Research Article

Hermite-Hadamard Type Inequalities for Superquadratic Functions via Fractional Integrals

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We use basic properties of superquadratic functions to obtain some new Hermite-Hadamard type inequalities via Riemann-Liouville fractional integrals. For superquadratic functions which are also convex, we get refinements of existing results.

1. Introduction

Let $f : I \subseteq \mathbb{R} \rightarrow \mathbb{R}$ be a convex function and $a, b \in I$ with $a < b$; then

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b f(t) dt \leq \frac{f(a) + f(b)}{2} \quad (1)$$

is known as the Hermite-Hadamard inequality.

This remarkable result is well known in the literature as the Hermite-Hadamard inequality. Recently, the generalizations, refinements, and improvements of the classical Hermite-Hadamard inequality have been the subject of intensive research.

Definition 1 (see [1]). A function $f : [0, \infty) \rightarrow \mathbb{R}$ is superquadratic provided that for all $x \geq 0$ there exists a constant $C_x \in \mathbb{R}$ such that

$$f(y) \geq f(x) + C_x(y-x) + f(|y-x|), \quad (2)$$

for all $y \geq 0$.

Theorem 2 (see [1]). The inequality

$$f\left(\int g d\mu\right) \leq \int \left[f(g(s)) - f\left(\left|g(s) - \int g d\mu\right|\right) \right] d\mu(s) \quad (3)$$

holds for all probability measures μ and all nonnegative, μ -integrable functions g , if and only if f is superquadratic.

The discrete version of the above theorem is also used in the sequel.

Lemma 3 (see [2]). Suppose that f is superquadratic. Let $x_i \geq 0$, $1 \leq i \leq n$, and let $\bar{x} = \sum_{i=1}^n \lambda_i x_i$, where $\lambda_i \geq 0$ and $\sum_{i=1}^n \lambda_i = 1$. Then

$$\sum_{i=1}^n \lambda_i f(x_i) \geq f(\bar{x}) + \sum_{i=1}^n \lambda_i f(|x_i - \bar{x}|). \quad (4)$$

Nonnegative superquadratic functions are much better behaved as we see next.

Lemma 4 (see [1]). Let f be a superquadratic function with $C_x \in \mathbb{R}$ as in Definition 1. Then one gets the following:

- (1) $f(0) \leq 0$;
- (2) if $f(0) = f'(0) = 0$, then $C_x = f'(x)$ whenever f is differentiable at $x > 0$;
- (3) if $f \geq 0$, then f is convex and $f(0) = f'(0) = 0$.

The Hermite-Hadamard inequalities for superquadratic functions are established by Banic et al. in [3].

Theorem 5 (see [3]). Let $f : [0, \infty) \rightarrow \mathbb{R}$ be a superquadratic function and $0 \leq a < b$; then

$$\begin{aligned} & f\left(\frac{a+b}{2}\right) + \frac{1}{b-a} \int_a^b f\left(\left|x - \frac{a+b}{2}\right|\right) dx \\ & \leq \frac{1}{b-a} \int_a^b f(x) dx \\ & \leq \frac{f(a) + f(b)}{2} \\ & \quad - \frac{1}{(b-a)^2} \int_a^b [(b-x)f(x-a) \\ & \quad + (x-a)f(b-x)] dx. \end{aligned} \quad (5)$$

It is remarkable that Sarikaya et al. [4] proved the following interesting inequalities of Hermite-Hadamard type involving Riemann-Liouville fractional integrals.

Theorem 6 (see [4]). Let $f : [a, b] \rightarrow \mathbb{R}$ be a positive function with $a < b$ and $f \in L_1[a, b]$. If f is a convex function on $[a, b]$, then the following inequalities for fractional integrals hold:

$$\begin{aligned} f\left(\frac{a+b}{2}\right) & \leq \frac{\Gamma(\alpha+1)}{2(b-a)^\alpha} [J_{a^+}^\alpha f(b) + J_{b^-}^\alpha f(a)] \\ & \leq \frac{f(a) + f(b)}{2}, \end{aligned} \quad (6)$$

with $\alpha > 0$.

We remark that the symbols $J_{a^+}^\alpha$ and $J_{b^-}^\alpha f$ denote the left-sided and right-sided Riemann-Liouville fractional integrals of the order $\alpha \geq 0$ with $a \geq 0$ which are defined by

$$\begin{aligned} J_{a^+}^\alpha f(x) &= \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t) dt, \quad x > a, \\ J_{b^-}^\alpha f(x) &= \frac{1}{\Gamma(\alpha)} \int_x^b (t-x)^{\alpha-1} f(t) dt, \quad x < b, \end{aligned} \quad (7)$$

respectively. Here, $\Gamma(\alpha)$ is the gamma function defined by $\Gamma(\alpha) = \int_0^\infty e^{-t} t^{\alpha-1} dt$.

Fractional integral operators are widely used to solve differential equations and integral equations. So a lot of work has been obtained on the theory and applications of fractional integral operators.

For more results concerning the fractional integral operators, we refer the reader to [5–10] and references cited therein.

In this paper, we establish some new Hermite-Hadamard type inequalities for superquadratic functions via Riemann-Liouville fractional integrals which refine the inequalities of (6) for superquadratic functions which are also convex.

2. Main Results

From Lemma 3 for $n = 2$, we get for $0 \leq m \leq z \leq M$, $m \leq M$, for the superquadratic function f that

$$\begin{aligned} f(z) & \leq \frac{M-z}{M-m} f(m) + \frac{z-m}{M-m} f(M) \\ & \quad - \frac{M-z}{M-m} f(z-m) - \frac{z-m}{M-m} f(M-z), \\ f(M+m-z) & \leq \frac{z-m}{M-m} f(m) + \frac{M-z}{M-m} f(M) \\ & \quad - \frac{M-z}{M-m} f(z-m) - \frac{z-m}{M-m} f(M-z) \end{aligned} \quad (8)$$

hold, and therefore

$$\begin{aligned} f(z) + f(M+m-z) & \leq f(m) + f(M) \\ & \quad - 2 \frac{M-z}{M-m} f(z-m) \\ & \quad - 2 \frac{z-m}{M-m} f(M-z). \end{aligned} \quad (9)$$

Let $0 \leq a \leq x \leq (a+b)/2$; we get that

$$a \leq x \leq \frac{a+b}{2} \leq a+b-x \leq b. \quad (10)$$

Therefore, by replacing in (9)

$$z = \frac{a+b}{2}, \quad M = a+b-x, \quad m = x, \quad (11)$$

we get that

$$2f\left(\frac{a+b}{2}\right) \leq f(x) + f(a+b-x) - 2f\left(\frac{a+b}{2} - x\right). \quad (12)$$

Theorem 7. Let f be a superquadratic integrable function on $[0, b]$ with $0 \leq a < b$. Then

$$\begin{aligned} & f\left(\frac{a+b}{2}\right) \\ & \leq \frac{\Gamma(\alpha+1)}{2(b-a)^\alpha} [J_{a^+}^\alpha f(b) + J_{b^-}^\alpha f(a)] \\ & \quad - \frac{\alpha}{2(b-a)^\alpha} \int_a^b f\left|\frac{a+b}{2} - x\right| \\ & \quad \times ((b-x)^{\alpha-1} + (x-a)^{\alpha-1}) dx \end{aligned} \quad (13)$$

with $\alpha > 0$.

Proof. By inequality (12), we get that

$$\begin{aligned}
 f\left(\frac{a+b}{2}\right) &= f\left(\frac{a+b}{2}\right) \frac{\alpha}{2(b-a)^\alpha} \\
 &\quad \times \int_a^b \left((b-x)^{\alpha-1} + (x-a)^{\alpha-1}\right) dx \\
 &= f\left(\frac{a+b}{2}\right) \frac{\alpha}{(b-a)^\alpha} \\
 &\quad \times \int_a^{(a+b)/2} \left((b-x)^{\alpha-1} + (x-a)^{\alpha-1}\right) dx \\
 &\leq \frac{\alpha}{2(b-a)^\alpha} \int_a^{(a+b)/2} [f(x) + f(a+b-x)] \\
 &\quad \times \left((b-x)^{\alpha-1} + (x-a)^{\alpha-1}\right) dx \\
 &\quad - \frac{\alpha}{(b-a)^\alpha} \int_a^{(a+b)/2} f\left(\frac{a+b}{2} - x\right) \\
 &\quad \times \left((b-x)^{\alpha-1} + (x-a)^{\alpha-1}\right) dx. \tag{14}
 \end{aligned}$$

By the change of variable $x \rightarrow a+b-x$, we get

$$\begin{aligned}
 &\frac{\alpha}{2(b-a)^\alpha} \int_a^{(a+b)/2} f(a+b-x) \\
 &\quad \times \left((b-x)^{\alpha-1} + (x-a)^{\alpha-1}\right) dx \\
 &= \frac{\alpha}{(b-a)^\alpha} \int_{(a+b)/2}^b f(x) \\
 &\quad \times \left((b-x)^{\alpha-1} + (x-a)^{\alpha-1}\right) dx, \tag{15} \\
 &\frac{\alpha}{(b-a)^\alpha} \int_a^{(a+b)/2} f\left(\frac{a+b}{2} - x\right) \\
 &\quad \times \left((b-x)^{\alpha-1} + (x-a)^{\alpha-1}\right) dx \\
 &= \frac{\alpha}{(b-a)^\alpha} \int_{(a+b)/2}^b f\left(x - \frac{a+b}{2}\right) \\
 &\quad \times \left((b-x)^{\alpha-1} + (x-a)^{\alpha-1}\right) dx.
 \end{aligned}$$

Therefore

$$\begin{aligned}
 f\left(\frac{a+b}{2}\right) &= f\left(\frac{a+b}{2}\right) \frac{\alpha}{2(b-a)^\alpha} \\
 &\quad \times \int_a^b \left((b-x)^{\alpha-1} + (x-a)^{\alpha-1}\right) dx \\
 &= f\left(\frac{a+b}{2}\right) \frac{\alpha}{(b-a)^\alpha} \\
 &\quad \times \int_a^{(a+b)/2} \left((b-x)^{\alpha-1} + (x-a)^{\alpha-1}\right) dx
 \end{aligned}$$

$$\begin{aligned}
 &\leq \frac{\alpha}{2(b-a)^\alpha} \int_a^b f(x) \\
 &\quad \times \left((b-x)^{\alpha-1} + (x-a)^{\alpha-1}\right) dx \\
 &\quad - \frac{\alpha}{2(b-a)^\alpha} \int_a^b f\left|\frac{a+b}{2} - x\right| \\
 &\quad \times \left((b-x)^{\alpha-1} + (x-a)^{\alpha-1}\right) dx \\
 &= \frac{\Gamma(\alpha+1)}{2(b-a)^\alpha} [J_{a^+}^\alpha f(b) + J_{b^-}^\alpha f(a)] \\
 &\quad - \frac{\alpha}{2(b-a)^\alpha} \int_a^b f\left|\frac{a+b}{2} - x\right| \\
 &\quad \times \left((b-x)^{\alpha-1} + (x-a)^{\alpha-1}\right) dx. \tag{16}
 \end{aligned}$$

We have completed the proof. $\square$

Corollary 8. Putting $\alpha = 1$ in Theorem 7 gives

$$\begin{aligned}
 f\left(\frac{a+b}{2}\right) &\leq \frac{1}{b-a} \int_a^b f(x) dx \\
 &\quad - \frac{1}{b-a} \int_a^b f\left|\frac{a+b}{2} - x\right| dx. \tag{17}
 \end{aligned}$$

Let $0 \leq a \leq x \leq (a+b)/2$; we get that

$$a \leq x \leq a+b-x \leq b. \tag{18}$$

Therefore, by replacing in (9)

$$z = x, \quad M = b, \quad m = a, \tag{19}$$

we get that

$$\begin{aligned}
 &f(x) + f(a+b-x) \\
 &\leq f(a) + f(b) \\
 &\quad - 2 \frac{b-x}{b-a} f(x-a) - 2 \frac{x-a}{b-a} f(b-x). \tag{20}
 \end{aligned}$$

Theorem 9. Let f be a superquadratic integrable function on $[0, b]$ with $0 \leq a < b$. Then

$$\begin{aligned}
 &\frac{\Gamma(\alpha+1)}{2(b-a)^\alpha} [J_{a^+}^\alpha f(b) + J_{b^-}^\alpha f(a)] \\
 &\leq \frac{f(a) + f(b)}{2} \\
 &\quad - \frac{\alpha}{2(b-a)^\alpha} \int_a^b \left[\frac{b-x}{b-a} f(x-a) + \frac{x-a}{b-a} f(b-x) \right] \\
 &\quad \times \left((b-x)^{\alpha-1} + (x-a)^{\alpha-1}\right) dx \tag{21}
 \end{aligned}$$

with $\alpha > 0$.

Proof. By inequality (20), we get that

$$\begin{aligned}
 & \frac{f(a) + f(b)}{2} \\
 &= \frac{f(a) + f(b)}{2} \frac{\alpha}{2(b-a)^\alpha} \\
 & \quad \times \int_a^b ((b-x)^{\alpha-1} + (x-a)^{\alpha-1}) dx \\
 &= [f(a) + f(b)] \frac{\alpha}{2(b-a)^\alpha} \\
 & \quad \times \int_a^{(a+b)/2} ((b-x)^{\alpha-1} + (x-a)^{\alpha-1}) dx, \\
 & \frac{\Gamma(\alpha+1)}{2(b-a)^\alpha} [J_{a^+}^\alpha f(b) + J_{b^-}^\alpha f(a)] \\
 &= \frac{\alpha}{2(b-a)^\alpha} \int_a^b f(x) \\
 & \quad \times ((b-x)^{\alpha-1} + (x-a)^{\alpha-1}) dx \\
 &= \frac{\alpha}{2(b-a)^\alpha} \int_a^{(a+b)/2} [f(x) + f(a+b-x)] \\
 & \quad \times ((b-x)^{\alpha-1} + (x-a)^{\alpha-1}) dx \\
 &\leq \frac{\alpha}{2(b-a)^\alpha} \int_a^{(a+b)/2} [f(a) + f(b)] \\
 & \quad \times ((b-x)^{\alpha-1} + (x-a)^{\alpha-1}) dx \\
 & \quad - \frac{\alpha}{(b-a)^\alpha} \int_a^{(a+b)/2} \frac{b-x}{b-a} f(x-a) \\
 & \quad \times ((b-x)^{\alpha-1} + (x-a)^{\alpha-1}) dx \\
 & \quad - \frac{\alpha}{(b-a)^\alpha} \int_a^{(a+b)/2} \frac{x-a}{b-a} f(b-x) \\
 & \quad \times ((b-x)^{\alpha-1} + (x-a)^{\alpha-1}) dx \\
 &= \frac{f(a) + f(b)}{2} \\
 & \quad - \frac{\alpha}{2(b-a)^\alpha} \int_a^b \left[\frac{b-x}{b-a} f(x-a) \right. \\
 & \quad \left. + \frac{x-a}{b-a} f(b-x) \right] \\
 & \quad \times ((b-x)^{\alpha-1} + (x-a)^{\alpha-1}) dx. \tag{22}
 \end{aligned}$$

The proof is completed. $\square$

Corollary 10. Choosing $\alpha = 1$ in Theorem 9, one has

$$\begin{aligned}
 \frac{1}{b-a} \int_a^b f(x) dx &\leq \frac{f(a) + f(b)}{2} \\
 &\quad - \frac{1}{(b-a)^2} \int_a^b [(b-x)f(x-a) \\
 &\quad + (x-a)f(b-x)] dx. \tag{23}
 \end{aligned}$$

Corollary 11. Let f be defined as in Theorem 7; one gets

$$\begin{aligned}
 & f\left(\frac{a+b}{2}\right) + \frac{\alpha}{2(b-a)^\alpha} \\
 & \quad \times \int_a^b f\left|\frac{a+b}{2} - x\right| ((b-x)^{\alpha-1} + (x-a)^{\alpha-1}) dx \\
 &\leq \frac{\Gamma(\alpha+1)}{2(b-a)^\alpha} [J_{a^+}^\alpha f(b) + J_{b^-}^\alpha f(a)] \\
 &\leq \frac{f(a) + f(b)}{2} - \frac{\alpha}{2(b-a)^\alpha} \\
 & \quad \times \int_a^b \left[\frac{b-x}{b-a} f(x-a) + \frac{x-a}{b-a} f(b-x) \right] \\
 & \quad \times ((b-x)^{\alpha-1} + (x-a)^{\alpha-1}) dx. \tag{24}
 \end{aligned}$$

Corollary 12. Taking $\alpha = 1$ in Corollary 11, one obtains

$$\begin{aligned}
 & f\left(\frac{a+b}{2}\right) + \frac{1}{b-a} \int_a^b f\left|\frac{a+b}{2} - x\right| dx \\
 &\leq \int_a^b f(x) dx \\
 &\leq \frac{f(a) + f(b)}{2} - \frac{1}{(b-a)^2} \\
 & \quad \times \int_a^b [(b-x)f(x-a) \\
 & \quad + (x-a)f(b-x)] dx. \tag{25}
 \end{aligned}$$

3. Conclusion

In this note, we obtain some new Hermite-Hadamard type inequalities for superquadratic functions via Riemann-Liouville fractional integrals. For superquadratic functions which are also convex, we get refinements of known results. The concept of superquadratic functions in several variables is introduced in [11]. An interesting topic is whether we can use the methods in this paper to establish the Hermite-Hadamard inequalities for superquadratic functions in several variables via Riemann-Liouville integrals.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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