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Research Article

Stability of the Exponential Functional Equation in Riesz Algebras

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We deal with the stability of the exponential Cauchy functional equation $F(x+y) = F(x)F(y)$ in the class of functions $F : G \to L$ mapping a group $(G, +)$ into a Riesz algebra L. The main aim of this paper is to prove that the exponential Cauchy functional equation is stable in the sense of Hyers-Ulam and is not superstable in the sense of Baker. To prove the stability we use the Yosida Spectral Representation Theorem.

1. Introduction

In 1979 Baker et al. (cf. [1]) proved that the exponential functional equation

$$f(x+y) = f(x)f(y) \quad \text{for } x, y \in V \tag{1}$$

in the class of functions mapping a vector space V to the real numbers $\mathbb{R}$ is superstable; that is, any function f satisfying, with given $\delta > 0$, the inequality

$$|f(x+y) - f(x)f(y)| \le \delta \quad \text{for } x, y \in V \tag{2}$$

is either bounded or exponential (satisfies (1)). Then Baker generalized this famous result in [2]. We quote this theorem here since it will be used in the sequel.

Theorem 1 (cf. [2, Theorem 1]). *Let $(S, +)$ be a semigroup and let $\delta > 0$ be given. If a function $f : S \to \mathbb{C}$ satisfies the inequality*

$$|f(x+y) - f(x)f(y)| \le \delta \tag{3}$$

for all $x, y \in S$, then either $|f(x)| \le (1 + \sqrt{1+4\delta})/2$ for all $x \in S$ or $f(x+y) = f(x)f(y)$ for all $x, y \in S$.

After that the stability of the exponential functional equation has been widely investigated (cf., e.g., [3–6]).

This paper will primarily be concerned with the question if similar result holds true in the class of functions taking values in Riesz algebra L with the common notion of the absolute value $|x| = \sup\{x, -x\}$ of an element $x \in L$ stemming from the order structure of L.

The main aim of the present paper is to show that the superstability phenomenon does not hold in such an order setting. However, we prove that the exponential functional equation (1) is stable in the Ulam-Hyers sense; that is, for any given $f : G \to L$ satisfying inequality (3) there exists an exponential function $g : G \to L$ which approximates f uniformly on G in the sense that the set $\{|f(x) - g(x)| : x \in G\}$ is bounded in L.

As a method of investigation we apply spectral representation theory for Riesz spaces; to be more precise, we use the Yosida Spectral Representation Theorem for Riesz spaces with a strong order unit.

For some recent results concerning stability of functional equations in vector lattices we refer the interested reader to [7–12].

2. Preliminaries

Throughout the paper $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{R}$, and $\mathbb{R}_+$ are used to denote the sets of all positive integers, integers, real numbers and nonnegative real numbers, respectively.

For the readers convenience we quote basic definitions and properties concerning Riesz spaces (cf. [13]).

Definition 2 (cf. [13, Definitions 11.1 and 22.1]). We say that a real linear space L, endowed with a partial order $\leq$, is a *Riesz space* if $\sup\{x, y\}$ exists for all $x, y \in L$ and

$$ax + y \leq az + y \quad x, y, z \in X, \ x \leq z, \ a \in \mathbb{R}_+. \tag{4}$$

We define the absolute value of $x \in L$ by the formula $|x| := \sup\{x, -x\} \geq 0$. A Riesz space L is called *Archimedean* if, for each $x \in L$, the inequality $x \leq 0$ holds whenever the set $\{nx : n \in \mathbb{N}\}$ is bounded above. We say that L is a *Riesz algebra* if L is a Riesz space endowed with the common algebra multiplication satisfying $xy \geq 0$ for $x, y \geq 0$. A Riesz algebra L is termed an *f-algebra*, whenever $\inf\{x, y\} = 0$ implies $\inf\{xz, y\} = \inf\{zx, y\} = 0$ for every $z \geq 0$.

There are several types of convergence that may be defined according to the order structure. One of them is the *relatively uniform convergence* defined as follows.

Definition 3 (cf. [13, Definition 39.1]). Let L be a Riesz space and let $u \in L_+ := \{x \in L : x \geq 0\}$. A sequence $\{f_n\}_{n\in\mathbb{N}}$ in L is said to *converge u-uniformly* to an element $f \in L$ whenever, for every $\varepsilon > 0$, there exists a positive integer n_0 such that $|f - f_n| \leq \varepsilon u$ holds for all $n \geq n_0$. A sequence $\{f_n\}_{n\in\mathbb{N}}$ in L is called *u-uniform Cauchy sequence* whenever, for every $\varepsilon > 0$, there exists a positive integer n_1 such that $|f_m - f_n| \leq \varepsilon u$ holds for all $m, n \geq n_1$.

Definition 4 (cf. [13, Definition 39.3]). A Riesz space L is called *u-uniformly complete* (with a given $u \in L_+$) whenever every u-uniform Cauchy sequence has a u-uniform limit in L. Furthermore L is called *uniformly complete* if it is u-uniformly complete with any $u \in L_+$.

There is a large class of spaces satisfying the above conditions. In particular every Dedekind σ-complete space (see Definition 5 below) is an Archimedean and uniformly complete Riesz space.

Definition 5 (cf. [13, Definition 1.1]). We say that a Riesz space L is *Dedekind σ-complete* if any non-empty at most countable subset which is bounded above has a supremum.

Definition 6 (cf. [13, Definition 21.4]). The element $e \in L_+$ is called a *strong unit* if for every $l \in L$ there exists $\alpha \in \mathbb{R}$ such that $|l| \leq \alpha e$.

For more detailed information and, in particular, examples of Riesz spaces posessing the above properties we refer the interested reader to [13].

In further considerations the Yosida Spectral Representation Theorem, which is quoted below, will be used.

Theorem 7. *Yosida Spectral Representation Theorem (cf. [13, Theorem 45.3]). Let L be an Archimedean Riesz space with a strong unit $e \in L_+$. Then there exists a topological space X and a Riesz subspace $\widehat{L}$ of the space $C(X)$ of all real continuous functions on X (with the pointwise order and pointwise operations of addition and scalar multiplication) and a Riesz isomorphism of L onto $\widehat{L}$.*

We will not distinguish $l \in L$ and its Yosida representant, if no confusion can occur.

Directly from the construction of the Yosida repesentatives one can deduce that the Yosida representative of a strong unit $e \in L_+$ is a constant function $e \equiv 1$. We omit the formal details here as they exceed the scope of the paper.

In general, the space $\widehat{L}$ of Yosida representatives is a Riesz subspace of $C(X)$. The following theorem gives us conditions under which $\widehat{L}$ is the whole $C(X)$.

Theorem 8 (cf. [13, Theorem 45.4]). *Let L be an Archimedean Riesz space with a strong unit $e \in L_+$ and let $\widehat{L}$ and $C(X)$ be as in the previous Theorem. The following conditions are now mutually equivalent.*

(i) *L is uniformly complete.*

(ii) $\widehat{L} = C(X)$.

(iii) *Every e-uniform Cauchy sequence in L has an e-uniform limit in L.*

Hence, if L is a Dedekind σ-complete space with a strong unit, then $\widehat{L} = C(X)$.

In the case where $\widehat{L} = C(X)$, the Yosida representation $\widehat{L}$ of L is not only a Riesz space but also a Riesz algebra with respect to the pointwise multiplication of functions in $C(X)$. But then, since L and $\widehat{L}$ are isomorphic as Riesz spaces, we may introduce ring multiplication for the elements $f, g \in L$ induced by the multiplication of representatives, that is,

$$h = fg \quad \text{if } \pi h = \pi f \pi g, \tag{5}$$

where $\pi : L \to C(X)$ is the Riesz isomorphism. Notice that $h \in L$ given by (5) is uniquely determined. Such a multiplication makes L into a commutative Riesz algebra with a unit element (a strong unit $e \in L_+$ is an algebra unit element, that is, $fe = ef = f$ for $f \in L$).

From now on a multiplication in a Riesz space L will be construed in the above sense.

3. The Main Result

We start with some, easy to prove, properties of exponential real functions on a 2-divisible group.

Remark 9. Let $(G, +)$ be a 2-divisible group and let $f : G \to \mathbb{R}$ satisfy exponential functional equation (1). Then the following conditions hold.

(i) $f(x) \geq 0$ for $x \in G$.

(ii) If there exists $x \in G \setminus \{0\}$ such that $f(x) = 0$, then $f \equiv 0$.

(iii) If $f \not\equiv 0$, then $f(0) = 1$.

(iv) If f is bounded, then $f \equiv 0$ or $f \equiv 1$.

(v) If $f \not\equiv 0$ then $f(-x) = 1/f(x)$ for $x \in G$.

Our main result reads us the following.

Theorem 10. *Let $(G, +)$ be an Abelian 2-divisible group and let L be an Archimedean Riesz space with a strong unit $e \in L_+$. We assume that L is e-uniformly complete. If a function $F : G \to L$ satisfies*

$$\left|F(x+y) - F(x)F(y)\right| \le u^2 - u \quad \text{for } x, y \in G, \tag{6}$$

with given $u \in L_+$, then there exists an exponential function $E : G \to L$ such that

$$|F(x) - E(x)| \le u + e \quad \text{for } x \in G. \tag{7}$$

Proof. The idea of the proof is based on the use of the Yosida Spectral Representation Theorem which enables us to apply Theorem 1 of Baker.

The proof runs in four steps.

Step 1. Consider $F : G \to L$ satisfying (6). According to the Yosida Spectral Representation Theorem, for every $x, y \in G$, we have $F(x), F(y), F(x+y), u^2 - u \in C(X)$. Therefore, by (6), one has

$$\left|F(x+y)(s) - F(x)(s)F(y)(s)\right| \le u^2(s) - u(s) \quad \text{for } x, y \in G,\ s \in X. \tag{8}$$

It means that, for any $s \in X$, $F(\cdot)(s)$ satisfies all the assumptions of Theorem 1. By Theorem 1 either $F(\cdot)(s)$ is bounded on G with $|F(x)(s)| \le u(s)$ for $x \in G$ or $F(\cdot)(s)$ is exponential on the whole G. Let

$$\mathscr{B} := \{s \in X : |F(x)(s)| \le u(s) \text{ for } x \in G\},$$
$$\mathscr{E} := \{s \in X : F(\cdot)(s) \text{ is unbounded and exponential on } G\}. \tag{9}$$

Of course we have $X = \mathscr{B} \cup \mathscr{E}$ and $\mathscr{B} \cap \mathscr{E} = \emptyset$.

We will prove that $\mathscr{E}$ is an open subset of X. For the indirect proof consider $s \in \mathscr{E}$ and suppose that each neighbourhood $\mathscr{U}$ of s has a nonempty intersection with $\mathscr{B}$. Let $m_{k,x} \in C(X)$ be given by

$$m_{k,x}(t) := F(kx)(t) - u(t) \quad \text{for } k \in \mathbb{Z},\ x \in G,\ t \in X. \tag{10}$$

Since $s \in \mathscr{E}$, there exist $x \in G \setminus \{0\}$ and $k \in \mathbb{Z}$ such that $m_{k,x}(s) > 0$. On the other hand, according to the supposition, in each neighbourhood $\mathscr{U}$ of s there exists t with $m_{k,x}(t) \le 0$, which brings a contradiction with the continuity of $m_{k,x}$.

Step 2. For given $x \in G$ we define $E(x) \in C(X)$ by

$$E(x)(s) = \begin{cases} F(x)(s), & \text{if } s \in \mathscr{E} \\ 1, & \text{if } s \in \mathscr{B} \end{cases} \quad \text{for } s \in X. \tag{11}$$

We shall prove the continuity of $E(x)$. First consider the case $s \in \mathscr{E}$. Take an arbitrary neighbourhood $\mathscr{V}$ of $E(x)(s)$. Since $\mathscr{E}$ is open, there exists a neighbourhood $\mathscr{U}_1$ of s with $\mathscr{U}_1 \subset \mathscr{E}$. By the choice of s we have $E(x)(s) = F(x)(s)$ and by the continuity of $F(x)$ at s there exists a neighbourhood $\mathscr{U}_2$ of s such that $F(x)(\mathscr{U}_2) \subset \mathscr{V}$. Then $\mathscr{U} := \mathscr{U}_1 \cap \mathscr{U}_2$ forms a neighbourhood of s with $E(x)(\mathscr{U}) \subset \mathscr{V}$.

Thus, it remains to consider $s \in \mathscr{B}$. For arbitrary $\varepsilon > 0$ let $\mathscr{V} := (1-\varepsilon, 1+\varepsilon)$ be a neighbourhood of $1 = E(x)(s)$. We will prove that there exists a neighbourhood $\mathscr{U}$ of s such that $F(x)(\mathscr{U} \cap \mathscr{E}) \subset \mathscr{V}$. Contrary, suppose that in each neighbourhood $\mathscr{U}$ of s there exists $t \in \mathscr{U} \cap \mathscr{E}$ with $F(x)(t) > 1+\varepsilon$ or $F(x)(t) < 1-\varepsilon$. Consider the case $F(x)(t) < 1-\varepsilon$. Then, taking into account the positivity of $F(x)(t)$, which follows from the fact that $F(\cdot)(t)$ is exponential and unbounded on G, we have

$$F(-x)(t) = \frac{1}{F(x)(t)} > \frac{1}{1-\varepsilon} > 1 + \varepsilon. \tag{12}$$

Let $\delta > 0$ be fixed. Then there exists $k = k(\varepsilon) \in \mathbb{N}$ such that $m_{k,x}(t) > \delta$ or $m_{k,-x}(t) > \delta$ depending on the case where $F(x)(t) > 1+\varepsilon$ or $F(x)(t) < 1-\varepsilon$, respectively. On the other hand, by the continuity of $m_{k,x}$ and the fact that $m_{k,x}(s) \le 0 < \delta$ there exists a neighbourhood $\mathscr{U}_1$ of s such that $m_{k,x}(t) < \delta$ for $t \in \mathscr{U}_1$. By the same reasons, there exists a neighbourhood $\mathscr{U}_2$ of s such that $m_{k,-x}(t) < \delta$ for $t \in \mathscr{U}_2$. Then $\mathscr{U}_3 := \mathscr{U}_1 \cap \mathscr{U}_2$ is a neighbourhood of s such that $m_{k,x}(t) < \delta$ and $m_{k,-x}(t) < \delta$ for all $t \in \mathscr{U}_3$, which brings a contradiction.

Consequently $E(x)(\mathscr{U}) \subset \mathscr{V}$ as $E(x)(\mathscr{U} \cap \mathscr{E}) = F(x)(\mathscr{U} \cap \mathscr{E}) \subset \mathscr{V}$ and $E(x)(\mathscr{U} \cap \mathscr{B}) = \{1\} \subset \mathscr{V}$. This completes the proof that $E(x) \in C(X)$.

Therefore, by Theorem 8 one may treat $E(x)$ as an element of L. Since $x \in G$ has been chosen arbitrarily, in fact formula (11) defines a function $E : G \to L$.

Step 3. We will prove that function E given by (11) is exponential.

Let $x, y \in G$. If $s \in \mathscr{E}$ then we have $E(x+y)(s) = F(x+y)(s) = F(x)(s)F(y)(s) = E(x)(s)E(y)(s)$. Else $s \in \mathscr{B}$ and then $E(x+y)(s) = 1 = E(x)(s)E(y)(s)$. As a consequence we have $E(x+y) = E(x)E(y)$.

Step 4. We will prove (7).

Let $x \in G$. Then $|F(x)(s) - E(x)(s)| = 0$ for $s \in \mathscr{E}$ and $|F(x)(s) - E(x)(s)| = |F(x)(s) - 1| \le u(s) + 1$ for $s \in \mathscr{B}$. It means that

$$|F(x)(s) - E(x)(s)| \le u(s) + 1, \quad \text{for } s \in X,\ x \in G \tag{13}$$

which yields

$$|F(x) - E(x)| \le u + e \quad \text{for } x \in G. \tag{14}$$

□

4. Final Remarks

Let us recall the following theorem, which provides us with the condition under which a Riesz homomorphism (as a homomorphism between Riesz spaces) is multiplicative.

Theorem 11 (cf. [14, Proposition 353P]). *Let L be an Archimedean f-algebra with multiplicative identity e. If V is another Archimedean f-algebra with multiplicative identity e', and $T : L \to V$ is a positive linear operator such that $T(e) = e'$, then T is a Riesz homomorphism if and only if $T(u \cdot v) = T(u) \cdot T(v)$ for all $u, v \in L$.*

By the above theorem and the Yosida Spectral Representation Theorem one can obtain the following corollary.

Corollary 12. *Let $(G, +)$ be an Abelian 2-divisible group and let L be an Archimedean f-algebra with a multiplicative identity $e \in L_+$ which is a strong order unit. We assume that L is e-uniformly complete. If a function $F : G \to L$ satisfies*

$$\left|F(x+y) - F(x)F(y)\right| \le u^2 - u \quad \text{for } x, y \in G, \tag{15}$$

with given $u \in L_+$, then there exist an exponential function $E : G \to L$ such that

$$|F(x) - E(x)| \le u + e \quad \text{for } x \in G. \tag{16}$$

Theorem 10 (Corollary 12) states that the exponential functional equation (1) in Riesz algebras is stable in the Ulam-Hyers sense. Taking into account Theorem 1 it is natural to ask if (1) is superstable in the sense of Baker. It appears that the superstability phenomenon in Riesz algebras fails to hold. In the next example we show that there exists a group G, an f-algebra L satisfying all the assertions of Theorem 10 and a function $F : G \to L$ which fulfills (6) but is neither exponential nor bounded.

Example 13. Let $B[-1, 1]$ be an Archimedean f-algebra of all bounded real functions on the interval $[-1, 1]$ with a strong unit $e \equiv 1$ with the pointwise order, pointwise addition and multiplication. Then $B[-1, 1]$ is e-uniformly complete. Let $u \in B[-1, 1]$ be given by

$$u(s) := \begin{cases} 1 - s, & \text{if } s \in [-1, 0) \\ 0, & \text{if } s \in [0, 1]. \end{cases} \tag{17}$$

We define $F : \mathbb{R} \to B[-1, 1]$ by

$$F(x)(s) := \begin{cases} s, & \text{if } s \in [-1, 0) \\ \exp(sx), & \text{if } s \in [0, 1] \end{cases} \quad \text{for } x \in \mathbb{R}. \tag{18}$$

Then such an F is, clearly, neither bounded nor exponential. To observe that F satisfies (6) fix $x, y \in \mathbb{R}$. If $s < 0$ then we have $|F(x+y)(s) - F(x)(s)F(y)(s)| = |s - s^2| = u^2(s) - u(s)$. If $s \ge 0$ then $|F(x+y)(s) - F(x)(s)F(y)(s)| = 0$.

Remark 14. In general the exponential function satisfying assertions of Theorem 10 is not unique. Indeed, consider $B[-1, 1]$, $F : \mathbb{R} \to B[-1, 1]$, and $u \in B[-1, 1]$ as defined in Example 13. Then the exponential functions $E_1, E_2 : \mathbb{R} \to B[-1, 1]$ given by

$$\begin{aligned} E_1(x)(s) &:= \begin{cases} 1, & \text{if } s \in [-1, 0) \\ \exp(sx), & \text{if } s \in [0, 1] \end{cases} \quad \text{for } x \in \mathbb{R}, \\ E_2(x)(s) &:= \begin{cases} 0, & \text{if } s \in [-1, 0) \\ \exp(sx), & \text{if } s \in [0, 1] \end{cases} \quad \text{for } x \in \mathbb{R} \end{aligned} \tag{19}$$

approximate F uniformly on $\mathbb{R}$, that is, satisfy (7).

Conflict of Interests

The author declares that there is no conflict of interests regarding the publication of this paper.

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