Research Article

Conservation Laws, Symmetry Reductions, and New Exact Solutions of the (2 + 1)-Dimensional Kadomtsev-Petviashvili Equation with Time-Dependent Coefficients

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The (2 + 1)-dimensional Kadomtsev-Petviashvili equation with time-dependent coefficients is investigated. By means of the Lie group method, we first obtain several geometric symmetries for the equation in terms of coefficient functions and arbitrary functions of \( t \). Based on the obtained symmetries, many nontrivial and time-dependent conservation laws for the equation are obtained with the help of Ibragimov’s new conservation theorem. Applying the characteristic equations of the obtained symmetries, the (2 + 1)-dimensional KP equation is reduced to (1 + 1)-dimensional nonlinear partial differential equations, including a special case of (2 + 1)-dimensional Boussinesq equation and different types of the KdV equation. At the same time, many new exact solutions are derived such as soliton and soliton-like solutions and algebraically explicit analytical solutions.

1. Introduction

The Lie group method is a powerful tool to perform Lie symmetry analysis, study conservation laws, and look for exact solutions of nonlinear partial differential equations (NLPDEs) [1–4]. The notion of conservation laws, which plays an important role in the study of nonlinear science, is used for the development of appropriate numerical methods and for mathematical analysis, in particular, existence, uniqueness, and stability analysis [5, 6]. In addition, the existence of a large number of conservation laws of a partial differential equation (system) is a strong indication of its integrability. On the other hand, seeking exact solutions of NLPDEs has become one central theme of perpetual interest in mathematical physics as explicit solutions will be helpful to better understand the phenomena described by the equations. To get exact solutions of NLPDEs, many effective methods have been presented such as inverse scattering method [7], Hirota’s bilinear method [8], and Painlevé expansion method [9]. Among them the Lie group method offers a systematic algorithmic procedure to find the symmetry reductions and exact solutions of a partial differential equation. In this paper, we use the Lie group method to consider a time-dependent Kadomtsev-Petviashvili equation:

\[
E_1 \equiv u_{tt} + 6u_x^2 + 6uu_{xx} + u_{xxxx} + e(t)u_x + n(t)u_{yy} = 0, \tag{1}
\]

with time-dependent coefficient functions \( e(t), n(t) \), and \( n(t) \neq 0 \).

The above equation was also called “a 2D KdV equation with time-dependent coefficients” by Hereman and Zhuang [10]; they performed Painlevé analysis for (1) and found that (1) was Painlevé integrable when \( e_t + 2e^2 = 0, n_t + 4n e = 0 \). Equation (1) can be reduced to the KdV equation \( e(t) = 0, n(t) = 0 \) or the KP equation \( e(t) = 0, n(t) = \pm 1 \). Equation (1) can also be reduced to the cylindrical KdV equation

\[
u_t + 6uu_x + u_{xxx} + \frac{1}{2t}u = 0, \tag{2a}
\]

when \( e(t) = 1/2t, n(t) = 0 \) or the cylindrical KP equation

\[
u_{xx} + 6u_x^2 + 6uu_{xx} + u_{xxxx} + \frac{1}{2t}u_x + \frac{3}{t^2}u_{yy} = 0, \tag{2b}
\]
when \( e(t) = 1/2t \), \( n(t) = \pm 3/t^2 \). The KdV and KP equations and their cylindrical generalizations (2a) and (2b) are all known to be completely integrable [10]. Zhang et al. [11] performed Painlevé analysis for (1) and constructed bilinear auto-Bäcklund, analytic solutions in the Wronskian form. Soliton-like solutions, Jacobi elliptic function-like solutions, and other exact solutions have been obtained by the method of auxiliary equations [12–15]. Elwakil et al. [16] used the homogeneous balance method to study the exact solutions of (1). Based on the homogeneous balance method and Clarkson-Kruskal method, direct reduction and exact solutions have been obtained in [17] by Moussa and El-Shiekh. The bilinear formalism, bilinear Bäcklund transformation, and other exact solutions have been obtained by the method of polynomial approach in [18]. As far as we know, conservation laws and symmetry reductions for (1) have not been studied.

The rest of the paper is organized as follows. In Section 2, the Lie group method is applied to the time-dependent Kadomtsev-Petviashvili equation (1) and thus Lie symmetries of (1) are obtained. In Section 3, using the obtained symmetries and the general theorem on conservation laws by Ibragimov, nontrivial and time-dependent conservation laws are derived. In Section 4, we use the symmetry to get symmetry reductions and new exact solutions of (1). The last section is a short summary and discussion.

2. Lie Symmetry Analysis of (1)

Generally speaking, Lie symmetry denotes a transformation that leaves the solution manifold of a system invariant; that is, it maps any solution of the system into a solution of the same system, so it is also called geometric symmetry. In this section, we will perform Lie symmetry analysis for (1) by the classical Lie group method. Suppose that Lie symmetry of (1) is expressed as follows:

\[
V = \xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y} + \tau \frac{\partial}{\partial t} + \phi \frac{\partial}{\partial u},
\]

where \( \xi, \eta, \tau, \) and \( \phi \) are undetermined functions with respect to \( x, y, t, \) and \( u \). According to the procedures of Lie group method, the vector field (3) can be determined by applying the fourth prolongation of \( V \) to (1) and thus the undetermined functions \( \xi, \eta, \tau, \) and \( \phi \) must satisfy the following invariant condition:

\[
\phi^{xx} + 12u_x \phi^x + 6u_x \phi + 6u \phi^{xx} + \phi^{xxxx} + e'(t) \tau u_x + e(t) \phi^x + n'(t) \tau u_{yy} + n(t) \phi^{yy} = 0,
\]

where

\[
\phi^x = D_x \left( \phi - \xi u_x - \eta u_y - \tau u_t \right) + \xi u_{xx} + \eta u_{xy} + \tau u_{xt},
\]

\[
\phi^{xx} = D_{xx} \left( \phi - \xi u_x - \eta u_y - \tau u_t \right) + \xi u_{xxx} + \eta u_{xxy} + \tau u_{xxt},
\]

\[
\phi^{yyyy} = D_{yyyy} \left( \phi - \xi u_x - \eta u_y - \tau u_t \right) + \xi u_{yyyy} + \eta u_{yxy} + \tau u_{yyt},
\]

\[
\phi^{xxxx} = D_{xxxx} \left( \phi - \xi u_x - \eta u_y - \tau u_t \right) + \xi u_{xxxx} + \eta u_{xxyy} + \tau u_{xxxx},
\]

Substituting (5) into (4) with \( u \) being a solution of (1), that is, \( u_{xxxx} = -u_{x} + 6u_{xx} - 6uu_{xx} - e(t)u_x - n(t)u_{yy} \), we obtain the determining equations of symmetry (3). Solving the determining equations with the aid of Maple, we can get the following cases.

Case 1. When \( e(t) \) and \( n(t) \) are arbitrary functions,

\[
\xi = -\frac{g_1 y}{2n(t)} + f(t), \quad \eta = g(t), \quad \tau = 0,
\]

where \( f(t) \) and \( g(t) \) are arbitrary functions. It shows that (1) admits an infinite-dimensional Lie algebra of symmetries

\[
V = V_f + V_g,
\]

where

\[
V_f = f(t) \frac{\partial}{\partial x} + \frac{g_1}{6} \frac{\partial}{\partial u},
\]

\[
V_g = -\frac{g_1 y}{2n(t)} \frac{\partial}{\partial x} + g(t) \frac{\partial}{\partial y} + \left( \frac{g_1 n_1}{12n(t)} y - \frac{g_1}{12n(t)} y \right) \frac{\partial}{\partial u}.
\]

Case 2. When \( e(t) = 0, n(t) = -(t - m)^p C_1, \quad p \neq 0, C_1 \neq 0, \quad C_2 \neq 0, \)

\[
\xi = \frac{C_2 x}{3p} - \frac{g_1 y}{2C_1(t - m)^p} + f(t), \quad \eta = \frac{-2C_2}{3p} - \frac{C_2}{2} y + g(t), \quad \tau = \frac{C_2}{p} (t - m),
\]

\[
\phi = \frac{2C_2}{3p} u + \frac{g_1}{12C_1(t - m)^p} y + \frac{g_1 n_1}{12C_1(t - m)^p} y + \frac{f_1}{6},
\]

where \( m, p, C_1, \) and \( C_2 \) are constants and \( f(t) \) and \( g(t) \) are arbitrary functions. This shows that the symmetries of equation

\[
u_{xt} + 6u_{x} + 6uu_{xx} + uu_{xxx} + C_1(t - m)^p u_{yy} = 0
\]
have the form of
\[ V = V_1 + V_f + V_g, \]  
(12)

where
\[ V_1 = \frac{x}{3p} \frac{\partial}{\partial x} + \left( \frac{2}{3p} + \frac{1}{2} \right) y \frac{\partial}{\partial y} + \frac{(t-m)}{p} \frac{\partial}{\partial t} - \frac{2}{3p} u \frac{\partial}{\partial u}, \]  
(13)

is a one-dimensional Lie algebra of symmetries and \( V_f \) and \( V_g \) are two infinite-dimensional Lie algebra of symmetries as expressed by (9) with \( n(t) = (t-m)^2 C_1 \).

Case 3. When \( e(t) = 0, n(t) = \text{Const.}, \) and \( \tau(t) \neq 0, \)
\[ \xi = \frac{r_1}{3} x - \frac{r_1}{6n} y^2 - \frac{g_1}{2n} y + f(t), \]
\[ \eta = \frac{2}{3} \tau y + g(t), \quad \tau = \tau(t), \]  
(14)
\[ \phi = -\frac{2r_1}{3} u + \frac{r_1}{18} x - \frac{r_1}{36n} y^2 - \frac{g_1}{12n} y + \frac{f(t)}{6}, \]
where \( f(t) \) and \( g(t) \) are arbitrary functions. It shows that the KP equation
\[ u_{xt} + 6u_x^2 + 6uu_{xx} + u_{xxxx} + Cu_{yy} = 0 \]  
(15)

admits an infinite-dimensional Lie algebra of symmetries
\[ V = V_f + V_g + V_r, \]  
(16)
where \( C \) is a constant and \( C \neq 0; V_f \) and \( V_g \) are expressed by (9) with \( n(t) = \text{Const.}, \)
\[ V_r = \left( \frac{r_1}{3} x - \frac{r_1}{6n} y^2 \right) \frac{\partial}{\partial x} + \frac{2}{3} \tau y \frac{\partial}{\partial y} + \tau \frac{\partial}{\partial t} \]  
(17)
\[ + \left( -\frac{2r_1}{3} u + \frac{r_1}{18} x - \frac{r_1}{36n} y^2 \right) \frac{\partial}{\partial u}. \]

Case 4. When \( e(t) = -n_i/4n + C_3/\tau(t), \) \( \tau(t) \neq 0, \) and \( n_i \neq 0, \)
\[ \xi = \frac{r_1}{3} x - \frac{r_1}{6n(t)} y^2 - \frac{g_1}{2n(t)} y - \frac{\tau r_1 n_i}{8n^2(t)} y^2 \]  
\[ - \frac{\tau(t) n_i}{8n^2(t)} y^2 + \frac{\tau(t) n_i}{8n^2(t)} y^2 + f(t), \]
\[ \eta = \left( \frac{\tau(t) n_i}{2n(t)} + \frac{2}{3} \tau_i \right) y + g(t), \]
\[ \phi = -\frac{2r_1}{3} u + \frac{r_1}{18} x + \frac{\tau(t) n_i n_j}{12n^2(t)} y^2 \]  
\[ - \frac{\tau(t) n_i^3}{48n^2(t)} y^2 + \frac{\tau(t) n_i^3}{16n^2(t)} y^2 + \frac{\tau r_1 n_i}{144n^2(t)} y^2 \]  
\[ - \frac{\tau r_1 n_i}{36n(t)} y^2 + \frac{\tau r_1 n_i}{24n^2(t)} y^2 \]  
\[ + \frac{f(t)}{6} - \frac{g_i}{12n(t)} y + \frac{g_i n_i}{12n^2(t)} y, \]  
(18)

where \( f(t) \) and \( g(t) \) are arbitrary functions, \( C_3 \) is an integral constant, and \( n(t) \) and \( \tau(t) \) satisfy the following ordinary differential equation:
\[ n_{ttt} + \frac{2n_i t_i}{\tau(t)} + \frac{3n_i^2}{n(t) \tau(t)} = \frac{3n_i^3}{n^2(t)} \]  
(19)
This shows that, under the condition (19), the equation
\[ u_{xt} + 6u_x^2 + 6uu_{xx} + u_{xxxx} + Cu_{yy} = 0 \]  
(20)

admits an infinite-dimensional Lie algebra of symmetries
\[ V = V_f + V_g + V_r, \]  
(21)
where \( V_f \) and \( V_g \) are expressed by (9):
\[ V_r = \left( \frac{r_1}{3} x - \frac{r_1}{6n(t)} y^2 \right) \frac{\partial}{\partial x} + \frac{2}{3} \tau y \frac{\partial}{\partial y} + \tau \frac{\partial}{\partial t} \]  
(22)
\[ + \left( -\frac{2r_1}{3} u + \frac{r_1}{18} x - \frac{r_1}{36n(t)} y^2 \right) \frac{\partial}{\partial u}. \]

3. Conservation Laws for (1)

3.1. A General Theorem on Conservation Laws. As expressed through the famous Noether theorem, for a given differential equation, there is a close connection between Lie symmetries and conservation laws. To derive conservation laws of (1), we use the following conclusion proved by Ibragimov in [19].
Theorem 1. Every Lie point, Lie-Bäcklund, and nonlocal symmetry

\[ V = \xi^i (x, u, u_{(1)}, \ldots) \frac{\partial}{\partial x^i} + \eta^j (x, u, u_{(1)}, \ldots) \frac{\partial}{\partial u^j} \]  

(23)
of a system of \( m \) equations

\[ F_s (x, u, u_{(1)}, \ldots, u_{(N)}) = 0, \quad s = 1, \ldots, m, \]  

(24)

with \( n \) independent variables \( x = (x^1, \ldots, x^n) \) and \( m \) dependent variables; \( u = (u^1, \ldots, u^m) \) provides a conservation law for system (24) and the corresponding adjoint system

\[ F^*_s (x, u, v, v_{(1)}, \ldots, u_{(N)}, v_{(N)}) = \frac{\delta (\mathbf{j} E^*_s)}{\delta u^s} = 0, \quad s = 1, \ldots, m. \]  

(25)

Then the elements of the conservation vector \( T = (T^1, \ldots, T^n) \) are defined by the following expression:

\[ T^i = \xi^i L + W^i \]

\[ \times \left[ \frac{\partial L}{\partial u^i_t} - D_x^i \left( \frac{\partial L}{\partial u^i_t} \right) + D_x^i D_x^k \left( \frac{\partial L}{\partial u^i_{jk}} \right) - \cdots \right] + D_x^i (W^i) \]

\[ \times \left[ \frac{\partial L}{\partial u^i_{ij}} - D_x^i \left( \frac{\partial L}{\partial u^i_{ij}} \right) + D_x^i D_x^j (W^i) \right] \]

\[ + D_x^i D_x^j (W^i) \left[ \frac{\partial L}{\partial u^i_{jk}} - D_x^i \left( \frac{\partial L}{\partial u^i_{jk}} \right) \right] + \cdots, \]  

(26)

with

\[ W^i = \eta^i - \xi^i u^i, \quad s = 1, \ldots, m. \]  

(27)

3.2. Conservation Laws for (1). To search for conservation laws of (1) by Theorem 1, adjoint equation and formal Lagrangian of (1) must be known. We first construct its adjoint equation. Following the idea in [19], the adjoint equation of (1) is

\[ E^*_1 \equiv v_{xt} + 6u v_{xx} + v_{xxxx} - e (t) v_x + n (t) v_{yy} = 0, \]  

(28)

where \( v \) is a new dependent variable with respect to \( x, y, \) and \( t. \)

According to the method of constructing Lagrangian in [19], the formal Lagrangian for the system consisting of (1) and (28) is

\[ L = v (u_{xt} + 6u^2 + 6u u_{xx} + u_{xxxx} + e (t) u_x + n (t) u_{yy}). \]  

(29)

Theorem 1. However, we are only interested in the conservation laws of (1). Therefore one has to eliminate the nonlocal variable \( v \) which is introduced in the adjoint equation. To solve this problem, the concepts of self-adjointness, quasi-self-adjointness, and nonlinear self-adjointness are developed [20–24]. In the following, we will discuss the adjointness and nonlinear adjointness using these definitions.

Equation (1) is said to be self-adjoint if the equation obtained from the adjoint equation (28) by the substitution \( v = u \) is identical with the original equation (1). It is easy to see that (28) is not identical with (1) when \( v = u \), so (1) is not a self-adjoint equation. According to the definition of nonlinear self-adjointness [24], (1) is said to be nonlinearly self-adjoint if its adjoint equation (28) is satisfied for all solutions \( u \) of (1) upon a substitution

\[ v = H (x, y, t, u), \quad H (x, y, t, u) \neq 0. \]  

(30)

In other words, (1) is nonlinearly self-adjoint if and only if

\[ E^*_1 |_{v=H(x,y,t,u)} = \lambda (x, y, t, u) E^*_1, \]  

(31)

where \( \lambda \) is an undetermined and smooth function.

From (31), we can get the following equation:

\[ (H_u - \lambda) u_{xxxx} + n (t) (H_u - \lambda) u_{yy} + (H_u - \lambda) u_{xt} + 4H_{ux} u_x u_{xxx} + 4H_{xu} u_{xxx} + 2n (t) H_{ux} u_y + u_x^2 (6u H_u + 6H_{uuu} + 6a H_{uu}) \]

\[ + u (12u H_{ux} + 6H_{ux} - 6a u_x + 6H_{xu}) + H_{uu} u_x - \lambda e (t) u_x \]  

\[ + e (t) u_x H_u + n (t) u^2 H_{ux}, \]  

\[ + 12H_{xuu} u_x u_{xx} + u_x u_t H_u + 6H_{xuu} u_{xx} + 4H_{xuu} u_x + H_{xuu} u_t + 4H_{xuu} u^3 + 3H_{xuu} u_{xx}^2 + H_{uu} u_{xx} u_x \]  

\[ + (H_x + n (t) H_{yy} + H_{xt} + H_{xxxx}) = 0. \]  

(32)

Solving the above system with the aid of Maple, the final results read as

\[ \lambda = 0, \]  

(33)

\[ H = (a (t) y + b (t)) x - a y^3 \frac{1}{6n (t)} - b y^2 \frac{1}{2n (t)} \]  

\[ + e (t) a (t) y^3 \frac{1}{6n (t)} \frac{e (t) b (t) y^2}{2n (t)} + k (t) y + l (t), \]  

(34)

where \( a(t), b(t), k(t), \) and \( l(t) \) are arbitrary functions. In summary, we have the following statements.

Theorem 2. The time-dependent KP equation (1) is nonlinearly self-adjoint.

In the following, we first construct the conservation laws for the system consisting of the initial equation (1) and its adjoint (28).
For the symmetry in Case 1, the corresponding components of the conservation laws are

\[
X_1 = f(t) u_{x} v_{t} + f_{1} u_{x} v - g(t) u_{xxx} v - g(t) u_{x} v_{xx}
\]

\[
- f_{1} v_{2} u + f(t) u_{x} v + g(t) u_{y} v_{xx} + \frac{f_{2} v(t)}{6} v
\]

\[
+ g(t) u_{x} x v_{x} + g(t) u_{y} v_{t}
\]

\[
+ f(t) u_{x} v_{xxx} + f(t) v_{u} u_{x}
\]

\[
- f(t) u_{x} v_{xx} + \frac{g_{y} y u_{xx} v_{x}}{2n(t)} - \frac{1}{6} f_{1} v_{xx} - \frac{g_{y} y u_{yy} v}{2}
\]

\[
+ f(t) n(t) u_{y} v + \frac{g_{y} y v_{t}}{12n(t)} + \frac{g_{y} y v_{xxx}}{2n(t)} + 6f(t) u_{x} v_{u} u
\]

\[
- g(t) e(t) u_{x} v - 6g(t) u_{x} u_{x} v + 6g(t) u_{y} v_{u}
\]

\[
- 6g(t) u_{y} x u_{y} u - \frac{1}{6} f_{1} v_{t} - \frac{g_{y} y v_{y} t}{12n(t)} - \frac{g_{y} y u_{xxx} v_{y}}{12n(t)}
\]

\[
- \frac{g_{y} y u_{xx} v_{x}}{2n(t)} - \frac{g_{y} y u_{yyyy} v}{2n(t)} - \frac{g_{y} y u_{xx} v_{xx}}{12n(t)}
\]

\[
+ \frac{g_{y} y u_{xx} x u_{xx}}{12n(t)} - \frac{g_{y} y u_{xxx} x v_{x}}{2n(t)}
\]

\[
+ \frac{g_{y} y u_{xxx} x y}{2n(t)} - \frac{g_{y} y u_{xxx} x x}{2n(t)}
\]

\[
+ \frac{g_{y} y u_{xxx} x x}{2n(t)}
\]

\[
T_1 = \frac{g_{y} y u_{xxx} v}{2n(t)} - f(t) u_{x} v_{x} - g(t) u_{y} x v.
\]

(35)

For the symmetry in Case 2, the corresponding components of the conservation laws are

\[
X_2 = - \frac{6C_{2}m_{u}v_{u}}{p} - \frac{1}{6} f_{1} v_{xxx} - \frac{1}{2} f_{1} v_{t} + \frac{g_{y} y}{2n(t)} u_{x} v_{xx}
\]

\[
+ \frac{C_{2}m_{u}u_{xx} x v}{p} - \frac{g_{y} y}{2n(t)} u_{x} v + \frac{g_{y} y}{12n(t)} v_{t} + \frac{g_{y} y}{12n(t)} v_{xxx}
\]

\[
+ \frac{C_{2} x}{3p} u_{x} v_{t} + \frac{C_{2} x}{3p} u_{x} v_{xxx} - \frac{3g_{y} y}{n(t)} u_{x} v_{u} - \frac{g_{y} y}{2n(t)} u_{x} v_{t}
\]

\[
+ \frac{4C_{2} x y}{2} u_{y} v_{u} + \frac{2C_{2} x y}{3p} u_{x} v_{t} + \frac{2C_{2} x y}{3p} u_{x} v_{xxx}
\]

\[
+ \frac{2C_{2} x y}{3p} u_{x} v_{u} + 2C_{2} x y u_{x} v_{t} + \frac{2C_{2} x y}{3p} u_{x} v_{xxx}
\]

\[
+ \frac{4C_{2} y}{2} u_{y} v_{u} + \frac{2C_{2} y}{3p} u_{x} v_{t} + \frac{2C_{2} y}{3p} u_{x} v_{xxx}
\]

\[
+ \frac{C_{2} y}{2} u_{y} v_{u} + \frac{4C_{2} y}{3p} u_{x} v_{t} - \frac{C_{2} m_{u} x v_{u}}{p}
\]

\[
+ \frac{C_{2} y}{2} u_{y} v_{u} + \frac{6C_{2} y}{p} u_{y} v_{u} + \frac{C_{2} y}{p} u_{y} v_{t} + \frac{C_{2} y}{p} u_{y} v_{xxx} + 6 \frac{C_{2} m_{u} u_{y} v}{p}
\]

\[
- \frac{C_{2} m_{u} v_{xxx}}{p} - \frac{C_{2} x}{3p} u_{x} v_{xx} - \frac{2C_{2} y}{3p} u_{x} v_{y}
\]

\[
- \frac{3C_{2} y u_{y} v_{u} + 3C_{2} y u_{y} v_{u} - \frac{6C_{2} y}{p} u_{y} v_{u}}{p}
\]

\[
+ \frac{C_{2} x}{p} u_{x} v_{u} + \frac{C_{2} x}{p} u_{x} v_{t} + \frac{C_{2} x}{p} u_{x} v_{xxx} + \frac{6C_{2} m u_{x} v}{p}
\]

\[
- \frac{C_{2} m u_{xxx}}{p} - \frac{2C_{2} y}{3p} u_{x} v_{xx}
\]

\[
- \frac{3C_{2} y u_{y} u_{x} v}{p} - \frac{C_{2} y}{p} u_{x} v_{x}
\]

\[
- \frac{g_{y} y p}{12n(t)(t - m)} v_{t} - \frac{10C_{2} u u_{y} v}{p} - \frac{4C_{2} y u_{y} u_{y} v}{p}
\]

\[
+ C_{2} x u_{x} v_{x} + C_{2} x n(t) u_{x} v_{y} - \frac{g_{y} y}{2n(t)} u_{x} v_{u}
\]

\[
+ f(t) n(t) u_{y} v + \frac{g_{y} y p}{3p} v_{y} u + \frac{g_{y} y}{2n(t) - m} u_{x} v
\]

\[
- \frac{g_{y} y p}{2n(t) - m} v_{u} + \frac{g_{y} y}{2n(t)} v_{u} u
\]

\[
- \frac{g_{y} y}{2n(t) - m} u_{x} v_{xxx} + \frac{C_{2} m}{p} u_{x} v_{xx}
\]

\[
+ \frac{C_{2} m}{p} u_{x} v_{xx} + \frac{C_{2} y}{3p} u_{x} v_{xxx} - \frac{g_{y} y}{2n(t) - m} u_{xx} v_{x}
\]

\[
+ \frac{2C_{2} y}{3p} u_{x} v_{xx} v_{x} + \frac{C_{2} y}{3p} u_{x} v_{xxx} - \frac{2C_{2} y u_{y} u_{x} v}{3p}
\]

\[
- \frac{C_{2} y}{3p} u_{x} v_{xx} v_{x} + \frac{C_{2} y}{3p} u_{x} v_{xxx} - \frac{g_{y} y}{2n(t) - m} u_{x} v_{u}
\]

\[
+ \frac{4C_{2} y}{2} u_{y} v_{u} - \frac{g_{y} y}{2} u_{y} v - \frac{C_{2} y u_{x} v_{x}}{p} + \frac{2C_{2} x y u_{x} v}{3p}
\]

\[
+ \frac{4C_{2} y u_{y} u_{x} v}{p} - \frac{C_{2} m u_{x} v}{p}
\]
\[ Y_2 = -\frac{g_{1y}v}{12} - \frac{C_2 t n(t) u_{yy}v}{p} + \frac{C_2 m n(t) u_{yy}v}{p} \]

\[- f(t) n(t) v u_{xy} + \frac{g_{1y}v p}{12(t - m)} + \frac{2 C_2 n(t) uv_y}{3 p} \]

\[- \frac{g_{1y}v p}{12(t - m)} + \frac{C_2 x n(t) u_x v_y}{3 p} + f(t) n(t) u_x v_y \]

\[ + \frac{2 C_2 y n(t) u_x v_y}{3 p} + \frac{C_2 y n(t) p}{2 u_y v_y} + g(t) n(t) u_y v_y \]

\[- \frac{g_{1y}v v_y}{12(t - m)} + \frac{C_2 x n(t) u_x v_y}{3 p} + f(t) n(t) u_x v_y \]

\[ + \frac{2 C_2 y n(t) u_x v_y}{3 p} + \frac{C_2 y n(t) p}{2 u_y v_y} + g(t) n(t) u_y v_y \]

\[ + \frac{C_2 z n(t) u_y v_y}{p} = - \frac{C_2 m n(t) u_y v_y}{p} + \frac{g_{1y} v v_y}{2} \]

\[ - g(t) m n(t) u_y v_y - \frac{2 C_2 y v n(t) p}{2 u_y v_y} - \frac{C_2 y v n(t)}{2 u_y v_y}, \]

\[ T_2 = - \frac{C_2 u_x v}{p} - \frac{C_2 u x v}{3 p} + \frac{g_{1y} v u x v}{2 n(t)} - f(t) u_x v \]

\[ - \frac{2 C_2 y u x v}{3 p} - \frac{C_2 y u x v}{2} - g(t) u_y v - \frac{C_2 t v u x}{p} \]

\[ + \frac{C_2 v m u x}{p}. \]

(36)

Here we should note that the coefficient function \( n(t) \) in the expression of \( X_2, Y_2, \) and \( T_2 \) satisfies \( n(t) = (t - m)^p C_1, m, p, \) and \( C_1 \) are constants, and \( p \neq 0, C_1 \neq 0. \)

For the symmetry in Case 3, the corresponding components of the conservation laws are

\[ X_3 = -\frac{\tau u_{xxx}}{18} + f(t) u_{xxx} v + g(t) u_{xxxx} v + \frac{\tau}{18} u_{xxx} v + \frac{\tau}{18} u_{xxxx} v \]

\[ + g(t) u_{y v} + \frac{2}{3} \tau u_{xx} v - \frac{2}{3} \tau u_{xxxx} v \]

\[ + \tau(t) u_{xxxx} + \frac{1}{3} \tau u_{xxxx} v + 4 \tau u_{xx} v \]

\[- f(t) u_{xx} v - g(t) u_{xy} v + \frac{1}{3} \tau u_{uv} v + f(t) u_{xx} v \]

\[ - f(t) u_{xy} v - \frac{1}{6} f(t) u_{xxxx} v - \frac{1}{6} f(t) u_{xxxx} v + f(t) u_{xx} v \]

\[ + g(t) v_{t u x} + \frac{\tau}{18} v_{t u x} + \frac{\tau}{18} v_{t u x x} \]

\[ + f(t) v_{t u x} + \frac{\tau}{18} v_{t u x} + \frac{\tau}{18} v_{t u x x} \]

\[ + \frac{y}{2 n} g_{1y} v u_x - \frac{y^2}{6 n} g_{1y} v u_x + \frac{y}{18} \tau u_{x v} + \frac{y^2}{36} \tau u_{x v} + \frac{y^2}{18} \tau u_{x v} + \frac{y^2}{36} \tau u_{x v} + \frac{y^2}{18} \tau u_{x v} \]

\[ + \frac{x}{3} \tau u_{xy} v + \frac{\tau}{18} u_{xy} v + \frac{\tau}{18} u_{xxxx} v + \frac{\tau}{18} u_{xxxx} v \]

\[ + \frac{y}{2 n} g_{1y} v u_x - \frac{y^2}{6 n} g_{1y} v u_x + \frac{y}{18} \tau u_{x v} + \frac{y^2}{36} \tau u_{x v} + \frac{y^2}{18} \tau u_{x v} + \frac{y^2}{36} \tau u_{x v} + \frac{y^2}{18} \tau u_{x v} \]

\[ + \frac{x}{3} \tau u_{xy} v + \frac{\tau}{18} u_{xy} v + \frac{\tau}{18} u_{xxxx} v + \frac{\tau}{18} u_{xxxx} v \]

\[ + 18 \tau u_{xx} v - \frac{y^2}{6 n} u_{xxxx} v + \frac{y}{18} \tau u_{x v} + \frac{y^2}{6 n} \tau u_{xy} v + \frac{y}{3} \tau u_{xy} v \]

\[ + \frac{x}{3} \tau u_{xy} v + \frac{y^2}{6 n} u_{xy} v - \frac{y}{3} \tau u_{xy} v + f(t) u_{xy} v \]

\[ - 10 \tau u_{xx} v + \frac{x}{3} \tau u_{xx} v + \frac{x}{3} \tau u_{xx} v + \frac{y^2}{6 n} \tau u_{xy} v \]

\[ + \frac{y^2}{36 n} \tau u_{xxxx} v + \frac{y}{12 n} \tau u_{xx} v + \frac{y}{12 n} \tau u_{xx} v + \frac{y}{3} \tau u_{xx} v \]

\[ + \frac{x}{3} \tau u_{xx} v + 6 f(t) u_{xx} v + \frac{2 x}{3} \tau u_{xx} v \]

\[ + \frac{2 y}{3} \tau u_{xx} v + 6 g(t) u_{xx} v + \frac{g(t)}{3} \tau u_{xx} v \]

\[ - 6 \tau(t) u_{xx} v + 6 \tau(t) u_{xx} v - \frac{x}{3} \tau u_{xx} v \]

\[ - \frac{y}{2 n} g_{1y} v u_x - \frac{y^2}{6 n} \tau u_{xx} v - \frac{y}{18} \tau u_{xx} v \]

\[ + \frac{x}{3} \tau u_{xx} v + \frac{y}{2 n} g_{1y} v u_x - \frac{y}{18} \tau u_{xx} v - \frac{y}{18} \tau u_{xx} v - \frac{y}{18} \tau u_{xx} v \]

\[ - 4 y \tau u_{xx} v + 4 y \tau u_{xx} v + \frac{y^2}{6 n} \tau u_{xx} v \]

\[ + \frac{y}{2 n} g_{1y} v u_x - 4 y \tau u_{xx} v + \frac{4}{3} \tau u_{xx} v \]

\[ Y_3 = \frac{1}{2} \frac{g_{1y} v u_x}{12} + \frac{y}{18} \tau u_{xx} v + \frac{y}{18} \tau u_{xx} v \]

\[ + \frac{1}{12} \tau u_{xx} v + \frac{x}{3} \tau u_{xx} v + f(t) u_{xx} v \]

\[ + \frac{2 y}{3} \tau u_{xx} v + \frac{2 y}{3} \tau u_{xx} v + \frac{y}{3} \tau u_{xx} v \]

\[ + \frac{1}{12} \tau u_{xx} v + \frac{x}{3} \tau u_{xx} v + \frac{y}{18} \tau u_{xx} v + \frac{y}{3} \tau u_{xx} v \]

\[ + \frac{2 y}{3} \tau u_{xx} v + \frac{2 y}{3} \tau u_{xx} v + \frac{y}{3} \tau u_{xx} v \]

\[ - g(t) u_{xx} v \].
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\( T_3 = - \tau(t) \nu_{x,t} - \tau_i \nu_{x} + \frac{1}{18} \tau_i \nu_{x,t} - \frac{3}{4} \tau_i \nu_{x,xx} \)

\[ + \frac{y^2}{6n} \tau_i \nu_{x,xx} + \frac{y}{2n} \nu_{x,xx} - f(t) \nu_{x,xx} \]

\[ - \frac{2y}{3} \tau_i \nu_{x,y} - \tau(t) \nu_{x,xy}. \]

(37)

For the fourth symmetry, the two functions \( \tau(t) \) and \( n(t) \) are determined by the differential equation (19) and they have many explicit solutions. For simplicity, we take \( \tau(t) = 1; \) then \( n(t) = 1 + \tan^2 t \) and \( e(t) = (\tan t/2) + C_3. \) When \( f(t) = g(t) = 0, \) the corresponding Lie symmetry is

\[ V = \frac{-y^2}{4} \frac{\partial}{\partial x} + y \tan t \frac{\partial}{\partial y} + \frac{\partial}{\partial t} + 0 \frac{\partial}{\partial u}, \]

(38)

and the components of the conservation laws are

\[ X_4 = - \nu_{u,xxx} - \frac{y^2}{4} \nu_{u,xx} \nu_x - 6u \nu_{u,x} + \frac{y^2}{4} u_x \nu_{xx} \]

\[ + 6u \nu_{u,x} - C_3 u_x + v \nu_{u,xxx} + u \nu_t \]

\[ + \frac{\tan t}{2} u_v - \frac{y^2}{4} u_x \nu_{xxx} - \frac{y^2}{4} u_v - \frac{3}{4} u_{y,y} \nu \]

\[ - \frac{y^2}{4} \nu \nu_x + \nu_{u,xxx} \tan t + \nu_{u,x} \nu_t + \nu_{y,x} \nu_{x,y} \tan t \]

\[ - 6y \nu \nu_{u,y} \tan t - C_3 \nu \nu_{y,y} \tan t - \frac{\tan^2 t}{4} \nu \nu_{y,yy} \]

\[ - u_{x,x} \nu_{xx} - \nu_{y,xxx} \tan t - \nu \nu_{x,xxx} \tan t \]

\[ - 6y \nu \nu_{u,x} \tan t + 6 \nu \nu_{u,y} \nu_t, \]

\[ Y_4 = - \frac{y^2}{4} u_x \nu_y - \frac{y^2}{4} v \nu_{u,y} \tan^2 t + \nu \nu_{u,y} \tan t \]

\[ + \nu \nu_{u,y} \tan^3 t + v \nu_{u,y} + v \nu_{u} \nu \tan^2 t - \frac{1}{2} \nu y \nu_{u} \]

\[ + \frac{y^2}{4} v \nu_{x,y} + \frac{y^2}{4} v \nu_{y,x} \tan^2 t - \nu \nu_{y,y} \tan t - \nu \nu_{y,y} \tan^2 t \]

\[ - \nu \nu_{y,y} \tan t + \nu \nu \tan^2 t + \nu \nu \tan^2 t \]

\[ - \nu \nu \tan^2 t + \nu \nu \tan^2 t, \]

\[ T_4 = \frac{y^2}{4} \nu_{u,xx} - \nu \nu \nu_{x,y} \tan t - \nu \nu_{x,xt}. \]

(39)

We should mention that in the above components of the conservation laws for (1) and (28), \( u \) is a solution of (1) and \( v \) is a solution of the adjoint equation (28). Making use of the explicit solutions of (28), local conservation laws for (1) can be obtained. For example, when \( a(t) = 0 \) and \( b(t) = 0 \) in (34),

\[ v = k(t) y + l(t), \]

(40)

where \( k(t) \) and \( l(t) \) are arbitrary functions, is an exact solution of (28). Substituting (40) into the above four conservation laws, we can obtain time-dependent and local conservation laws for (1). Here we take \( (X_4, Y_4, T_4) \) as an illustrative example; when \( v = k(t) y + l(t) \), the components of the conservation laws \( (X_4, Y_4, T_4) \) become

\[ \tilde{X}_3 = C_3 y^2 u_x k(t) \tan t - C_3 l(t) y u_y \tan t \]

\[ - 6k(t) y^2 u_x u_y \tan t - 6l(t) y u_y u_x \tan t + \frac{y^3}{4} k(t) u_y y \]

\[ - \frac{y^2}{4} l(t) u_x - C_3 l(t) u_t + \frac{1}{2} l(t) u_t \tan t - 6l(t) u_x u_y \]

\[ - 6l(t) u_x u_y + \frac{y^2}{4} l(t) u_y y - k(t) y u_x \nu_{xxx} - \frac{y^3}{4} k(t) u_y y \]

\[ + l(t) y u_y y(y t) + \frac{y^2}{2} k(t) u_y \tan t \]

\[ - y l(t) u_{xxx} \tan t - k(t) y^2 u_{xxy} \tan t \]

\[ - \frac{y^3}{4} k(t) u_y y + \frac{y^2}{4} l(t) u_y y - \frac{y^3}{4} k(t) u_y y \]

\[ - 6k(t) y u_x u_y \tan t - 6l(t) y u_y u_x \tan t \]

\[ - 6k(t) y u_x u_y - C_3 k(t) y u_y - \frac{y^3}{4} k(t) u_y y \]

\[ + k(t) y^2 u_y y \tan t - 6l(t) y u_y u_x \tan t, \]

\[ \tilde{Y}_4 = - l(t) y u_y y \tan^2 t - l(t) y u_y y \tan t - l(t) u_y y \tan t + k(t) u_t \]

\[ - k(t) y^2 \tan^3 t u_y y + l(t) y \tan^3 t u_y y + \frac{y^3}{2} l(t) u_x \tan^3 t \]

\[ + \frac{y^2}{4} l(t) u_{xy} \tan^3 t + \frac{y^3}{4} k(t) u_{xy} + k(t) u_t \tan^3 t \]

\[ - \frac{y^2}{4} k(t) u_x - l(t) u_y y \tan^3 t - y k(t) u_{yy} y - l(t) u_y y \tan^3 t \]

\[ - l(t) u_y y \tan t + \frac{y^3}{4} l(t) u_y y + \frac{y^2}{2} l(t) u_x \]

\[ + \frac{y^2}{4} k(t) u_x \tan^3 t + \frac{y^3}{4} k(t) u_{xy} \tan^3 t \]

\[ - k(t) y^2 u_y y \tan t, \]

\[ \tilde{T}_4 = \frac{1}{4} (k(t) y + l(t)) \left( y^2 u_{xx} - 4 y u_{xy} y \tan t - 4 u_{st} \right). \]

(41)
These are local and explicit conservation laws of (1). Next we show that the above conservation laws (X̄₄, Ȳ₄, ̄T₄) are nontrivial:

\[
\begin{align*}
D_x(X₄) + D_y(Y₄) + D_z(T₄) & = -C₃y^2k(t)uₓₓtan²t - l(t)uₓ+t(t)uₓₓ - k(t)yuₓₓ - 12l(t)uₓuₓx - 2l(t)uₓyₜtan²t + \frac{1}{2}l(t)uₓtan²t - 2l(t)uₓyₜtan²t + \frac{1}{2}l(t)u_xtxt + C₃l(t)u_xt - l(t)uₓyₜtan²t + \frac{1}{2}l(t)u_xtxt + 6l(t)u_xt - 6l(t)u_uₓx + \frac{1}{2}l(t)u_xtxt + C₃l(t)u_xt - l(t)uₓyₜtan²t
\end{align*}
\]

From the above equation, we can obtain an algebraically explicit analytical solution for (1):

\[
\begin{align*}
Ω &= Ω(θ, t) \\
&= Ω(x, t) + F(y, t) + G(z) \\
&= Ω(x, t) + F(y, t) + G(z) \\
&= Ω(x, t) + F(y, t) + G(z) \\
&= Ω(x, t) + F(y, t) + G(z)
\end{align*}
\]

By the characteristic equations of the symmetry, we have

\[
\begin{align*}
u &= Ω(θ, t), \quad θ = y²/2 + 2uxt. Substituting it into (1), we get a symmetry reduction of (1):
\end{align*}
\]

\[
\begin{align*}
Ω₁ + \frac{θ}{t}Ω₂ + 12nt(Ω₃)θ + 8n²r³(Ω₄)ₘ = 0.
\end{align*}
\]

If the coefficient functions \( e(t) = 0, n(t) = \text{Const} \), the obtained symmetry reduction can be simplified to

\[
\begin{align*}
Ω₁ + \frac{θ}{t}Ω₂ + 12nt(Ω₃)θ + 8n²r³(Ω₄)ₘ = 0.
\end{align*}
\]

Integrating (50) with respect to θ and taking the constant of integration to zero, we get the following equation:

\[
\begin{align*}
Ω₁ + 12nt(Ω₂ + 8n²r³(Ω₄)ₘ) + \frac{θ}{t}Ω₃ + \frac{1}{2r}Ω = 0.
\end{align*}
\]
Equation (51) is the (1 + 1)-dimensional generalized KdV equation with variable coefficients. To the best of our knowledge, exact solutions of (51) have not been studied up to now. Solving (51) by the method in [25], we can get the following solutions for (1):

\[ u = \Omega(\theta, t) = \frac{\theta}{24nt^2} + \frac{M_1}{24nt M_1} - \frac{8n^2 M_1^2 c_2}{3t} - \frac{8n^2 M_1^2 c_4}{t} \]

\[ \varphi = M_1 \theta^{-1/2} + M_2 t^{-1/2} + M_3, \]

where \(M_1, M_2,\) and \(M_3\) are arbitrary constants and the function \(P(\varphi)\) satisfies

\[ P'^2 = c_0 + c_2 P^2 + c_4 P^4, \]

where \(c_0, c_2,\) and \(c_4\) are constants; solutions of (53) have been given in [26]. By means of the solutions of (53), plenty of solutions for (1) can be obtained; for example,

\[ u_1 = \frac{y^2/2 + 2nt}{24nt^2} + \frac{M_3}{24nt M_1} - \frac{8n^2 M_1^2}{3t} - \frac{8n^2 M_1^2 s^2(\varphi)}{t}, \]

\[ \left( c_0 = 1, c_2 = -1 - k^2, c_4 = k^2 \right), \]

\[ u_2 = \frac{y^2/2 + 2nt}{24nt^2} + \frac{M_3}{24nt M_1} - \frac{8n^2 M_1^2}{3t} - \frac{8n^2 M_1^2 s^2(\varphi)}{t}, \]

\[ \left( c_0 = k^2, c_2 = -1 - k^2, c_4 = 1 \right), \]

\[ u_3 = \frac{y^2/2 + 2nt}{24nt^2} + \frac{M_3}{24nt M_1} - \frac{8n^2 M_1^2 c_2}{3t} \]

\[ + \frac{8n^2 M_1^2 c_2 \text{sech}^2(\varphi)}{t}, \quad \left( c_0 = 0, c_2 > 0, c_4 < 0 \right), \]

\[ u_4 = \frac{y^2/2 + 2nt}{24nt^2} + \frac{M_3}{24nt M_1} - \frac{8n^2 M_1^2 c_2}{3t} \]

\[ + \frac{4n^2 M_1^2 c_2 \text{tanh}^2(\varphi)}{t}, \quad \left( c_0 = \frac{c_2^2}{4c_4}, c_2 < 0, c_4 > 0 \right), \]

where \(k \quad (0 < k < 1)\) denotes the modulus of the Jacobi elliptic function.

(iii) When \(e(t) = 0, n(t) = (t-\rho)^2 C_1, p \neq 0, C_1 \neq 0, f(t) = M_0,\) and \(g(t) = 1,\) we can get

\[ u = \Omega(\theta, t), \quad \theta = x - M_0 y. \]

And \(\Omega(\theta, t)\) satisfies the following reduction equation:

\[ \Omega_{\theta t} + 6 \left( \Omega_{\theta}^2 + \Omega \Omega_{\theta \theta} \right) + \Omega_{\theta \theta \theta} + e(t) \Omega_{\theta} \]

\[ + M_2^2 C_1 (t - m)^p \Omega_{\theta} = 0. \]

The above equation can be integrated by \(\theta\) and, when we take the constant of integration to zero, we get a reduced reduction equation:

\[ \Omega_{\theta} + 6 \Omega_{\theta \theta} + \Omega_{\theta \theta \theta} + M_0^2 C_1 (t - m)^p \Omega_{\theta} \Omega = 0. \]

Equation (57) is variable coefficient KdV equation and soliton-like solutions have been obtained in [27]. By means of the known solutions, many explicit solutions of (1) can be obtained. For example,

\[ u_1 = k_1 + 2ck_4 \text{sech} \left( \sqrt{c} \varphi \right), \]

\[ \varphi = k_4 (x - M_0 y) - 6k_4 k_4 t - 4k_4^3 t \]

\[ - \frac{M_0^2 C_1 k_4}{p + 1}, \]

\[ u_2 = k_1 - 2ck_4 \text{tanh}^2 (\varphi), \]

\[ \varphi = k_4 (x - M_0 y) - 6k_4 k_4 t + 8k_4^3 t \]

\[ - \frac{M_0^2 C_1 k_4}{p + 1}, \]

where \(k_1, k_4,\) and \(c\) are constants.

(iv) When \(e(t) \neq 0\) and \(n(t) = N_0 \exp((\int (e_0 - 2e^2) e) dt),\)

\[ f(t) = N_1, g(t) = 1. \] By the corresponding characteristic equation of the symmetry, we have

\[ u = \Omega(\theta, t), \quad \theta = x - N_1 y. \]

Substituting it into (1), we get the following symmetry reduction of (1):

\[ \Omega_{\theta t} + 6 \left( \Omega_{\theta}^2 + \Omega \Omega_{\theta \theta} \right) + \Omega_{\theta \theta \theta} + e(t) \Omega_{\theta} \]

\[ + N_0^2 N_0 \exp \left( \int \left( \frac{e_0^2 - 2e^2}{e} \right) dt \right) \Omega_{\theta \theta} = 0. \]

Integrating the above equation with respect to \(\theta\) and taking the constant of integration to zero, the obtained reduction equation becomes

\[ \Omega_{\theta} + 6 \Omega_{\theta \theta} + \Omega_{\theta \theta \theta} + e(t) \Omega \]

\[ + N_0^2 N_0 \exp \left( \int \left( \frac{e_0^2 - 2e^2}{e} \right) dt \right) \Omega_{\theta \theta} = 0. \]

Equation (61) is a variable coefficient KdV equation [28, 29].

4.2. For the Symmetry in Case 2, \(e(t)=0, n(t)=(t-m)^p C_1, p \neq 0, C_1 \neq 0.\) When \(f(t) = g(t) = 0, m = 0, p = C_2 = 2/3, then
n(t) = C_1t^{2/3}, and C_1 ≠ 0; the corresponding symmetry of (1) is

\[ V = \frac{x}{3} \frac{\partial}{\partial x} + y \frac{\partial}{\partial y} + t \frac{\partial}{\partial t} - \frac{2}{3} u \frac{\partial}{\partial u}. \]  

(62)

By the characteristic equations of the symmetry, we can get the explicit solutions for (1)

\[ u = \Omega(\theta, \delta) t^{-2/3}, \quad \theta = \frac{x^3}{t}, \quad \delta = \frac{y}{t}. \]  

(63)

where the function \( \Omega(\theta, \delta) \) satisfies the following reduction equation:

\[ -36^{1/3} \Omega_{\theta \theta} - 36^{2/3} \delta \Omega_{\delta \delta} - 50^{2/3} \Omega_{\theta} 
+ 54\theta^{1/3} \left( \Omega_{\delta}^2 + \Omega_{\theta} \Omega_{\delta} \right) + 36\theta^{1/3} \Omega_{\phi} \Omega 
+ 81\theta^{1/3} \Omega_{\phi \theta \phi} + 324\theta^{1/3} \Omega_{\phi \theta} + 180\theta^{2/3} \Omega_{\theta \theta} 
+ C_1 \Omega_{\delta \delta} = 0. \]  

(64)

Equation (64) is difficult to solve and we will study its exact solutions in a future paper.

4.3. For the Symmetry in Case 3, e(t)=0, n(t)=Const., and \( \tau(t) \neq 0 \). When \( f(t) = 0 \), \( g(t) = 0 \), the corresponding symmetry is

\[ V = \left( \frac{\tau_x}{3} - \frac{\tau_u}{6n} \right) \frac{\partial}{\partial x} + y \frac{\partial}{\partial y} + \tau(t) \frac{\partial}{\partial t} + \left( \frac{2\tau_u}{3} \right) \frac{\partial}{\partial u} \]  

+ \left( \frac{2\tau_u}{3} \right) \frac{\partial}{\partial u} \]  

(65)

By the characteristic equation of the symmetry, we have

\[ u = \frac{1}{18\tau} \tau_x - \frac{1}{36n} y \tau_u + \frac{1}{54n} y^2 \tau_y + \Omega(\theta, \delta) t^{-2/3}, \]

\[ \theta = x \tau^{-1/3} \]  

\[ \delta = \frac{y}{t}. \]  

(66)

Substituting it into (1), we get a symmetry reduction of (1):

\[ 6\Omega_{\theta \delta} + 6\Omega_{\theta \phi} \Omega_{\phi \theta} + n\Omega_{\delta \delta} = 0. \]  

(67)

Equation (67) is the special case of (2 + 1)-dimensional Boussinesq equation and exact solutions of (67) have been studied by Chen and Zhang in [30] (with \( a = 0, b = 0, r = -3/n, \) and \( s = -1/n \)). With the help of the known solutions in [30], many explicit solutions of (1) can be obtained. We list the following solitons solutions (\( u_1-u_4 \)) and Jacobi elliptic function solutions (\( u_5-u_9 \)):

\[ u_1 = \left( -\frac{n\omega^2}{6\alpha^2} + \frac{4}{3} \right) \tau^{-2/3} - 2\alpha^2 \tau^{-2/3} \tanh^2 (\varphi) \]

\[ + \frac{\tau_x x}{18 \tau} - \frac{\tau_u y^2}{36n \tau} + \frac{\tau_y^2 y^2}{54n \tau^2}, \]

\[ u_2 = \left( -\frac{n\omega^2}{6\alpha^2} - \frac{2}{3} \right) \tau^{-2/3} + 2\alpha^2 \tau^{-2/3} \text{sech}^2 (\varphi) \]

\[ + \frac{\tau_x x}{18 \tau} - \frac{\tau_u y^2}{36n \tau} + \frac{\tau_y^2 y^2}{54n \tau^2}, \]

\[ u_3 = \left( -\frac{n\omega^2}{6\alpha^2} + \frac{1}{3} \right) \tau^{-2/3} \]

\[ - \frac{\alpha^2}{2} \tau^{-2/3} \varepsilon \tanh^2 (\varphi) + \beta (1 + \text{sech} (\varphi))^4 \]

\[ + \frac{\tau_x x}{18 \tau} - \frac{\tau_u y^2}{36n \tau} + \frac{\tau_y^2 y^2}{54n \tau^2}, \]

\[ u_4 = \left( -\frac{n\omega^2}{6\alpha^2} - \frac{2}{3} \right) \tau^{-2/3} - 2\alpha^2 \tau^{-2/3} \text{sech}^2 (\varphi) \]

\[ \times \varepsilon + \beta m^2 \text{sn}^4 (\varphi) + \frac{\tau_x x}{18 \tau} - \frac{\tau_u y^2}{36n \tau} + \frac{\tau_y^2 y^2}{54n \tau^2}, \]

\[ u_5 = \left( -\frac{n\omega^2}{6\alpha^2} + \frac{2}{3} \right) \tau^{-2/3} - 2\alpha^2 \tau^{-2/3} \]

\[ \times \varepsilon + \beta m^2 \text{sn}^4 (\varphi) + \frac{\tau_x x}{18 \tau} - \frac{\tau_u y^2}{36n \tau} + \frac{\tau_y^2 y^2}{54n \tau^2}, \]

\[ u_6 = \left( -\frac{n\omega^2}{6\alpha^2} - \frac{2}{3} \right) \tau^{-2/3} - 2\alpha^2 \tau^{-2/3} \]

\[ \times \varepsilon \text{dn}^4 (\varphi) + \beta m^2 \text{cn}^4 (\varphi) \]

\[ + \frac{\tau_x x}{18 \tau} - \frac{\tau_u y^2}{36n \tau} + \frac{\tau_y^2 y^2}{54n \tau^2}, \]

\[ u_7 = \left( -\frac{n\omega^2}{6\alpha^2} - \frac{4}{3} \right) \tau^{-2/3} - 2\alpha^2 \tau^{-2/3} \]

\[ \times \varepsilon \left( 1 - m^2 \right) - \beta m^2 \text{sn}^4 (\varphi) \]

\[ + \frac{\tau_x x}{18 \tau} - \frac{\tau_u y^2}{36n \tau} + \frac{\tau_y^2 y^2}{54n \tau^2}, \]

\[ u_8 = \left( -\frac{n\omega^2}{6\alpha^2} - \frac{4}{3} \right) \tau^{-2/3} + 2\alpha^2 \tau^{-2/3} \]

\[ \times \varepsilon \left( 1 - m^2 \right) + \beta \text{dn}^4 (\varphi) \]

\[ + \frac{\tau_x x}{18 \tau} - \frac{\tau_u y^2}{36n \tau} + \frac{\tau_y^2 y^2}{54n \tau^2}. \]
$u_9 = \left(-\frac{n\omega^2}{6\alpha^2} - \frac{4}{3} \alpha^2 + \frac{2}{3} \alpha^2 m^2\right) r^{-2/3} - 2\alpha^2 r^{-2/3} \frac{e(1 - m^2) \text{sn}^4(\phi) + \beta \text{cn}^4(\phi)}{\text{sn}^2(\phi) \text{cn}^2(\phi)} + \frac{\tau_x}{18r} - \frac{\tau_y^2}{36nr} + \frac{\tau_z^2}{54nr^2}$

$u_{10} = \left(-\frac{n\omega^2}{6\alpha^2} - \frac{4}{3} \alpha^2 + \frac{2}{3} \alpha^2 m^2\right) r^{-2/3} - 2\alpha^2 r^{-2/3} \frac{e \text{dn}^4(\phi) - \beta m^2 (1 - m^2) \text{sn}^4(\phi)}{\text{sn}^2(\phi) \text{dn}^2(\phi)} + \frac{\tau_x}{18r} - \frac{\tau_y^2}{36nr} + \frac{\tau_z^2}{54nr^2}$

$u_{11} = \left(-\frac{n\omega^2}{6\alpha^2} - \frac{1}{3} \alpha^2 + \frac{2}{3} \alpha^2 m^2\right) r^{-2/3} - \frac{\alpha^2}{2} r^{-2/3} \frac{\text{cn}^4(\phi) + \beta(1 \pm \text{cn}(\phi))^4}{\text{cn}^2(\phi) (\sqrt{1 - m^2} \pm \text{sn}(\phi))^2} + \frac{\tau_x}{18r} - \frac{\tau_y^2}{36nr} + \frac{\tau_z^2}{54nr^2}$

$u_{12} = \left(-\frac{n\omega^2}{6\alpha^2} - \frac{1}{3} \alpha^2 + \frac{2}{3} \alpha^2 m^2\right) r^{-2/3} - \frac{\alpha^2}{2} r^{-2/3} \frac{\text{cn}^4(\phi) + \beta(1 \pm \text{cn}(\phi))^4}{\text{cn}^2(\phi) (\sqrt{1 - m^2} \pm \text{sn}(\phi))^2} + \frac{\tau_x}{18r} - \frac{\tau_y^2}{36nr} + \frac{\tau_z^2}{54nr^2}$

$u_{13} = \left(-\frac{n\omega^2}{6\alpha^2} - \frac{1}{3} \alpha^2 - \frac{1}{3} \alpha^2 m^2\right) r^{-2/3} + \frac{\alpha^2}{2} (1 - m^2) r^{-2/3} \frac{e \text{dn}^4(\phi) + \beta(1 \pm \text{sn}(\phi))^4}{\text{dn}^2(\phi) (1 \pm \text{sn}(\phi))^2} + \frac{\tau_x}{18r} - \frac{\tau_y^2}{36nr} + \frac{\tau_z^2}{54nr^2}$

$u_{14} = \left(-\frac{n\omega^2}{6\alpha^2} - \frac{1}{3} \alpha^2 - \frac{1}{3} \alpha^2 m^2\right) r^{-2/3} - \frac{\alpha^2}{2} (1 - m^2) r^{-2/3} \frac{e \text{cn}^4(\phi) + \beta(1 \pm \text{sn}(\phi))^4}{\text{cn}^2(\phi) (1 \pm \text{sn}(\phi))^2} + \frac{\tau_x}{18r} - \frac{\tau_y^2}{36nr} + \frac{\tau_z^2}{54nr^2}$

$u_{15} = \left(-\frac{n\omega^2}{6\alpha^2} - \frac{1}{3} \alpha^2 - \frac{1}{3} \alpha^2 m^2\right) r^{-2/3} + \frac{\alpha^2}{2} r^{-2/3} \frac{(1 - m^2)^2 + \beta(m \text{cn}(\phi) \pm \text{dn}(\phi))^4}{(m \text{cn}(\phi) \pm \text{dn}(\phi))^2} + \frac{\tau_x}{18r} - \frac{\tau_y^2}{36nr} + \frac{\tau_z^2}{54nr^2}$

where $\phi = \alpha(x r^{-1/3} + (1/6n) y^2 r^{-4/3}) + \omega(y r^{-2/3})$, $\alpha$ and $\omega$ are constants, $k(0 < k < 1)$ denotes the modulus of the Jacobi elliptic function, and $e$ and $\beta$ are arbitrary elements of $[0, 1)$. We should mention that the soliton solution $u_5$ is the limit of $u_5$ when $m \to 1, e = 0, \beta = 1$. The solutions $u_1, u_3, u_4$ are the limit of $u_7, u_{11},$ and $u_9$, respectively, when $m \to 1, \beta = 1$.

4.4. For the Symmetry in Case 4, $e(t) = -n/4n + C_3 / \tau(t), n(t)$, and $\tau(t)$ Satisfy (19). For simplicity, we take $f(t) = g(t) = 0, \tau(t) = 1; \text{then} n(t) = 1 + \tan^2 t$ and $e(t) = - \tan^2 t/2 + C_3$. Solving the corresponding characteristic equation, we get

\[ u = \Omega(\theta, \delta), \quad \theta = x + \frac{y^2}{4} \sin t \cos t, \quad \delta = y \cos t. \]

Substituting it into (1), we get a symmetry reduction of (1):

\[ \frac{\delta^2}{4} \Omega_{\theta\theta} + 6\Omega_{\theta\theta} \Omega + 6\Omega_\theta^2 + \Omega_{\theta\theta\theta} + C_3 \Omega_\theta + \Omega_{\delta\delta} = 0. \] (70)

Obviously, $\Omega = -(C_3 / 6) \theta + N_1 \delta + N_2$ is a solution of (70). From that, we can get an algebraically explicit analytical solution for (1) as follows:

\[ u = \frac{C_3}{6} \left(x + \frac{y^2}{4} \sin t \cos t\right) + N_1 y \cos t + N_2. \] (71)

where $N_1$ and $N_2$ are integral constants. And, if $C_3 = 0$, (70) becomes the following $(2 + 1)$-dimensional variable coefficient Boussinesq equation:

\[ \frac{\delta^2}{4} \Omega_{\theta\theta} + 6\Omega_{\theta\theta} \Omega + 6\Omega_\theta^2 + \Omega_{\theta\theta\theta} + \Omega_{\theta\delta} = 0. \] (72)
Remark 3. To the best of our knowledge, the symmetry reductions obtained in this paper have not been reported in the existent literature, so they are completely new. The exact solutions of (1) obtained here are all different from the known solutions and they are also new. All the solutions and conservation laws obtained in this paper for (1) have been checked by Maple software.

5. Conclusions

In summary, by performing Lie symmetry analysis to (1), four cases of geometric symmetries are obtained when the coefficient functions satisfy four different constraint conditions. According to the relationship between symmetry and conservation laws given by Ibragimov, many explicit and nontrivial conservation laws, which includes arbitrary functions of \( t \), are derived. These conservation laws may be useful for the explanation of some practical physical problems. Using the associated vector fields of the obtained symmetry, (1) is reduced to \((1 + 1)\)-dimensional nonlinear partial differential equations including different types of variable coefficient KdV equation (see (51), (57), and (61)), special case of \((2 + 1)\)-dimensional Boussinesq equation (see (67) and (72)), and other reduction equations (see (64) and (70)). Many new explicit solutions of (1) have been derived by solving the reduction equations. These solutions, including soliton solutions, Jacobi doubly periodic solutions, and algebraically explicit analytical solutions, can make one discuss the behavior of solutions and also provide mathematical foundation for the explanation of some interesting physical phenomena.

Conflict of Interests

The author declares that there is no conflict of interests regarding the publication of this paper.

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References


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