Research Article

Complete Moment Convergence for Arrays of Rowwise $\varphi$-Mixing Random Variables

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We investigate the complete moment convergence for maximal partial sum of arrays of rowwise $\varphi$-mixing random variables under some more general conditions. The results obtained in the paper generalize and improve some known ones.

1. Introduction

Let $\{X_n, n \geq 1\}$ be a sequence of random variables defined on a fixed probability space $(\Omega, \mathcal{F}, P)$. Let $n$ and $m$ be positive integers. Write $\mathcal{F}_m^m = \sigma (X_i, n \leq i \leq m)$. Given $\sigma$-algebras $\mathcal{B}, \mathcal{R}$ in $\mathcal{F}$, let

$$\varphi(\mathcal{B}, \mathcal{R}) = \sup_{A \in \mathcal{B}, B \in \mathcal{R}, P(A) > 0} |P(B \mid A) - P(B)|.$$  

(1)

Define the $\varphi$-mixing coefficients by

$$\varphi(n) = \sup_{k \geq 1} \varphi(\mathcal{F}_k^\infty, \mathcal{F}_{k+n}^\infty), \quad n \geq 0.$$  

(2)

A random variable sequence $\{X_n, n \geq 1\}$ is said to be $\varphi$-mixing if $\varphi(n) \downarrow 0$ as $n \rightarrow \infty$. $\varphi(n)$ is called mixing coefficient. A triangular array of random variables $\{X_{nk}, k \geq 1, n \geq 1\}$ is said to be an array of rowwise $\varphi$-mixing random variables if, for every $n \geq 1$, $\{X_{nk}, k \geq 1\}$ is a $\varphi$-mixing sequence of random variables. The notion of $\varphi$-mixing random variables was introduced by Dobrushin [1] and many applications have been found. See, for example, Utev [2] for central limit theorem, Gan and Chen [3] for limit theorem, Peligrad [4] for weak invariance principle, Shao [5] for almost sure invariance principles, Chen and Wang [6], Shen et al. [7, 8], Wu [9], and Wang et al. [10] for complete convergence, Hu and Wang [11] for large deviations, and so forth. When these are compared with corresponding results of independent random variable sequences, there still remains much to be desired.

Definition 1. A sequence of random variables $\{U_n, n \geq 1\}$ is said to converge completely to a constant $a$ if, for any $\varepsilon > 0$,

$$\sum_{n=1}^{\infty} P(|U_n - a| > \varepsilon) < \infty.$$  

(3)

In this case, one writes $U_n \rightarrow a$ completely. This notion was given first by Hsu and Robbins [12].

Definition 2. Let $\{Z_n, n \geq 1\}$ be a sequence of random variables and $a_n > 0, b_n > 0, q > 0$. If

$$\sum_{n=1}^{\infty} a_n E\left[b_n^{-1} |Z_n| - \frac{\varepsilon}{q}\right] < \infty \quad \forall \varepsilon > 0,$$  

(4)

then the above result was called the complete moment convergence by Chow [13].

Let $\{X_{nk}, k \geq 1, n \geq 1\}$ be an array of rowwise $\varphi$-mixing random variables with mixing coefficients $\{\varphi(n), n \geq 1\}$ in each row, let $\{a_n, n \geq 1\}$ be a sequence of positive real numbers such that $a_n \uparrow \infty$, and let $\{\Psi_k(t), k \geq 1\}$ be a sequence of positive even functions such that

$$\frac{\psi_k(|t|)}{|t|^q} \uparrow, \quad \frac{\Psi_k(|t|)}{|t|^q} \downarrow \quad \text{as} \quad |t| \uparrow$$  

(5)
for some $1 \leq q < p$ and each $k \geq 1$. In order to prove our results, we mention the following conditions:

$$E X_{n k} = 0, \ k \geq 1, \ n \geq 1,$$  \hspace{1cm} (6)

$$\sum_{n=1}^{\infty} \sum_{k=1}^{n} \frac{E \Psi_{k}(X_{n k})}{\Psi_{k}(a_n)} < \infty,$$  \hspace{1cm} (7)

$$\sum_{n=1}^{\infty} \left( \frac{E X_{n k}^{2}}{a_n} \right)^{v/2} < \infty,$$  \hspace{1cm} (8)

where $v \geq p$ is a positive integer.

The following are examples of function $\Psi_{k}(t)$ satisfying assumption (5): $\Psi_{k}(t) = |t|^p$ for some $q < \beta < p$ or $\Psi_{k}(t) = |t|^\beta \log(1 + |t|^{p-\beta})$ for $t \in (-\infty, +\infty)$. Note that these functions are nonmonotone on $t \in (-\infty, +\infty)$, while it is simple to show that, under condition (5), the function $\Psi_{k}(t)$ is an increasing function for $t > 0$. In fact, $\Psi_{k}(t) = (\Psi_{k}(t)/|t|^p) \cdot |t|^p, \ t > 0$, and $|t|^p \uparrow [t] \uparrow$; then we have $\Psi_{k}(t) \uparrow$.

Recently Gan et al. [14] obtained the following complete convergence for \( \varphi \)-mixing random variables.

**Theorem A.** Let $\{X_{n}, n \geq 1\}$ be a sequence of \( \varphi \)-mixing mean zero random variables with $\sum_{n=1}^{\infty} \varphi(1/2)(n) < \infty$, let $\{a_n, n \geq 1\}$ be a sequence of positive real numbers with $a_n \uparrow \infty$, and let $\{\Psi_{n}(t), n \geq 1\}$ be a sequence of nonnegative even functions such that $\Psi_{n}(t) > 0$ as $t > 0$ and $(\Psi_{n}(t/|t|) \uparrow$ and $(\Psi_{n}(|t|/|t|^p) \downarrow$ as $|t| \uparrow \infty$, where $p \geq 2$. If the following conditions are satisfied:

$$\sum_{n=1}^{\infty} \sum_{k=1}^{n} \frac{E \Psi_{k}(X_{n k})}{\Psi_{k}(a_n)} < \infty,$$  \hspace{1cm} (9)

$$\sum_{n=1}^{\infty} \sum_{k=1}^{n} E \left[ \frac{X_{n k}^{r}}{a_n^r} \right] < \infty,$$  \hspace{1cm} (10)

where $0 < r \leq 2, s > 0$, then

$$\frac{1}{a_n} \max_{1 \leq j \leq n} \left\{ \frac{1}{k} \sum_{k=1}^{j} X_{k} \right\} \rightarrow 0 \ \text{completely}. \hspace{1cm} (11)$$

For more details about this type of complete convergence, one can refer to Gan and Chen [3], Wu et al. [15], Wu [16], Huang et al.[17], Shen [18], Shen et al. [19, 20], and so on. The purpose of this paper is extending Theorem A to the complete moment convergence, which is a more general version of the complete convergence, and making some improvements such that the conditions are more general. In this work, the symbol $C$ always stands for a generic positive constant, which may vary from one place to another.

### 2. Preliminary Lemmas

In this section, we give the following lemma which will be used to prove our main results.

**Lemma 3** (cf. Wang et al. [10]). Let $\{X_{n}, n \geq 1\}$ be a sequence of \( \varphi \)-mixing random variables satisfying $\sum_{n=1}^{\infty} \varphi(1/2)(n) < \infty$, $p \geq 2$. Assume that $EX_{n} = 0$, and $E|X_{n}|^p < \infty$, for each $n \geq 1$.

Then there exists a constant $C$ depending only on $p$ and $\varphi(\cdot)$ such that

$$E \left[ \max_{1 \leq j \leq n} \left( \frac{1}{k} \sum_{k=1}^{j} X_{k} \right)^{p/2} \right] \leq C \left[ \sum_{j=1}^{n} E |X_{j}|^{p} + \left( \sum_{j=1}^{n} EX_{j}^{2} \right)^{p/2} \right],$$  \hspace{1cm} (12)

for every $a \geq 0$ and $n \geq 1$. In particular, one has

$$E \left[ \max_{1 \leq j \leq n} \left( \frac{1}{k} \sum_{k=1}^{j} X_{k} \right)^{p/2} \right] \leq C \left[ \sum_{j=1}^{n} E |X_{j}|^{p} + \left( \sum_{j=1}^{n} EX_{j}^{2} \right)^{p/2} \right],$$  \hspace{1cm} (13)

for every $n \geq 1$.

### 3. Main Results and Their Proofs

Let $\{X_{n}, k \geq 1, n \geq 1\}$ be an array of rowwise \( \varphi \)-mixing random variables and let $\varphi(\cdot)$ be the mixing coefficient of $\{X_{n}, k \geq 1\}$ for any $n \geq 1$. Our main results are as follows.

**Theorem 4.** Let $\{X_{n}, k \geq 1, n \geq 1\}$ be an array of rowwise \( \varphi \)-mixing random variables satisfying $\sup_{n \geq 1} \sum_{k=1}^{\infty} \varphi(1/2)(k) < \infty$ and let $\{a_n, n \geq 1\}$ be a sequence of positive real numbers such that $a_n \uparrow \infty$. Also, let $\{\Psi_{n}(t), k \geq 1\}$ be a positive even function satisfying (5) for $1 \leq q < p \leq 2$. Then under conditions (6) and (7), one has

$$\sum_{n=1}^{\infty} a_n^{-q} E \left[ \max_{1 \leq j \leq n} \left( \frac{1}{k} \sum_{k=1}^{j} X_{k} \right)^{q} \right] < \infty, \ \forall \varepsilon > 0. \hspace{1cm} (14)$$

**Proof.** Firstly, let us prove the following statements from conditions (5) and (7).

(i) For $r \geq 1, 0 < u \leq q$,

$$\sum_{n=1}^{\infty} \left( \sum_{k=1}^{n} E \left| X_{n k} \right|^r I \left( \left| X_{n k} \right| > a_n \right) \right)^{u} \leq \sum_{n=1}^{\infty} \left( \sum_{k=1}^{n} E \left| X_{n k} \right|^{r} I \left( \left| X_{n k} \right| > a_n \right) \right)^{u} \leq \sum_{n=1}^{\infty} \left( \frac{n}{a_n^r} \sum_{k=1}^{n} \frac{E \Psi_{k}(X_{n k})}{\Psi_{k}(a_n)} \right)^{u} \leq \left( \sum_{n=1}^{\infty} \frac{n}{a_n^r} \sum_{k=1}^{n} \frac{E \Psi_{k}(X_{n k})}{\Psi_{k}(a_n)} \right)^{u} \leq \left( \sum_{n=1}^{\infty} \frac{n}{a_n^r} \sum_{k=1}^{n} \frac{E \Psi_{k}(X_{n k})}{\Psi_{k}(a_n)} \right)^{u} < \infty. \hspace{1cm} (15)$$

(ii) For $v \geq p$,

$$\sum_{n=1}^{\infty} \sum_{k=1}^{n} E \left| X_{n k} \right|^r I \left( \left| X_{n k} \right| \leq a_n \right) \leq \sum_{n=1}^{\infty} \sum_{k=1}^{n} E \left| X_{n k} \right|^{r} I \left( \left| X_{n k} \right| \leq a_n \right) \leq \sum_{n=1}^{\infty} \sum_{k=1}^{n} \frac{E \Psi_{k}(X_{n k})}{\Psi_{k}(a_n)} < \infty. \hspace{1cm} (16)$$
For \( n \geq 1 \), denote \( M_n(X) = \max_{1 \leq j \leq n} \sum_{k=1}^{j} X_{nk} \). It is easy to check that

\[
\sum_{n=1}^{\infty} a_n^{-q} E[ M_n(X) - a_n ]^q \leq \sum_{n=1}^{\infty} a_n^{-q} \left( \int_0^{\infty} P \{ M_n(X) > a_n + t^{1/q} \} dt \right)
= \sum_{n=1}^{\infty} a_n^{-q} \left( \int_0^{\infty} P \{ M_n(X) > a_n + t^{1/q} \} dt \right) + \sum_{n=1}^{\infty} a_n^{-q} \int_{a_n}^{\infty} P \{ M_n(X) > a_n + t^{1/q} \} dt
\]

\[
\leq \sum_{n=1}^{\infty} P \{ M_n(X) > a_n \}
+ \sum_{n=1}^{\infty} a_n^{-q} \int_{a_n}^{\infty} P \{ M_n(X) > a_n + t^{1/q} \} dt \leq \sum_{n=1}^{\infty} P \{ M_n(X) > a_n \}
\]

From (19) and (20), it follows that, for \( n \) large enough,

\[
P \left( \max_{1 \leq j \leq n} \left| \frac{1}{a_n} \sum_{k=1}^{j} X_{nk} \right| > \varepsilon \right)
\leq \sum_{n=1}^{\infty} P \{ |X_{nk}| > a_n \} + P \left( \max_{1 \leq j \leq n} \left| t_j^{(n)} \right| > \frac{\varepsilon}{2} \right).
\]

Hence we only need to prove that

\[
I \equiv \sum_{n=1}^{\infty} \sum_{k=1}^{n} P \{ |X_{nk}| > a_n \} < \infty,
\]

(22)

For \( I \), it follows by (15) that

\[
I = \sum_{n=1}^{\infty} \sum_{k=1}^{n} E I \{ |X_{nk}| > a_n \}
\leq \sum_{n=1}^{\infty} \sum_{k=1}^{n} E X_{nk}^q I \{ |X_{nk}| > a_n \}
\]

(23)

For \( II \), take \( r \geq 2 \). Since \( p \leq 2 \), \( r \geq p \), we have by Markov inequality, Lemma 3, \( C_r \)-inequality, and (16) that

\[
II \leq \sum_{n=1}^{\infty} \left( \frac{\varepsilon}{2} \right)^r E \max_{1 \leq j \leq n} \left[ X_j^{(n)} \right]^r
\leq C \sum_{n=1}^{\infty} \left( \frac{\varepsilon}{2} \right)^r \frac{1}{a_n^q} \left[ n \left( \sum_{k=1}^{n} E X_{nk}^q \right)^{r/2} + \left( \sum_{k=1}^{n} E X_{nk}^q \right)^{r/2} \right]
\]

\[
\leq C \sum_{n=1}^{\infty} \left( \frac{\varepsilon}{2} \right)^r \frac{1}{a_n^q} \left[ n \left( \sum_{k=1}^{n} E X_{nk}^q \right)^{r/2} + \left( \sum_{k=1}^{n} E X_{nk}^q \right)^{r/2} \right]
\]

\[
\leq C \sum_{n=1}^{\infty} \left( \frac{\varepsilon}{2} \right)^r \frac{1}{a_n^q} \left[ n \left( \sum_{k=1}^{n} E X_{nk}^q \right)^{r/2} + \left( \sum_{k=1}^{n} E X_{nk}^q \right)^{r/2} \right]
\]

\[
\leq C \sum_{n=1}^{\infty} \left( \frac{\varepsilon}{2} \right)^r \frac{1}{a_n^q} \left[ n \left( \sum_{k=1}^{n} E X_{nk}^q \right)^{r/2} + \left( \sum_{k=1}^{n} E X_{nk}^q \right)^{r/2} \right]
\]

\[
\leq C \sum_{n=1}^{\infty} \left( \frac{\varepsilon}{2} \right)^r \frac{1}{a_n^q} \left[ n \left( \sum_{k=1}^{n} E X_{nk}^q \right)^{r/2} + \left( \sum_{k=1}^{n} E X_{nk}^q \right)^{r/2} \right]
\]

(24)

Next we prove that \( I_2 < \infty \). Denote \( Y_{nk} = X_{nk} I \{ |X_{nk}| \leq t^{1/q} \} \), \( Z_{nk} = X_{nk} - Y_{nk} \), and \( M_n(Y) = \max_{1 \leq j \leq n} \sum_{k=1}^{j} Y_{nk} \). Obviously,

\[
P \left( M_n(X) > t^{1/q} \right)
\leq \sum_{n=1}^{\infty} P \left( \max_{1 \leq j \leq n} \left| t_j^{(n)} \right| > \frac{\varepsilon}{2} \right) + P \left( M_n(Y) > t^{1/q} \right).
\]

(25)
Hence,
\[
I_2 \leq \sum_{n=1}^{\infty} \sum_{k=1}^{n} a_n^{-q} \int_{a_n^q}^\infty P \{ |X_{nk}| > t^{1/q} \} \, dt \\
+ \sum_{n=1}^{\infty} a_n^{-q} \int_{0}^{a_n^q} P \{ M_n(Y) > t^{1/q} \} \, dt \\
\leq I_3 + I_4.
\]

For \( I_3 \), by (15), we have
\[
I_3 = \sum_{n=1}^{\infty} \sum_{k=1}^{n} a_n^{-q} \int_{a_n^q}^\infty P \{ |X_{nk}| I(\{ |X_{nk}| > a_n \}) > t^{1/q} \} \, dt \\
\leq \sum_{n=1}^{\infty} \sum_{k=1}^{n} a_n^{-q} \int_{0}^{a_n^q} P \{ |X_{nk}| I(\{ |X_{nk}| > a_n \}) > t^{1/q} \} \, dt \\
= \sum_{n=1}^{\infty} \sum_{k=1}^{n} E |X_{nk}|^q I(\{ |X_{nk}| > a_n \}) \, dt < \infty.
\]

Now let us prove that \( I_4 < \infty \). Firstly, it follows by (6) and (15) that
\[
\max_{t \geq a_n^q} \frac{1}{t^{1/q}} \left| \sum_{k=1}^{n} E Y_{nk} \right| = \max_{t \geq a_n^q} \frac{1}{t^{1/q}} \left| \sum_{k=1}^{n} E Z_{nk} \right| \\
\leq \max_{t \geq a_n^q} t^{-1/q} \sum_{k=1}^{n} E |X_{nk}| I(\{ |X_{nk}| > a_n \}) \\
\leq \frac{\sum_{k=1}^{n} a_n^{-1} E |X_{nk}| I(\{ |X_{nk}| > a_n \})}{a_n^q} \\
\leq \frac{\sum_{k=1}^{n} E |X_{nk}|^q I(\{ |X_{nk}| > a_n \})}{a_n^q} \rightarrow 0 \quad \text{as} \quad n \rightarrow \infty.
\]

Therefore, for \( n \) sufficiently large,
\[
\max_{t \geq a_n^q} \frac{1}{t^{1/q}} \left| \sum_{k=1}^{n} E Y_{nk} \right| \leq \frac{t^{1/q}}{2}, \quad t \geq a_n^q.
\]

Then for \( n \) sufficiently large,
\[
P \{ M_n(Y) > t^{1/q} \} \\
\leq P \left\{ \max_{1 \leq j \leq n} \left( Y_{nk} - E Y_{nk} \right) > \frac{t^{1/q}}{2} \right\}, \quad t \geq a_n^q.
\]

Let \( d_n = \lfloor a_n \rfloor + 1 \). By (30), Lemma 3, and \( C_r \)-inequality, we can see that
\[
I_4 \leq C \sum_{n=1}^{\infty} \sum_{k=1}^{n} t^{-2/q} E \left\{ \left( \max_{1 \leq j \leq n} \sum_{k=1}^{n} (Y_{nk} - E Y_{nk}) \right)^2 \right\} \, dt \\
\leq C \sum_{n=1}^{\infty} \sum_{k=1}^{n} t^{-2/q} (E Y_{nk} - E Y_{nk})^2 \, dt \\
\leq C \sum_{n=1}^{\infty} \sum_{k=1}^{n} a_n^{-q} \int_{0}^{a_n^q} t^{-2/q} E Y_{nk}^2 \, dt \\
= C \sum_{n=1}^{\infty} \sum_{k=1}^{n} a_n^{-q} \int_{0}^{a_n^q} t^{-2/q} E Y_{nk}^2 \, dt \\
+ C \sum_{n=1}^{\infty} \sum_{k=1}^{n} a_n^{-q} \int_{a_n^q}^{\infty} t^{-2/q} E Y_{nk}^2 \, dt \\
\leq I_{41} + I_{42}.
\]

For \( I_{41} \), since \( q < 2 \), we have
\[
I_{41} = \sum_{n=1}^{\infty} \sum_{k=1}^{n} a_n^{-q} E X_{nk}^2 I(\{ X_{nk} \leq a_n \}) \int_{a_n^q}^{\infty} t^{-2/q} \, dt \\
\leq C \sum_{n=1}^{\infty} \sum_{k=1}^{n} E X_{nk}^2 I(\{ X_{nk} \leq a_n \}) \frac{a_n^q}{a_n^q} \\
= C \sum_{n=1}^{\infty} \sum_{k=1}^{n} E X_{nk}^2 I(\{ X_{nk} \leq a_n \}) \\
+ C \sum_{n=1}^{\infty} \sum_{k=1}^{n} E X_{nk}^2 I(\{ a_n < X_{nk} \leq d_n \}) \\
\leq I_{41} + I_{42}.
\]

Since \( p \leq 2 \), by (16), it implies \( I_{41}^p < \infty \). Now we prove that \( I_{41}^p < \infty \). Since \( q < 2 \) and \( (a_n + 1)/a_n \to 1 \) as \( n \to \infty \), by (15) we have
\[
I_{41}^p \leq C \sum_{n=1}^{\infty} \sum_{k=1}^{n} \frac{a_n^{-q}}{a_n^q} E |X_{nk}|^q I(\{ X_{nk} \leq a_n \}) \\
\leq C \sum_{n=1}^{\infty} \sum_{k=1}^{n} \left( \frac{a_n + 1}{a_n} \right)^{-q} E |X_{nk}|^q I(\{ X_{nk} \leq a_n \}) \\
\leq C \sum_{n=1}^{\infty} \sum_{k=1}^{n} E |X_{nk}|^q I(\{ X_{nk} \leq a_n \}) < \infty.
\]
Let $t = u^q$ in $I_{42}$. Note that, for $q < 2$,

$$\int_{d_n}^{\infty} u^{q-3} E X_{nk}^2 I \left( |X_{nk}| < u \right) du$$

$$= \int_{d_n}^{\infty} u^{q-3} E X_{nk}^2 I \left( |X_{nk}| > d_n \right) \cdot I \left( |X_{nk}| < u \right) du$$

$$= E \left[ X_{nk}^2 I \left( |X_{nk}| > d_n \right) \right] \int_{d_n}^{\infty} u^{q-3} I \left( |X_{nk}| < u \right) du$$

$$= E \left[ X_{nk}^2 I \left( |X_{nk}| > d_n \right) \right] \int_{d_n}^{\infty} u^{q-3} du$$

$$\leq CE |X_{nk}|^q I \left( |X_{nk}| > d_n \right).$$

Then by (15) and $d_n > a_n$, we have

$$I_{42} = C \sum_{n=1}^{\infty} \sum_{k=1}^{n} a_n^{-q} \int_{d_n}^{\infty} u^{q-3} E X_{nk}^2 I \left( |X_{nk}| < u \right) du$$

$$\leq C \sum_{n=1}^{\infty} \sum_{k=1}^{n} a_n^{-q} E |X_{nk}|^q I \left( |X_{nk}| > a_n \right) < \infty.$$ (35)

This completes the proof of Theorem 4. \qed

**Theorem 5.** Let $\{X_{nk}, k \geq 1, n \geq 1\}$ be an array of rowwise $\varphi$-mixing random variables satisfying $\sup_{n \geq 1} \sum_{k=1}^{n} \varphi_{nk}^{1/2} < \infty$ and let $\{a_n, n \geq 1\}$ be a sequence of positive real numbers such that $a_n \uparrow \infty$. Also, let $\varphi_k(t), k \geq 1$ be a positive even function satisfying (5) for $1 \leq q < p$ and $p > 2$. Then conditions (6)-(8) imply (14).

**Proof.** Following the notation, by a similar argument as in the proof of Theorem 4, we can easily prove that $I_1 < \infty$, $I_3 < \infty$ and that (19) and (20) hold. To complete the proof, we only need to prove that $I_4 < \infty$.

Let $\delta \geq p$ and $d_n = [a_n] + 1$. By (30), Markov inequality, Lemma 3, and the $C_r$-inequality we can get

$$I_4 \leq C \sum_{n=1}^{\infty} d_n^{-q} \int_{d_n}^{\infty} t^{-\delta/q} E \left[ \max_{1 \leq j \leq n} \sum_{k=1}^{j} \left( Y_{nk} - E Y_{nk} \right) \right] dt$$

$$\leq C \sum_{n=1}^{\infty} d_n^{-q} \int_{d_n}^{\infty} t^{-\delta/q} \left[ \sum_{k=1}^{n} E |Y_{nk}|^\delta + \left( \sum_{k=1}^{n} E Y_{nk}^2 \right)^{\delta/2} \right] dt$$

$$= C \sum_{n=1}^{\infty} \sum_{k=1}^{n} a_n^{-q} \int_{d_n}^{\infty} t^{-\delta/q} E |Y_{nk}|^\delta dt$$

$$+ C \sum_{n=1}^{\infty} a_n^{-q} \int_{d_n}^{\infty} t^{-\delta/q} \left( \sum_{k=1}^{n} E Y_{nk}^2 \right)^{\delta/2} dt$$

$$\leq I_{43} + I_{44}.$$ (36)

For $I_{43}$, we have

$$I_{43} = C \sum_{n=1}^{\infty} \sum_{k=1}^{n} a_n^{-q} \int_{d_n}^{\infty} t^{-\delta/q} E |X_{nk}|^\delta I \left( |X_{nk}| \leq d_n \right) dt$$

$$+ C \sum_{n=1}^{\infty} a_n^{-q} \int_{d_n}^{\infty} t^{-\delta/q} E |X_{nk}|^\delta I \left( |X_{nk}| \leq t^{1/q} \right) dt$$

$$\leq I'_{43} + I''_{43}.$$ (37)

By a similar argument as in the proof of $I_{41} < \infty$ and $I_{42} < \infty$ (replacing the exponent 2 by $\delta$), we can get $I_{43} < \infty$ and $I''_{43} < \infty$.

For $I_{44}$, since $\delta > 2$, we can see that

$$I_{44} = C \sum_{n=1}^{\infty} \sum_{k=1}^{n} a_n^{-q} \int_{d_n}^{\infty} t^{-\delta/q} \left( \sum_{k=1}^{n} E X_{nk}^2 I \left( |X_{nk}| \leq a_n \right) \right) dt$$

$$+ C \sum_{n=1}^{\infty} a_n^{-q} \int_{d_n}^{\infty} t^{-\delta/q} \left( \sum_{k=1}^{n} E X_{nk}^2 I \left( a_n < |X_{nk}| \leq t^{1/q} \right) \right) dt$$

$$\leq I'_{44} + I''_{44}.$$ (38)

Since $\delta > p > q$, from (8) we have

$$I'_{44} = C \sum_{n=1}^{\infty} a_n^{-q} \left( \sum_{k=1}^{n} E X_{nk}^2 I \left( |X_{nk}| \leq a_n \right) \right) \int_{d_n}^{\infty} t^{-\delta/q} dt$$

$$\leq C \sum_{n=1}^{\infty} \left( \sum_{k=1}^{n} E X_{nk}^2 I \left( |X_{nk}| \leq a_n \right) \right)^{\delta/2}$$

$$\leq C \sum_{n=1}^{\infty} \left( \frac{E X_{nk}^2 I \left( |X_{nk}| \leq a_n \right)}{a_n} \right)^{\delta/2} < \infty.$$ (39)

Next we prove that $I''_{44} < \infty$. To start with, we consider the case $1 \leq q \leq 2$. Since $\delta > 2$, by (15), we have

$$I''_{44} = C \sum_{n=1}^{\infty} \sum_{k=1}^{n} a_n^{-q} \int_{d_n}^{\infty} t^{-\delta/q} \left( \sum_{k=1}^{n} E X_{nk}^2 I \left( a_n < |X_{nk}| \leq t^{1/q} \right) \right)^{\delta/2} dt$$

$$\leq C \sum_{n=1}^{\infty} a_n^{-q} \int_{d_n}^{\infty} t^{-\delta/q} \left( \sum_{k=1}^{n} E X_{nk}^2 I \left( |X_{nk}| > a_n \right) \right)^{\delta/2} dt$$
Finally, we prove that $I''_{4\epsilon} < C$ in the case $2 < q < p$. Since $\delta > q$ and $\delta > 2$, we have by (15) that

$$
I''_{4\epsilon} \leq C \sum_{n=1}^{\infty} \int_{\delta/2}^{\infty} \left( \sum_{k=1}^{n} E\left[ |X_{n,k}|^q \right] \right)^{\delta/2} dt
$$

Combining Theorem 5 and (45)–(47), we can prove Corollary 6 immediately.

**Remark 7.** Noting that in this paper we consider the case $1 \leq q \leq p$, which has a more wide scope than the case $q = 1$, $p > 2$ in Gan et al. [14]. In addition, compared with $\varphi$-mixing random variables, the arrays of $\varphi$-mixing random variables not only have many related properties, but also have a wide range of application. So it is very significant to study it.

**Remark 8.** Under the condition of Theorem 4, we have

$$
\lim_{n \to \infty} \sum_{n=1}^{\infty} a_n^{-q} E \left\{ \max_{1 \leq j \leq n} \sum_{k=1}^{j} |X_{n,k}| - \epsilon a_n \right\}^q = C
$$

Thus we get the desired result immediately. The proof is completed.

**Corollary 6.** Let $\{X_{n,k}, k \geq 1, n \geq 1\}$ be an array of row-wise $\varphi$-mixing mean zero random variables with $\sup_{n \geq 1} \Psi_{1/2}(k) < \infty$, $q \geq 1$. If, for some $\alpha > 0$ and $v \geq 2$,

$$
\max_{1 \leq j \leq n} E\left| X_{n,j} \right|^\alpha = O(n^\alpha),
$$

where $v/\alpha \geq \max\{v,2\}$, $v \geq 2$, then, for any $\epsilon > 0$,

$$
\sum_{n=1}^{\infty} \epsilon^{-1} E \left\{ \max_{1 \leq j \leq n} \sum_{k=1}^{j} |X_{n,k}| - \epsilon n^{1/2} \right\}^q < \infty.
$$

**Proof.** Put $\Psi_k(t) = |t|^v \Psi(t)^{1/2}$, $p = v + \delta$, $\delta > 0$, and $a_n = n^{1/2}$. Since $v \geq 2$, $(v/\alpha - \alpha > \max\{v,2\})$ and $a_n = n^{1/2}$, then

$$
\Psi_k(t) = \frac{|t|^v \Psi(t)^{1/2}}{|t|^p} = \frac{|t|^v}{|t|^p} \downarrow \text{ as } |t| \uparrow \infty.
$$

It follows by (42) and $(v/\alpha - \alpha > 2)$ that

$$
\sum_{n=1}^{\infty} E\left( \frac{E\left| X_{n,k} \right|^v}{n^{1/2} \Psi_k(a_n)} \right) = \sum_{n=1}^{\infty} E\left( \frac{\left| X_{n,k} \right|^v}{n^{1/2} \Psi_k(a_n)} \right) \leq C \sum_{n=1}^{\infty} \frac{1}{n^{(v/\alpha - \alpha - 1)} \Psi_k(a_n)} < \infty.
$$

Since $v \geq 2$, by Jensen's inequality it follows that

Clearly $2/q - (2\alpha/v) \geq 1 > 0$. Take $s > p$ such that $(s/2)(2/q - (2\alpha/v) - 1) > 1$. Therefore,

$$
\sum_{n=1}^{\infty} \left( \sum_{k=1}^{n} E\left| X_{n,k} \right|^2 / n^{2s/q} \right)^{s/2} \leq C \sum_{n=1}^{\infty} \frac{1}{n^{(s/2)/2\alpha/v - 1}}.
$$

References


