_{\nu})^{l_{\lambda,\nu}}))/(\sum_{\mu \in J} \beta_{\mu}(z)(\prod_{\nu=1}^n f(z + c_{\nu})^{m_{\mu,\nu}})) = Q(z, f(p(z)))$, where $I = \{\lambda = (l_{\lambda,1}, l_{\lambda,2}, \dots, l_{\lambda,n}) \mid l_{\lambda,\nu} \in \mathbb{N} \cup \{0\}, \nu = 1, 2, \dots, n\}$ and $J = \{\mu = (m_{\mu,1}, m_{\mu,2}, \dots, m_{\mu,n}) \mid m_{\mu,\nu} \in \mathbb{N} \cup \{0\}, \nu = 1, 2, \dots, n\}$ are two finite index sets, c_{ν} ($\nu = 1, 2, \dots, n$) are distinct complex numbers, $\alpha_{\lambda}(z)$ ($\lambda \in I$) and $\beta_{\mu}(z)$ ($\mu \in J$) are small functions relative to $f(z)$, and $Q(z, u)$ is a rational function in u with coefficients which are small functions of $f(z)$, $p(z) = p_k z^k + p_{k-1} z^{k-1} + \dots + p_0 \in \mathbb{C}[z]$ of degree $k \geq 1$. We also give some examples to show that our results are sharp.

1. Introduction and Main Results

Let $f(z)$ be a function meromorphic in the complex plane $\mathbb{C}$. We assume that the reader is familiar with the standard notations and results in Nevanlinna's value distribution theory of meromorphic functions such as the characteristic function $T(r, f)$, proximity function $m(r, f)$, counting function $N(r, f)$, and the first and second main theorems (see, e.g., [1–4]). We also use $\overline{N}(r, f)$ to denote the counting function of the poles of $f(z)$ whose every pole is counted only once. The notations $\rho(f)$ and $\mu(f)$ denote the order and the lower order of $f(z)$, respectively. $S(r, f)$ denotes any quantity that satisfies the condition: $S(r, f) = o(T(r, f))$ as $r \rightarrow \infty$ possibly outside an exceptional set of r of finite linear measure. A meromorphic function $a(z)$ is called a small function of $f(z)$ or a small function relative to $f(z)$ if and only if $T(r, a(z)) = S(r, f)$.

Recently, some papers (see, e.g., [5–7]) focusing on complex difference and functional difference equations emerged. In 2005, Laine et al. [5] firstly considered the growth of meromorphic solutions of the complex functional difference equations by utilizing Nevanlinna theory. They obtained the following result.

Theorem A. Suppose that f is a transcendental meromorphic solution of the equation

$$\sum_{\{J\}} \alpha_J(z) \left(\prod_{j \in J} f(z + c_j) \right) = f(p(z)), \quad (1)$$

where $\{J\}$ is a collection of all subsets of $\{1, 2, \dots, n\}$, c_j 's are distinct complex constants, and $p(z)$ is a polynomial of degree $k \geq 2$. Moreover, we assume that the coefficients $\alpha_J(z)$ are small functions relative to f and that $n \geq k$. Then

$$T(r, f) = O((\log r)^{\alpha+\epsilon}), \quad (2)$$

where $\alpha = \log n / \log k$.

In 2007, Rieppo [6] gave an estimation of growth of meromorphic solutions of complex functional equations as follows.

Theorem B. Suppose that f is a transcendental meromorphic function. Let $Q(z, f)$, $R(z, f)$ be rational functions in f with small meromorphic coefficients relative to f such that $0 < q := \deg_f Q \leq d := \deg_f R$ and $p(z) = p_k z^k + p_{k-1} z^{k-1} +$

$\dots + p_0 \in \mathbb{C}[z]$ of degree $k > 1$. If f is a solution of the functional equation

$$R(z, f(z)) = Q(z, f(p(z))), \quad (3)$$

then $qk \leq d$, and for any $\varepsilon, 0 < \varepsilon < 1$, there exist positive real constants K_1 and K_2 such that

$$K_1(\log r)^{\alpha-\varepsilon} \leq T(r, f) \leq K_2(\log r)^{\alpha+\varepsilon}, \quad \alpha = \frac{\log d - \log q}{\log k}, \quad (4)$$

when r is large enough.

Rieppo [6] also considered the growth order of meromorphic solutions of functional equation (3) when $k = 1$ and got the following.

Theorem C. Suppose that f is a transcendental meromorphic solution of (3), where $p(z) = az + b$, $a, b \in \mathbb{C}$, $a \neq 0$ and $|a| \neq 1$. Then

$$\mu(f) = \rho(f) = \frac{\log d - \log q}{\log |a|}. \quad (5)$$

Two years later, Zheng et al. [7] extended Theorem A to more general type and obtained a similar result of Theorem C. In fact, they got the following two results.

Theorem D. Suppose that f is a transcendental meromorphic solution of the equation

$$\sum_{\{J\}} \alpha_J(z) \left(\prod_{j \in J} f(z + c_j) \right) = Q(z, f(p(z))), \quad (6)$$

where $\{J\}$ is a collection of all nonempty subsets of $\{1, 2, \dots, n\}$, c_j ($j = 1, \dots, n$) are distinct complex constants, $p(z) = p_k z^k + p_{k-1} z^{k-1} + \dots + p_0 \in \mathbb{C}[z]$ of degree $k > 1$, and $Q(z, u)$ is a rational function in u of $\deg_u Q = q(> 0)$. Also suppose that all the coefficients of (6) are small functions relative to f . Then $qk \leq n$, and

$$T(r, f) = O((\log r)^{\alpha+\varepsilon}), \quad (7)$$

where $\alpha = (\log n - \log q)/\log k$.

Theorem E. Suppose that f is a transcendental meromorphic solution of (6), where $\{J\}$ is a collection of all nonempty subsets of $\{1, 2, \dots, n\}$, c_j ($j = 1, \dots, n$) are distinct complex constants, $p(z) = az + b$, $a, b \in \mathbb{C}$, and $Q(z, u)$ is a rational function in u of $\deg_u Q = q(> 0)$. Also suppose that all the coefficients of (6) are small functions relative to f .

(i) If $0 < |a| < 1$, then we have

$$\mu(f) \geq \frac{\log q - \log n}{-\log |a|}. \quad (8)$$

(ii) If $|a| > 1$, then we have $q \leq n$ and

$$\rho(f) \leq \frac{\log n - \log q}{\log |a|}. \quad (9)$$

(iii) If $|a| = 1$, $q > n$, then we have $\rho(f) = \mu(f) = \infty$.

In this paper, we will consider a more general class of complex functional difference equations. We prove the following results, which generalize the above related results.

Theorem 1. Suppose that $f(z)$ is a transcendental meromorphic solution of the functional difference equation

$$\frac{\sum_{\lambda \in I} \alpha_\lambda(z) \left(\prod_{v=1}^n f(z + c_v)^{l_{\lambda,v}} \right)}{\sum_{\mu \in J} \beta_\mu(z) \left(\prod_{v=1}^n f(z + c_v)^{m_{\mu,v}} \right)} = Q(z, f(p(z))), \quad (10)$$

where c_v ($v = 1, \dots, n$) are distinct complex constants, $I = \{\lambda = (l_{\lambda,1}, l_{\lambda,2}, \dots, l_{\lambda,n}) \mid l_{\lambda,v} \in \mathbb{N} \cup \{0\}, v = 1, 2, \dots, n\}$ and $J = \{\mu = (m_{\mu,1}, m_{\mu,2}, \dots, m_{\mu,n}) \mid m_{\mu,v} \in \mathbb{N} \cup \{0\}, v = 1, 2, \dots, n\}$ are two finite index sets, $p(z) = p_k z^k + p_{k-1} z^{k-1} + \dots + p_0 \in \mathbb{C}[z]$ of degree $k > 1$, and $Q(z, u)$ is a rational function in u of $\deg_u Q = q(> 0)$. Also suppose that all the coefficients of (10) are small functions relative to f . Denoting

$$\sigma_v = \max_{\lambda, \mu} \{l_{\lambda,v}, m_{\mu,v}\} \quad (v = 1, 2, \dots, n), \quad \sigma = \sum_{v=1}^n \sigma_v. \quad (11)$$

Then $qk \leq \sigma$, and

$$T(r, f) = O((\log r)^{\alpha+\varepsilon}), \quad (12)$$

where $\alpha = (\log \sigma - \log q)/\log k$.

Theorem 2. Suppose that f is a transcendental meromorphic solution of the equation

$$\frac{\sum_{\lambda \in I} \alpha_\lambda(z) \left(\prod_{v=1}^n f(z + c_v)^{l_{\lambda,v}} \right)}{\sum_{\mu \in J} \beta_\mu(z) \left(\prod_{v=1}^n f(z + c_v)^{m_{\mu,v}} \right)} = Q(z, f(az + b)), \quad (13)$$

where c_v ($v = 1, \dots, n$) are distinct complex constants, $I = \{\lambda = (l_{\lambda,1}, l_{\lambda,2}, \dots, l_{\lambda,n}) \mid l_{\lambda,v} \in \mathbb{N} \cup \{0\}, v = 1, 2, \dots, n\}$ and $J = \{\mu = (m_{\mu,1}, m_{\mu,2}, \dots, m_{\mu,n}) \mid m_{\mu,v} \in \mathbb{N} \cup \{0\}, v = 1, 2, \dots, n\}$ are two finite index sets, $a, b \in \mathbb{C}$, and $Q(z, u)$ is a rational function in u of $\deg_u Q = q(> 0)$. Also suppose that all the coefficients of (10) are small functions relative to f . Denoting

$$\sigma_v = \max_{\lambda, \mu} \{l_{\lambda,v}, m_{\mu,v}\} \quad (v = 1, 2, \dots, n), \quad \sigma = \sum_{v=1}^n \sigma_v. \quad (14)$$

(i) If $0 < |a| < 1$, then we have

$$\mu(f) \geq \frac{\log q - \log \sigma}{-\log |a|}. \quad (15)$$

(ii) If $|a| > 1$, then we have $q \leq \sigma$ and

$$\rho(f) \leq \frac{\log \sigma - \log q}{\log |a|}. \quad (16)$$

(iii) If $|a| = 1$ and $q > \sigma$, then we have $\mu(f) = \rho(f) = \infty$.

Next we will give some examples to show that our results are best in some extent.

Example 3. Let $c_1 = \arctan 2$, $c_2 = -\pi/4$. Then it is easy to check that $f(z) = \tan z$ solves the following equation:

$$\begin{aligned} & \frac{f(z+c_1)^2 f(z+c_2)}{f(z+c_1) + f(z+c_2)^2} \\ &= \left(-4f\left(\frac{z}{2}\right)^8 + 8f\left(\frac{z}{2}\right)^7 + 28f\left(\frac{z}{2}\right)^6 - 56f\left(\frac{z}{2}\right)^5 \right. \\ & \quad \left. - 32f\left(\frac{z}{2}\right)^4 + 56f\left(\frac{z}{2}\right)^3 + 28f\left(\frac{z}{2}\right)^2 - 8f\left(\frac{z}{2}\right) - 4 \right) \\ & \quad \times \left(3f\left(\frac{z}{2}\right)^8 + 10f\left(\frac{z}{2}\right)^7 + 16f\left(\frac{z}{2}\right)^6 + 122f\left(\frac{z}{2}\right)^5 \right. \\ & \quad \left. - 6f\left(\frac{z}{2}\right)^4 - 122f\left(\frac{z}{2}\right)^3 + 16f\left(\frac{z}{2}\right)^2 - 10f\left(\frac{z}{2}\right) + 3 \right)^{-1}. \end{aligned} \quad (17)$$

Obviously, we have

$$\mu(f) = \rho(f) = 1 = \frac{\log q - \log \sigma}{-\log |a|}, \quad (18)$$

where $q = 8$, $\sigma = 4$ and $a = 1/2$.

Example 3 shows that the estimate in Theorem 2(i) is sharp.

Example 4. It is easy to check that $f(z) = \tan z$ satisfies the equation

$$\begin{aligned} & \frac{f(z+(\pi/3))^2 f(z+(\pi/6)) - f(z+(\pi/6))}{f(z+(\pi/3)) f(z+(\pi/6))^2 - f(z+(\pi/3))} \\ &= \frac{\sqrt{3}f(2z)^2 + 4f(2z) + \sqrt{3}}{-\sqrt{3}f(2z)^2 + 4f(2z) - \sqrt{3}}. \end{aligned} \quad (19)$$

Clearly, we have

$$\mu(f) = \rho(f) = 1 = \frac{\log \sigma - \log q}{\log |a|}, \quad (20)$$

where $\sigma = 4$, $q = 2$ and $a = 2$.

Example 4 shows that the estimate in Theorem 2(ii) is sharp.

Example 5. $f(z) = \tan z$ satisfies the equation of the form

$$\begin{aligned} & \frac{f(z+(\pi/4))^2}{f(z+(\pi/4)) + f(z-(\pi/4))^2} \\ &= \frac{-(f(z/2)^2 - 2f(z/2) - 1)^3}{8f(z/2)(f(z/2)^2 - 1)(f(z/2)^2 + 2f(z/2) - 1)}, \end{aligned} \quad (21)$$

where $\sigma = 4$, $q = 6$, and $a = 1/2$. $\rho(f) = \mu(f) = 1 > \log(3/2)/\log 2 = (\log q - \log \sigma)/-\log |a|$.

Example 5 shows that the strict inequality in Theorem 2 may occur. Therefore, we do not have the same estimation as in Theorem C for the growth order of meromorphic solutions of (13).

The following Example shows that the restriction $q > \sigma$ in case (iii) in Theorem 2 is necessary.

Example 6. Meromorphic function $f(z) = \tan z$ solves the following equation:

$$\frac{f(z+(\pi/4))^2}{f(z+(\pi/4)) + f(z-(\pi/4))^2} = \frac{(f(z)+1)^3}{4f(z)(1-f(z))}, \quad (22)$$

where $a = 1$ and $4 = \sigma > q = 3$, but $\rho(f) = \mu(f) = 1$.

Next, we give an example to show that case (iii) in Theorem 2 may hold.

Example 7. Function $f(z) = ze^{e^z}$ satisfies the following equation:

$$\begin{aligned} & \frac{(z+\log 6)(z+\log 2)^5 [f(z+\log 4)^4 + f(z+\log 4)]}{(z+\log 4)f(z+\log 6)} \\ &= \frac{(z+\log 4)^3 f(z+\log 2)^6 + (z+\log 2)^6}{f(z+\log 2)}, \end{aligned} \quad (23)$$

where $a = 1$ and $q = 6 > 5 = \sigma$. Obviously, $\rho(f) = \mu(f) = \infty$.

2. Main Lemmas

In order to prove our results, we need the following lemmas.

Lemma 1 (see [4, 8]). *Let $f(z)$ be a meromorphic function. Then for all irreducible rational functions in f ,*

$$R(z, f) = \frac{P(z, f)}{Q(z, f)} = \frac{\sum_{i=0}^p a_i(z) f^i}{\sum_{j=0}^q b_j(z) f^j}, \quad (24)$$

such that the meromorphic coefficients $a_i(z)$, $b_j(z)$ satisfy

$$\begin{aligned} T(r, a_i) &= S(r, f), \quad i = 0, 1, \dots, p, \\ T(r, b_j) &= S(r, f), \quad j = 0, 1, \dots, q; \end{aligned} \quad (25)$$

then one has

$$T(r, R(z, f)) = \max\{p, q\} \cdot T(r, f) + S(r, f). \quad (26)$$

From the proof of Theorem 1 in [9], we have the following estimate for the Nevanlinna characteristic.

Lemma 2. *Let $f_1, f_2, \dots, f_n$ be distinct meromorphic functions and*

$$F(z) = \frac{P(z)}{Q(z)} = \frac{\sum_{\lambda \in I} \alpha_\lambda(z) f_1^{l_{\lambda,1}} f_2^{l_{\lambda,2}} \dots f_n^{l_{\lambda,n}}}{\sum_{\mu \in J} \beta_\mu(z) f_1^{m_{\mu,1}} f_2^{m_{\mu,2}} \dots f_n^{m_{\mu,n}}}. \quad (27)$$

Then

$$T(r, F(z)) \leq \sum_{\nu=1}^n \sigma_{\nu} T(r, f_{\nu}) + S(r, f), \quad (28)$$

where $I = \{\lambda = (l_{\lambda,1}, l_{\lambda,2}, \dots, l_{\lambda,n}) \mid l_{\lambda,\nu} \in \mathbb{N} \cup \{0\}, \nu = 1, 2, \dots, n\}$ and $J = \{\mu = (m_{\mu,1}, m_{\mu,2}, \dots, m_{\mu,n}) \mid m_{\mu,\nu} \in \mathbb{N} \cup \{0\}, \nu = 1, 2, \dots, n\}$ are two finite index sets, $\sigma_{\nu} = \max_{\lambda \in I} \{l_{\lambda,\nu}, m_{\mu,\nu}\}$ ($\nu = 1, 2, \dots, n$), $\alpha_{\lambda}(z) = o(T(r, f_{\nu}))$ ($\lambda \in I$) and $\beta_{\mu}(z) = o(T(r, f_{\nu}))$ ($\mu \in J$) hold for all $\nu \in \{1, 2, \dots, n\}$ and satisfy $T(r, \alpha_{\lambda}) = S(r, f)$ ($\lambda \in I$) and $T(r, \beta_{\mu}) = S(r, f)$ ($\mu \in J$).

Lemma 3 (see [7]). Let c be a complex constant. Given $\varepsilon > 0$ and a meromorphic function f , one has

$$T(r, f(z \pm c)) \leq (1 + \varepsilon) T(r + |c|, f), \quad (29)$$

for all $r > r_0$, where r_0 is some positive constant.

Lemma 4 (see [4]). Let $g : (0, +\infty) \rightarrow \mathbb{R}$, $h : (0, +\infty) \rightarrow \mathbb{R}$ be monotone increasing functions such that $g(r) \leq h(r)$ outside of an exceptional set E of finite linear measure. Then, for any $\alpha > 1$, there exists $r_0 > 0$ such that $g(r) \leq h(\alpha r)$ for all $r > r_0$.

Lemma 5 (see [10]). Let f be a transcendental meromorphic function, and $p(z) = a_k z^k + a_{k-1} z^{k-1} + \dots + a_1 z + a_0$, $a_k \neq 0$, be a nonconstant polynomial of degree k . Given $0 < \delta < |a_k|$, denote $\lambda = |a_k| + \delta$ and $\mu = |a_k| - \delta$. Then given $\varepsilon > 0$ and $a \in \mathbb{C} \cup \{\infty\}$, one has

$$kn(\mu r^k, a, f) \leq n(r, a, f(p(z))) \leq kn(\lambda r^k, a, f)$$

$$\begin{aligned} N(\mu r^k, a, f) + O(\log r) &\leq N(r, a, f(p(z))) \\ &\leq N(\lambda r^k, a, f) + O(\log r) \end{aligned}$$

$$(1 - \varepsilon) T(\mu r^k, f) \leq T(r, f(p(z))) \leq (1 + \varepsilon) T(\lambda r^k, f), \quad (30)$$

for all r large enough.

Lemma 6 (see [11]). Let $\phi : [r_0, +\infty) \rightarrow (0, +\infty)$ be positive and bounded in every finite interval, and suppose that $\phi(\mu r^m) \leq A\phi(r) + B$ holds for all r large enough, where $\mu > 0$, $m > 1$, $A > 1$ and B are real constants. Then

$$\phi(r) = O((\log r)^{\alpha}), \quad (31)$$

where $\alpha = \log A / \log m$.

Lemma 7 (see [6]). Let $\phi : (r_0, \infty) \rightarrow (1, \infty)$, where $r_0 \geq 1$, be a monotone increasing function. If for some real constant $\alpha > 1$, there exists a real number $K > 1$ such that $\phi(\alpha r) \geq K\phi(r)$, then

$$\lim_{r \rightarrow \infty} \frac{\log \phi(r)}{\log r} \geq \frac{\log K}{\log \alpha}. \quad (32)$$

Lemma 8 (see [12]). Let $\phi : (1, \infty) \rightarrow (0, \infty)$ be a monotone increasing function and let f be a nonconstant meromorphic

function. If, for some real constant $\alpha \in (0, 1)$, there exist real constants $K_1 > 0$ and $K_2 \geq 1$ such that

$$T(r, f) \leq K_1 \phi(\alpha r) + K_2 T(\alpha r, f) + S(\alpha r, f), \quad (33)$$

then

$$\rho(f) \leq \frac{\log K_2}{-\log \alpha} + \overline{\lim}_{r \rightarrow \infty} \frac{\log \phi(r)}{\log r}. \quad (34)$$

3. Proof of Theorems

Proof of Theorem 1. We assume $f(z)$ is a transcendental meromorphic solution of (10). Denoting $C = \max\{|c_1|, |c_2|, \dots, |c_n|\}$. According to Lemmas 1, 2, and 3 and the last assertion of Lemma 5, we get that for any $\varepsilon_1 > 0$,

$$\begin{aligned} &q(1 - \varepsilon_1) T(\mu r^k, f) + S(r, f) \\ &\leq qT(r, f(p(z))) + S(r, f) \\ &= T(r, Q(z, f(p(z)))) \\ &= T\left(r, \frac{\sum_{\lambda \in I} \alpha_{\lambda}(z) \left(\prod_{\nu=1}^n f(z + c_{\nu})^{l_{\lambda,\nu}}\right)}{\sum_{\mu \in J} \beta_{\mu}(z) \left(\prod_{\nu=1}^n f(z + c_{\nu})^{m_{\mu,\nu}}\right)}\right) \\ &\leq \sum_{\nu=1}^n \sigma_{\nu} T(r, f(z + c_{\nu})) + S(r, f) \\ &\leq \sum_{\nu=1}^n \sigma_{\nu} (1 + \varepsilon_1) T(r + C, f(z)) + S(r, f) \\ &= \left(\sum_{\nu=1}^n \sigma_{\nu}\right) (1 + \varepsilon_1) T(r + C, f(z)) + S(r, f) \\ &= \sigma(1 + \varepsilon_1) T(r + C, f(z)) + S(r, f), \end{aligned} \quad (35)$$

where r is large enough and $\mu = |p_k| - \delta$ for some $0 < \delta < |p_k|$. Since $T(r + C, f) \leq T(\beta r, f)$ holds for r large enough for $\beta > 1$, we may assume r to be large enough to satisfy

$$q(1 - \varepsilon_1) T(\mu r^k, f) \leq \sigma(1 + \varepsilon_1) T(\beta r, f) \quad (36)$$

outside a possible exceptional set of finite linear measure. By Lemma 4, we know that whenever $\gamma > 1$,

$$q(1 - \varepsilon_1) T(\mu r^k, f) \leq \sigma(1 + \varepsilon_1) T(\gamma \beta r, f) \quad (37)$$

holds for all r large enough. Denote $t = \gamma \beta r$; thus the inequality (37) may be written in the form

$$T\left(\frac{\mu}{(\gamma \beta)^k} t^k, f\right) \leq \frac{\sigma(1 + \varepsilon_1)}{q(1 - \varepsilon_1)} T(t, f). \quad (38)$$

By Lemma 6, we have

$$T(r, f) = O((\log r)^{\alpha_1}), \quad (39)$$

where

$$\begin{aligned}\alpha_1 &= \frac{\log(\sigma(1+\varepsilon_1)/q(1-\varepsilon_1))}{\log k} \\ &= \frac{\log \sigma - \log q}{\log k} + \frac{\log((1+\varepsilon_1)/(1-\varepsilon_1))}{\log k}.\end{aligned}\quad (40)$$

Denoting now $\alpha = (\log \sigma - \log q)/\log k$ and $\varepsilon = \log((1+\varepsilon_1)/(1-\varepsilon_1))/\log k$; thus we obtain the required form.

Finally, we show that $qk \leq \sigma$. If $qk > \sigma$, then we have $\alpha < 1$. For sufficiently small $\varepsilon > 0$, we have $\alpha + \varepsilon < 1$, which contradicts with the transcendency of f . Thus Theorem 1 is proved. $\square$

Proof of Theorem 2. Suppose $f(z)$ is a transcendental meromorphic solution of (13). Denoting $C = \max\{|c_1|, |c_2|, \dots, |c_n|\}$.

(i) $0 < |a| < 1$. We may assume that $q > \sigma$, since the case $q \leq \sigma$ is trivial by the fact that $\mu(f) \geq 0$. By Lemmas 1–3, we have for any $\varepsilon > 0$ and $\beta > 1$,

$$\begin{aligned}qT(r, f(p(z))) + S(r, f) &= T(r, Q(z, f(p(z)))) \\ &= T\left(r, \frac{\sum_{\lambda \in I} \alpha_\lambda(z) \left(\prod_{v=1}^n f(z+c_v)^{l_{\lambda,v}}\right)}{\sum_{\mu \in J} \beta_\mu(z) \left(\prod_{v=1}^n f(z+c_v)^{m_{\mu,v}}\right)}\right) \\ &\leq \sum_{v=1}^n \sigma_v T(r, f(z+c_v)) + S(r, f) \\ &\leq \sum_{v=1}^n \sigma_v (1+\varepsilon) T(r+C, f(z)) + S(r, f) \\ &= \left(\sum_{v=1}^n \sigma_v\right) (1+\varepsilon) T(r+C, f(z)) + S(r, f) \\ &= \sigma(1+\varepsilon) T(r+C, f(z)) + S(r, f) \\ &\leq \sigma(1+\varepsilon) T(\beta r, f) + S(r, f),\end{aligned}\quad (41)$$

where r is large enough.

By the last assertion of Lemma 5 and (41), we obtain that, for $\mu = |a| - \delta$ ($0 < \delta < |a|, 0 < \mu < 1$), the following inequality

$$q(1-\varepsilon)T(\mu r, f) \leq \sigma(1+\varepsilon)T(\beta r, f) \quad (42)$$

holds, where r is large enough outside of a possible set of finite linear measure. By Lemma 4, we get that for any $\gamma > 1$ and sufficiently large r ,

$$q(1-\varepsilon)T(\mu r, f) \leq \sigma(1+\varepsilon)T(\gamma \beta r, f). \quad (43)$$

Therefore,

$$\frac{q(1-\varepsilon)}{\sigma(1+\varepsilon)}T(r, f) \leq T\left(\frac{\gamma \beta}{\mu}r, f\right). \quad (44)$$

Since $\beta > 1, \gamma > 1, 0 < \mu < 1$ and $q > \sigma$, we have $\beta\gamma/\mu > 1$ and $q(1-\varepsilon)/\sigma(1+\varepsilon) > 1$ when ε is small enough. Using Lemma 7, we see that

$$\mu(f) \geq \frac{\log q(1-\varepsilon) - \log \sigma(1+\varepsilon)}{\log \gamma \beta - \log \mu}. \quad (45)$$

Letting $\varepsilon \rightarrow 0, \delta \rightarrow 0, \beta \rightarrow 1$ and $\gamma \rightarrow 1$, we have

$$\mu(f) \geq \frac{\log q - \log \sigma}{-\log |a|}. \quad (46)$$

(ii) $|a| > 1$. By the similar reasoning as is (i), we easily obtain that

$$\begin{aligned}q(1-\varepsilon)T(\mu r, f) &\leq qT(r, f(p(z))) \\ &\leq \sigma(1+\varepsilon)T(r+C, f(z)) + S(r, f)\end{aligned}\quad (47)$$

for all r large enough. We may select sufficiently small numbers $\delta > 0$ and $\varepsilon > 0$, such that $\mu = |a| - \delta > 1$ and $(1/\mu) + \varepsilon < 1$. Thus we have

$$T(\mu r, f) \leq \frac{\sigma(1+\varepsilon)}{q(1-\varepsilon)}T(r+C, f(z)) + S(r, f); \quad (48)$$

namely,

$$T(\mu r, f) \leq \frac{\sigma(1+\varepsilon)}{q(1-\varepsilon)}T(r+C, f(z)), \quad (49)$$

where r is large enough possibly outside of a set of finite linear measure. By Lemma 4, we have for any $1 < \gamma < \mu$,

$$T(\mu r, f) \leq \frac{\sigma(1+\varepsilon)}{q(1-\varepsilon)}T(\gamma r, f(z)); \quad (50)$$

that is,

$$T(r, f) \leq \frac{\sigma(1+\varepsilon)}{q(1-\varepsilon)}T\left(\frac{\gamma}{\mu}r, f(z)\right) \quad (51)$$

holds for all sufficiently large r . By Lemma 8, we obtain

$$\rho(f) \leq \frac{\log \sigma - \log q + \log(1+\varepsilon) - \log(1-\varepsilon)}{-\log(\gamma/\mu)}. \quad (52)$$

Letting $\varepsilon \rightarrow 0, \delta \rightarrow 0$ and $\gamma \rightarrow 1$, we have

$$\rho(f) \leq \frac{\log \sigma - \log q}{\log |a|}. \quad (53)$$

(iii) $|a| = 1$ and $q > \sigma$. The proof of this case is completely similar as in the case in (i). In fact, we set $\mu = |a| - \delta = 1 - \delta$ ($0 < \delta < 1, 0 < \mu < 1$). Similarly, we can get

$$\mu(f) \geq \frac{\log q - \log \sigma}{-\log |a|}. \quad (54)$$

Since $|a| = 1$, we have $\mu(f) = \rho(f) = \infty$. $\square$

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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References

- [1] W. Cherry and Z. Ye, *Nevanlinna's Theory of Value Distribution*, Springer Monographs in Mathematics, Springer, Berlin, Germany, 2001.
- [2] W. K. Hayman, *Meromorphic Functions*, Oxford Mathematical Monographs, Clarendon Press, Oxford, UK, 1964.
- [3] Y. Z. He and X. Z. Xiao, *Algebroid Functions and Ordinary Differential Equations*, Beijing, China, 1988.
- [4] I. Laine, *Nevanlinna Theory and Complex Differential Equations*, vol. 15 of *de Gruyter Studies in Mathematics*, Walter de Gruyter, Berlin, Germany, 1993.
- [5] I. Laine, J. Rieppo, and H. Silvennoinen, "Remarks on complex difference equations," *Computational Methods and Function Theory*, vol. 5, no. 1, pp. 77–88, 2005.
- [6] J. Rieppo, "On a class of complex functional equations," *Annales Academiæ Scientiarum Fennicæ*, vol. 32, no. 1, pp. 151–170, 2007.
- [7] X.-M. Zheng, Z.-X. Chen, and J. Tu, "Growth of meromorphic solutions of some difference equations," *Applicable Analysis and Discrete Mathematics*, vol. 4, no. 2, pp. 309–321, 2010.
- [8] A. Z. Mokhon'ko, "The Nevanlinna characteristics of certain meromorphic functions," *Teoriya Funkcii, Funkcional'nyi Analiz i ih Prilozhenija*, vol. 14, pp. 83–87, 1971 (Russian).
- [9] A. A. Mokhon'ko and V. D. Mokhon'ko, "Estimates of the Nevanlinna characteristics of certain classes of meromorphic functions, and their applications to differential equations," *Akademija Nauk SSSR*, vol. 15, pp. 1305–1322, 1974.
- [10] R. Goldstein, "Some results on factorisation of meromorphic functions," *Journal of the London Mathematical Society*, vol. 4, pp. 357–364, 1971.
- [11] R. Goldstein, "On meromorphic solutions of certain functional equations," *Aequationes Mathematicæ*, vol. 18, no. 1-2, pp. 112–157, 1978.
- [12] G. G. Gundersen, J. Heittokangas, I. Laine, J. Rieppo, and D. Yang, "Meromorphic solutions of generalized Schröder equations," *Aequationes Mathematicæ*, vol. 63, no. 1-2, pp. 110–135, 2002.