Research Article

Solving Nonstiff Higher-Order Ordinary Differential Equations Using 2-Point Block Method Directly

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We describe the development of a 2-point block backward difference method (2PBBD) for solving system of nonstiff higher-order ordinary differential equations (ODEs) directly. The method computes the approximate solutions at two points simultaneously within an equidistant block. The integration coefficients that are used in the method are obtained only once at the start of the integration. Numerical results are presented to compare the performances of the method developed with 1-point backward difference method (1PBD) and 2-point block divided difference method (2PBDD). The result indicated that, for finer step sizes, this method performs better than the other two methods, that is, 1PBD and 2PBDD.

1. Introduction

In this paper, we consider the system of d th order ODEs of the form

$$y_i^{(d_i)} = f_i(x, \tilde{Y}), \quad i = 1, 2, \dots, s, \quad (1)$$

with $\tilde{Y}(a) = \tilde{\eta}$ in the interval $a \leq x \leq b$, where

$$\begin{aligned} \tilde{Y}(x) &= (y_1, \dots, y_1^{(d_1-1)}, \dots, y_s, \dots, y_s^{(d_s-1)}), \\ \tilde{\eta} &= (\eta_1, \dots, \eta_1^{(d_1-1)}, \dots, \eta_s, \dots, \eta_s^{(d_s-1)}). \end{aligned} \quad (2)$$

For simplicity of discussion and without loss of generality, we consider the single equation

$$y^{(d)} = f(x, \tilde{Y}), \quad \tilde{Y}(a) = \tilde{\eta}, \quad (3)$$

where

$$\tilde{Y}^T = (y, y', \dots, y^{(d-1)}), \quad \tilde{\eta}^T = (\eta, \eta', \dots, \eta^{(d-1)}). \quad (4)$$

As shown in Figure 1, here the 2-point block method, the interval $[a, b]$, is divided into series of blocks with each block containing two points; that is, x_{n-1} and x_n is the first block while x_{n+1} and x_{n+2} is the second block, where solutions to (3) are to be computed.

Previous works on block method for solving (3) directly are given by Milne [1], Rosser [2], Shampine and Watts [3], and Chu and Hamilton [4]. According to Omar [5], both implicit and explicit block Adams methods in their divided difference form are developed for the solution of higher-order ODEs. Majid [6] has derived a code based on the variable step size and order of fully implicit block method to solve nonstiff higher-order ODEs directly. Ibrahim [7] has developed a new block backward differentiation formula method of variable step size for solving first- and second-order ODEs directly. Suleiman et al. [8] have introduced one-point backward difference methods for solving higher-order ODEs. Hence, this motivates us to extend the method to block method in solving nonstiff higher-order ODEs.

FIGURE 1: 2-point method.

2. The Formulation of the Predict-Evaluate-Correct-Evaluate (PECE) Multistep Block Method in Its Backward Difference Form (MSBBD) for Nonstiff Higher-Order ODEs

The code developed will be using the PECE mode with constant stepsize. The predictor and corrector for first and second point will have the following form.

Predictor:

$$\text{pr } y_{n+r}^{(d-t)} = \sum_{i=0}^{t-1} \frac{h^i}{i!} y_n^{(d-t+i)} + h^t \sum_{i=0}^{k-1} \gamma_{r,t,i} \nabla^i f_n, \quad (5)$$

where $\gamma_{r,t,i}$ is coefficient for predictor for $r = 1, 2$ and $t = 1, 2, \dots, d$.

Corrector:

$$y_{n+r}^{(d-t)} = \sum_{i=0}^{t-1} \frac{h^i}{i!} y_n^{(d-t+i)} + h^t \sum_{i=0}^k \gamma_{r,t,i}^* \nabla^i f_{n+r}, \quad (6)$$

where $\gamma_{r,t,i}^*$ is coefficient for corrector for $r = 1, 2$ and $t = 1, 2, \dots, d$.

We also formulate the corrector in terms of the predictor. Both points y_{n+1} and y_{n+2} can be written as

$$y_{n+1}^{(d-t)} = \text{pr } y_{n+1}^{(d-t)} + h [\gamma_{1,t,k} \nabla^k f_{n+1}], \quad (7)$$

$$y_{n+2}^{(d-t)} = \text{pr } y_{n+2}^{(d-t)} + h [\gamma_{2,t,k} \nabla^k f_{n+2} - \gamma_{2,t,k-1} \nabla^{k+1} f_{n+2}]. \quad (8)$$

We derived the formulation for both the predictor and corrector.

3. Derivation for Higher-Order Explicit Integration Coefficients

3.1. For the First Point. The derivation for up to third-order explicit integration coefficients for the first point y_{n+1} has been given by Suleiman et al. [8].

3.2. For the Second Point. Integrating (3) once yields

$$y^{(d-1)}(x_{n+2}) = y^{(d-1)}(x_n) + \int_{x_n}^{x_{n+2}} f(x, y, y', y'', \dots, y^{(d-1)}) dx. \quad (9)$$

Let $P_n(x)$ be the interpolating polynomial which interpolates the k values $(x_n, f_n), (x_{n-1}, f_{n-1}), \dots, (x_{n-k+1}, f_{n-k+1})$; then

$$P_n(x) = \sum_{i=0}^{k-1} (-1)^i \binom{-s}{i} \nabla^i f_n. \quad (10)$$

Approximating f in (6) with $P_n(x)$ and letting

$$x = x_n + sh \quad \text{or} \quad s = \frac{x - x_n}{h} \quad (11)$$

gives

$$y^{(d-1)}(x_{n+2}) = y^{(d-1)}(x_n) + \int_0^2 \sum_{i=0}^{k-1} (-1)^i \binom{-s}{i} \nabla^i f_n h ds \quad (12)$$

or

$$y^{(d-1)}(x_{n+2}) = y^{(d-1)}(x_n) + h \sum_{i=0}^{k-1} \gamma_{2,1,i} \nabla^i f_n, \quad (13)$$

where

$$\gamma_{2,1,i} = (-1)^i \int_0^2 \binom{-s}{i} ds. \quad (14)$$

Define the generating function $G_1(t)$ for the coefficient $\gamma_{2,1,i}$ as follows:

$$G_1(t) = \sum_{i=0}^{\infty} \gamma_{2,1,i} t^i. \quad (15)$$

Substituting $\gamma_{2,1,i}$ in (14) into $G_1(t)$ gives

$$\begin{aligned} G_1(t) &= \sum_{i=0}^{\infty} (-t)^i \int_0^2 \binom{-s}{i} ds, \\ G_1(t) &= \int_0^2 (1-t)^{-s} ds, \\ G_1(t) &= \int_0^2 e^{-s \log(1-t)} ds, \end{aligned} \quad (16)$$

which leads to

$$G_1(t) = - \left[\frac{(1-t)^{-2}}{\log(1-t)} - \frac{1}{\log(1-t)} \right]. \quad (17)$$

Equation (17) can be written as

$$- \left(\sum_{i=0}^{\infty} \gamma_{2,1,i} t^i \right) \log(1-t) = (2-t) \left[\frac{t}{(1-t)^2} \right] \quad (18)$$

or

$$\begin{aligned} &(\gamma_{2,1,0} + \gamma_{2,1,1}t + \gamma_{2,1,2}t^2 + \dots) \left(t + \frac{t^2}{2} + \frac{t^3}{3} + \dots \right) \\ &= (2-t) (t + 2t^2 + 3t^3 + \dots). \end{aligned} \quad (19)$$

Hence, the coefficients of $\gamma_{2,1,i}$ are given by

$$\sum_{i=0}^k \left(\frac{\gamma_{2,1,i}}{k-i+1} \right) = k+2, \quad \gamma_{2,1,k} = (k+2) - \sum_{i=0}^{k-1} \frac{\gamma_{2,1,i}}{(k-i+1)}, \quad k=1,2,\dots, \quad \gamma_{2,1,0} = 2. \quad (20)$$

Integrating (1) twice yields

$$y^{(d-2)}(x_{n+2}) = y^{(d-2)}(x_n) + hy^{(d-1)}(x_n) + h^2 \sum_{i=0}^{k-1} \gamma_{2,2,i} \nabla^i f_n. \quad (21)$$

Substituting x with s gives

$$\gamma_{2,2,i} = (-1)^i \int_0^2 \frac{(2-s)}{1!} \binom{-s}{i} ds. \quad (22)$$

The generating function of the coefficient $\gamma_{2,2,i}$ is defined as follows:

$$G_2(t) = \sum_{i=0}^{\infty} \gamma_{2,2,i} t^i. \quad (23)$$

Substituting (22) into $G_2(t)$ above gives

$$G_2(t) = \int_0^2 \frac{(2-s)}{1!} e^{-s \log(1-t)} ds. \quad (24)$$

Substituting $G_1(t)$ into (24) yields

$$G_2(t) = \frac{1}{1!} \left[\frac{2}{\log(1-t)} - \frac{1!G_1(t)}{\log(1-t)} \right]. \quad (25)$$

Equation (25) can be written as

$$\left(\sum_{i=0}^{\infty} \gamma_{2,2,i} t^i \right) \log(1-t) = \frac{1}{1!} [2 - 1!G_1(t)] \quad (26)$$

or

$$\begin{aligned} & (\gamma_{2,2,0} + \gamma_{2,2,1}t + \gamma_{2,2,2}t^2 + \dots) \left(t + \frac{t^2}{2} + \frac{t^3}{3} + \dots \right) \\ &= \frac{1}{1!} [-2 + 1! (\gamma_{2,1,0} + \gamma_{2,1,1}t + \gamma_{2,1,2}t^2 + \dots)]. \end{aligned} \quad (27)$$

Hence the coefficients of $\gamma_{2,2,k}$ in relation to coefficients of the previous order $\gamma_{2,1,k}$ are given by

$$\sum_{i=0}^k \frac{\gamma_{2,2,i}}{k-i+1} = \gamma_{2,1,k+1}, \quad (28)$$

$$\gamma_{2,2,0} = \gamma_{2,1,1},$$

$$\gamma_{2,2,k} = \gamma_{2,1,k+1} - \sum_{i=0}^{k-1} \frac{\gamma_{2,2,i}}{k-i+1}, \quad k=1,2,\dots \quad (29)$$

By using the same process previously, we note that for integrating $(d-1)$ times yield

$$G_{(d-1)}(t) = \int_0^2 \frac{(2-s)^{(d-2)}}{(d-2)!} e^{-s \log(1-t)} ds, \quad (30)$$

$$G_{(d-1)}(t) = \frac{1}{(d-2)!} \left[\frac{2^{(d-2)}}{\log(1-t)} - \frac{(d-2)!G_{(d-2)}(t)}{\log(1-t)} \right], \quad (31)$$

and, from (29), we get

$$\gamma_{2,(d-1),0} = \gamma_{2,(d-2),1},$$

$$\gamma_{2,(d-1),k} = \gamma_{2,(d-2),k+1} - \sum_{i=0}^{k-1} \frac{\gamma_{2,(d-1),i}}{k-i+1}, \quad k=1,2,\dots \quad (32)$$

Integrating (d) times yield

$$\begin{aligned} y(x_{n+2}) &= y(x_n) + hy'(x_n) + \dots + \frac{h^{(d-1)}}{(d-1)!} y^{(d-1)}(x_n) \\ &+ \int_{x_n}^{x_{n+2}} \frac{(x_{n+2}-x)^{(d-1)}}{(d-1)!} \\ &\times f(x, y, y', y'', \dots, y^{(d-1)}) dx \end{aligned} \quad (33)$$

or, in the backward difference formulation, given by

$$y(x_{n+2}) = y(x_n) + hy'(x_n) + \dots + \frac{h^{(d-1)}}{(d-1)!} y^{(d-1)}(x_n)$$

$$+ h^{(d)} \sum_{i=0}^{k-1} \gamma_{2,(d),i} \nabla^i f_n, \quad (34)$$

where

$$\gamma_{2,(d),i} = (-1)^i \int_0^2 \frac{(2-s)^{(d-1)}}{(d-1)!} \binom{-s}{i} ds. \quad (35)$$

The generating function

$$G_{(d)}(t) = \sum_{i=0}^{\infty} \gamma_{2,(d),i} t^i. \quad (36)$$

Substituting (35) into $G_{(d)}(t)$ above yields

$$G_{(d)}(t) = \int_0^2 \frac{(2-s)^{(d-1)}}{(d-1)!} e^{-s \log(1-t)} ds. \quad (37)$$

As in (30), we now substitute $G_{(d-1)}(t)$ in (37) giving

$$G_{(d)}(t) = \frac{1}{(d-1)!} \left[\frac{2^{(d-1)}}{\log(1-t)} - \frac{(d-1)!G_{(d-1)}(t)}{\log(1-t)} \right]. \quad (38)$$

Equation (38) can be written as

$$\begin{aligned} & \left(\sum_{i=0}^{\infty} \gamma_{2,(d),i} t^i \right) \log(1-t) \\ &= \frac{1}{(d-1)!} \left[2^{(d-1)} - (d-1)!G_{(d-1)}(t) \right] \end{aligned} \quad (39)$$

or

$$\begin{aligned} & (\gamma_{2,(d),0} + \gamma_{2,(d),1}t + \gamma_{2,(d),2}t^2 + \dots) \left(t + \frac{t^2}{2} + \frac{t^3}{3} + \dots \right) \\ &= \frac{1}{(d-1)!} \\ & \times \left[+ (d-1)! \left(\gamma_{2,(d-1),0} + \gamma_{2,(d-1),1}t + \gamma_{2,(d-1),2}t^2 + \dots \right) \right]. \end{aligned} \quad (40)$$

Hence the coefficients of $\gamma_{2,(d),k}$ in relation to coefficients of the previous order $\gamma_{2,(d-1),k}$ are given by

$$\begin{aligned} & \sum_{i=0}^k \frac{\gamma_{2,(d),i}}{k-i+1} = \gamma_{2,(d-1),k+1}, \\ & \gamma_{2,(d),0} = \gamma_{2,(d-1),1}, \\ & \gamma_{2,(d),k} = \gamma_{2,(d-1),k+1} - \sum_{i=0}^{k-1} \frac{\gamma_{2,(d),i}}{k-i+1}, \quad k = 1, 2, \dots \end{aligned} \quad (41)$$

4. Derivation for Higher-Order Implicit Integration Coefficients

4.1. For the First Point. The derivation for up to third-order implicit integration coefficients for the first point y_{n+1} has been given by Suleiman et al. [8].

4.2. For the Second Point. Integrating (3) once yields

$$\begin{aligned} & y^{(d-1)}(x_{n+2}) \\ &= y^{(d-1)}(x_n) + \int_{x_n}^{x_{n+2}} f(x, y, y', y'', \dots, y^{(d-1)}) dx. \end{aligned} \quad (42)$$

Let $P_n(x)$ be the interpolating polynomial which interpolates the k values $(x_n, f_n), (x_{n-1}, f_{n-1}), \dots, (x_{n-k+1}, f_{n-k+1})$; then

$$P_n(x) = \sum_{i=0}^k (-1)^i \binom{-s}{i} \nabla^i f_{n+2}. \quad (43)$$

As in the previous derivation, we choose

$$x = x_{n+2} + sh \quad \text{or} \quad s = \frac{x - x_{n+2}}{h}. \quad (44)$$

Replacing x by s yields

$$y^{(d-1)}(x_{n+2}) = y^{(d-1)}(x_n) + \int_{-2}^0 \sum_{i=0}^k (-1)^i \binom{-s}{i} \nabla^i f_{n+2} h ds. \quad (45)$$

Simplify

$$y^{(d-1)}(x_{n+2}) = y^{(d-1)}(x_n) + h \sum_{i=0}^k \gamma_{2,1,i}^* \nabla^i f_{n+2}, \quad (46)$$

where

$$\gamma_{2,1,i}^* = (-1)^i \int_{-2}^0 \binom{-s}{i} ds. \quad (47)$$

Define the generating function $G_1^*(t)$ for the coefficient $\gamma_{2,1,i}^*$ as follows:

$$G_1^*(t) = \sum_{i=0}^{\infty} \gamma_{2,1,i}^* t^i. \quad (48)$$

or

$$G_1^*(t) = \sum_{i=0}^{\infty} (-t)^i \int_{-2}^0 \binom{-s}{i} ds, \quad (49)$$

$$G_1^*(t) = \int_{-2}^0 (1-t)^{-s} ds, \quad (50)$$

$$G_1^*(t) = \int_{-2}^0 e^{-s \log(1-t)} ds, \quad (51)$$

which leads to

$$G_1^*(t) = - \left[\frac{1}{\log(1-t)} - \frac{(1-t)^2}{\log(1-t)} \right]. \quad (52)$$

For the case $t = 2$, the approximate solution of y has the form

$$\begin{aligned} & y^{(d-2)}(x_{n+2}) = y^{(d-2)}(x_n) + h y^{(d-1)}(x_n) \\ & + \int_{x_n}^{x_{n+2}} \frac{(x_{n+2} - x)^{(1)}}{1!} \\ & \times f(x, y, y', y'', \dots, y^{(d-1)}) dx. \end{aligned} \quad (53)$$

The coefficients are given by

$$\gamma_{2,2,i}^* = (-1)^i \int_{-2}^0 \frac{(-s)}{1!} \binom{-s}{i} ds, \quad (54)$$

where $\gamma_{2,2,i}^*$ are the coefficients of the backward difference formulation of (54) which can be represented by

$$y^{(d-2)}(x_{n+2}) = y^{(d-2)}(x_n) + h y^{(d-1)}(x_n) + h^2 \sum_{i=0}^k \gamma_{2,2,i}^* \nabla^i f_{n+2}. \quad (55)$$

Define the generating function of the coefficient $\gamma_{2,2,i}^*$ as follows:

$$G_2^*(t) = \sum_{i=0}^{\infty} \gamma_{2,2,i}^* t^i. \quad (56)$$

Substituting (54) into $G_2^*(t)$ above gives

$$G_2^*(t) = \int_{-2}^0 \frac{(-s)}{1!} e^{-s \log(1-t)} ds. \quad (57)$$

Solving (57) with the substitution of (51) produces the relationship

$$G_2^*(t) = \frac{1}{1!} \left[\frac{2(1-t)^2}{\log(1-t)} - \frac{1!G_1^*(t)}{\log(1-t)} \right]. \quad (58)$$

By using the same process previously, we note that for integrating $(d-1)$ times yield

$$G_{(d-1)}^*(t) = \int_{-2}^0 \frac{(-s)^{(d-2)}}{(d-2)!} e^{-s \log(1-t)} ds, \quad (59)$$

$$G_{(d-1)}^*(t) = \frac{1}{(d-2)!} \left[\frac{2^{(d-2)}(1-t)^2}{\log(1-t)} - \frac{(d-2)!G_{(d-2)}^*(t)}{\log(1-t)} \right]. \quad (60)$$

Integrating (d) times yield

$$\begin{aligned} y(x_{n+2}) &= y(x_n) + hy'(x_n) + \cdots + \frac{h^{(d-1)}}{(d-1)!} y^{(d-1)}(x_n) \\ &+ \int_{x_n}^{x_{n+2}} \frac{(x_{n+2} - x)^{(d-1)}}{(d-1)!} \\ &\times f(x, y, y', y'', \dots, y^{(d-1)}) dx. \end{aligned} \quad (61)$$

The coefficients are given by

$$\gamma_{2,(d),i}^* = (-1)^i \int_{-2}^0 \frac{(-s)^{(d-1)}}{(d-1)!} \binom{-s}{i} ds, \quad (62)$$

where $\gamma_{2,(d),i}^*$ are the coefficients of the backward difference formulation of (62) which can be represented by

$$\begin{aligned} y(x_{n+2}) &= y(x_n) + hy'(x_n) + \cdots + \frac{h^{(d-1)}}{(d-1)!} y^{(d-1)}(x_n) \\ &+ h^{(d)} \sum_{i=0}^k \gamma_{2,(d),i}^* \nabla^i f_{n+2}. \end{aligned} \quad (63)$$

Define the generating function $G_{(d)}^*(t)$ of the coefficient $\gamma_{2,(d),i}^*$ as follows:

$$G_{(d)}^*(t) = \sum_{i=0}^{\infty} \gamma_{2,(d),i}^* t^i. \quad (64)$$

Substituting (62) into $G_{(d)}^*(t)$ above gives

$$G_{(d)}^*(t) = \int_{-2}^0 \frac{(-s)^{(d-1)}}{(d-1)!} e^{-s \log(1-t)} ds. \quad (65)$$

Solving (65) with the substitution of (59) produces the relationship

$$G_{(d)}^*(t) = \frac{1}{(d-1)!} \left[\frac{2^{(d-1)}(1-t)^2}{\log(1-t)} - \frac{(d-1)!G_{(d-1)}^*(t)}{\log(1-t)} \right]. \quad (66)$$

5. The Relationship between the Explicit and Implicit Coefficients

5.1. For the First Point. Calculating the integration coefficients directly is time consuming when large numbers of integration are involved. An efficient technique of computing the coefficients is by formulating a recursive relationship between them. With this recursive relationship, we are able to obtain the implicit integration coefficient with minimal effort. The relationship between the explicit and implicit coefficients for the first point y_{n+1} is already given by Suleiman et al. [8].

5.2. For the Second Point. For first-order coefficients,

$$G_1^*(t) = - \left[\frac{1}{\log(1-t)} - \frac{(1-t)^2}{\log(1-t)} \right]. \quad (67)$$

It can be written as

$$G_1^*(t) = -(1-t)^2 \left[\frac{1}{(1-t)^2 \log(1-t)} - \frac{1}{\log(1-t)} \right]. \quad (68)$$

By substituting

$$G_1(t) = - \left[\frac{1}{(1-t)^2 \log(1-t)} - \frac{1}{\log(1-t)} \right] \quad (69)$$

into (68), we have

$$\begin{aligned} G_1^*(t) &= (1-t)^2 G_1(t), \\ \left(\sum_{i=0}^{\infty} \gamma_{2,1,i}^* t^i \right) &= (1-t)^2 \left(\sum_{i=0}^{\infty} \gamma_{2,1,i} t^i \right). \end{aligned} \quad (70)$$

Expanding the equation yields

$$\begin{aligned} &(\gamma_{2,1,0}^* + \gamma_{2,1,1}^* t + \gamma_{2,1,2}^* t^2 + \cdots) \\ &= \frac{1}{(1+2t+3t^2+\cdots)} (\gamma_{2,1,0} + \gamma_{2,1,1} t + \gamma_{2,1,2} t^2 + \cdots), \end{aligned}$$

TABLE 1: The explicit integration coefficients for k from 0 to 6 (for y_{n+2}).

k	0	1	2	3	4	5	6
$\gamma_{2,1,k}$	2	2	7/3	8/3	269/90	33/10	13613/3780
$\gamma_{2,2,k}$	2	4/3	4/3	62/45	43/30	94/63	1466/945
$\gamma_{2,3,k}$	4/3	2/3	3/5	26/45	359/630	179/315	16159/28350

TABLE 2: The implicit integration coefficients for k from 0 to 6 (for y_{n+2}).

k	0	1	2	3	4	5	6
$\gamma_{2,1,k}^*$	2	-2	1/3	0	-1/90	-1/90	-8/945
$\gamma_{2,2,k}^*$	2	-8/3	2/3	4/90	1/90	1/315	1/1890
$\gamma_{2,3,k}^*$	4/3	-2	3/5	2/45	1/70	2/315	47/14175

$$\begin{aligned} & (\gamma_{2,1,0}^* + \gamma_{2,1,1}^* t + \gamma_{2,1,2}^* t^2 + \dots) (1 + 2t + 3t^2 + \dots) \\ &= (\gamma_{2,1,0} + \gamma_{2,1,1} t + \gamma_{2,1,2} t^2 + \dots). \end{aligned} \quad (71)$$

This gives the recursive relationship

$$\sum_{i=0}^k (k-i+1) \gamma_{2,1,i}^* = \gamma_{2,1,k}. \quad (72)$$

For second-order coefficient,

$$G_2^*(t) = \frac{1}{1!} \left[\frac{2(1-t)^2}{\log(1-t)} - \frac{1!G_1^*(t)}{\log(1-t)} \right]. \quad (73)$$

It can be written as

$$G_2^*(t) = \frac{(1-t)^2}{1!} \left[\frac{2}{\log(1-t)} - \frac{1!G_1^*(t)}{(1-t)^2 \log(1-t)} \right]. \quad (74)$$

Substituting (70) into the equation above gives

$$G_2^*(t) = \frac{(1-t)^2}{1!} \left[\frac{2}{\log(1-t)} - \frac{1!(1-t)^2 G_1(t)}{(1-t)^2 \log(1-t)} \right] \quad (75)$$

or

$$G_2^*(t) = \frac{(1-t)^2}{1!} \left[\frac{2}{\log(1-t)} - \frac{1!G_1(t)}{\log(1-t)} \right]. \quad (76)$$

Substituting (25) into (76) gives

$$\begin{aligned} G_2^*(t) &= (1-t)^2 G_2(t), \\ \left(\sum_{i=0}^{\infty} \gamma_{2,2,i}^* t^i \right) &= (1-t)^2 \left(\sum_{i=0}^{\infty} \gamma_{2,2,i} t^i \right). \end{aligned} \quad (77)$$

Expanding the equation yields

$$\begin{aligned} & (\gamma_{2,2,0}^* + \gamma_{2,2,1}^* t + \gamma_{2,2,2}^* t^2 + \dots) \\ &= \frac{1}{(1+2t+3t^2+\dots)} (\gamma_{2,2,0} + \gamma_{2,2,1} t + \gamma_{2,2,2} t^2 + \dots), \end{aligned}$$

$$\begin{aligned} & (\gamma_{2,2,0}^* + \gamma_{2,2,1}^* t + \gamma_{2,2,2}^* t^2 + \dots) (1 + 2t + 3t^2 + \dots) \\ &= (\gamma_{2,2,0} + \gamma_{2,2,1} t + \gamma_{2,2,2} t^2 + \dots). \end{aligned} \quad (78)$$

This gives the recursive relationship

$$\sum_{i=0}^k (k-i+1) \gamma_{2,2,i}^* = \gamma_{2,2,k}. \quad (79)$$

By using the same process previously, we note that, for $(d-1)$ -order coefficient, we have

$$G_{(d-1)}^*(t) = (1-t)^2 G_{(d-1)}(t), \quad (80)$$

which leads to a recursive relationship

$$\sum_{i=0}^k (k-i+1) \gamma_{2,(d-1),i}^* = \gamma_{2,(d-1),k}. \quad (81)$$

For (d) -order coefficient, we have

$$G_{(d)}^*(t) = \frac{1}{(d-1)!} \left[\frac{2^{(d-1)}(1-t)^2}{\log(1-t)} - \frac{(d-1)!G_{(d-1)}^*(t)}{\log(1-t)} \right]. \quad (82)$$

It can be written as

$$G_{(d)}^*(t) = \frac{(1-t)^2}{(d-1)!} \left[\frac{2^{(d-1)}}{\log(1-t)} - \frac{(d-1)!G_{(d-1)}^*(t)}{(1-t)^2 \log(1-t)} \right]. \quad (83)$$

Substituting (80) into (83) gives

$$G_{(d)}^*(t) = \frac{(1-t)^2}{(d-1)!} \left[\frac{2^{(d-1)}}{\log(1-t)} - \frac{(d-1)!(1-t)^2 G_{(d-1)}(t)}{(1-t)^2 \log(1-t)} \right] \quad (84)$$

TABLE 3: List of test problems.

	Problem	Initial value	Interval
1	$y' = y$ Exact solution: $y(x) = e^x$ Source: artificial problem	$y(0) = 1$	$0 \leq x \leq 20$
2	$y'' = -2y' + 3y$ Exact solution: $y(x) = e^x + e^{-3x}$ Source: Suleiman [9]	$y(0) = 2$ $y'(0) = -2$	$0 \leq x \leq 64$
3	$y_1'' = -\frac{y_1}{r^3}$ $y_2'' = -\frac{y_2}{r^3}$ $r = (y_1^2 + y_2^2)^2$ Exact solution: $y_1(x) = \cos x$ $y_2(x) = \sin x$ Source: Shampine and Gordon [10]	$y_1(0) = 1$ $y_1'(0) = 0$ $y_2(0) = 0$ $y_2'(0) = 1$	$0 \leq x \leq 16\pi$
4	$y_1'' = -2y_1' - 5y_2 + 3$ $y_2' = y_1' + 2y_2$ Exact solution: $y_1(x) = 2 \cos x + 6 \sin x - 6x - 2$ $y_1(x) = 2 \sin x - 2 \cos x + 3$ Source: Suleiman [9]	$y_1(0) = 0$ $y_1'(0) = 0$ $y_2(0) = 1$	$0 \leq x \leq 16\pi$
5	$y''' = -\frac{1}{x}y'' + \frac{1}{x^2}y' + \frac{1}{x}$ Exact solution: $y(x) = \frac{x^2}{8} \left(2 \ln\left(\frac{x}{2}\right) - \left(\frac{33}{13}\right) - \frac{2}{3} \ln(2) \right) + \left(\frac{1}{3} - \frac{26}{21} \ln\left(\frac{x}{2}\right) \right) \ln(2) + \frac{33}{26}$ Source: Russell and Shampine [11]	$y(1) = \frac{26}{21} \ln^2(2) + \frac{99}{104}$ $y'(1) = -\frac{40}{21} \ln(2) - \frac{5}{13}$ $y''(1) = \frac{3}{26} + \frac{4}{7} \ln(2)$	$1 \leq x \leq 50$

TABLE 4: Numerical result for Problem 1.

H	Method	NS	$\text{Log}_{10}(\text{MAXE})$	Time
10^{-2}	2PBBD	1000	-5.40549	6914
	2PBDD	1000	-5.88584	9509
	1PBD	2000	-6.77769	6912
10^{-3}	2PBBD	10000	-7.34272	49146
	2PBDD	10000	-8.87606	54025
	1PBD	20000	-9.77432	50095
10^{-4}	2PBBD	100000	-9.40044	195037
	2PBDD	100000	-10.67248	256339
	1PBD	200000	-10.48025	197800
10^{-5}	2PBBD	1000000	-9.37408	1759047
	2PBDD	1000000	-9.37799	2055758
	1PBD	2000000	-9.23059	1638426
10^{-6}	2PBBD	10000000	-8.81572	17500121
	2PBDD	10000000	-8.81572	19917218
	1PBD	20000000	-8.28209	14653209
10^{-7}	2PBBD	100000000	-7.78104	176146062
	2PBDD	100000000	-7.78104	199668362
	1PBD	200000000	-7.39425	145626635

TABLE 5: Numerical result for Problem 2.

H	Method	NS	$\text{Log}_{10}\text{MAXE}$	Time
10^{-2}	2PBBD	3200	-4.87654	21516
	2PBDD	3200	-4.58643	23350
	1PBD	6400	-5.83463	22890
10^{-3}	2PBBD	32000	-7.86398	86824
	2PBDD	32000	-7.56556	111374
	1PBD	64000	-8.80976	87426
10^{-4}	2PBBD	320000	-9.96338	776833
	2PBDD	320000	-9.82447	828900
	1PBD	640000	-8.99888	725855
10^{-5}	2PBBD	3200000	-8.68100	7769459
	2PBDD	3200000	-8.68099	8208180
	1PBD	6400000	-8.04024	6474481
10^{-6}	2PBBD	32000000	-7.64880	77008100
	2PBDD	32000000	-7.64879	81918617
	1PBD	64000000	-7.15759	63364050
10^{-7}	2PBBD	320000000	-6.53269	773213500
	2PBDD	320000000	-6.53269	817874121
	1PBD	640000000	-6.28394	616850084

or

$$G_{(d)}^*(t) = \frac{(1-t)^2}{(d-1)!} \left[\frac{2^{(d-1)}}{\log(1-t)} - \frac{(d-1)!G_{(d-1)}(t)}{\log(1-t)} \right]. \quad (85)$$

Substituting

$$G_{(d)}(t) = \frac{1}{(d-1)!} \left[\frac{2^{(d-1)}}{\log(1-t)} - \frac{(d-1)!G_{(d-1)}(t)}{\log(1-t)} \right] \quad (86)$$

into (85) leads to

$$\begin{aligned} G_{(d)}^*(t) &= (1-t)^2 G_{(d)}(t), \\ \left(\sum_{i=0}^{\infty} \gamma_{2,(d),i}^* t^i \right) &= (1-t)^2 \left(\sum_{i=0}^{\infty} \gamma_{2,(d),i} t^i \right). \end{aligned} \quad (87)$$

Expanding the equation yields

$$\begin{aligned} &(\gamma_{2,(d),0}^* + \gamma_{2,(d),1}^* t + \gamma_{2,(d),2}^* t^2 + \cdots) \\ &= \frac{1}{(1+2t+3t^2+\cdots)} \\ &\times (\gamma_{2,(d),0} + \gamma_{2,(d),1} t + \gamma_{2,(d),2} t^2 + \cdots), \end{aligned}$$

$$\begin{aligned} &(\gamma_{2,(d),0}^* + \gamma_{2,(d),1}^* t + \gamma_{2,(d),2}^* t^2 + \cdots) (1+2t+3t^2+\cdots) \\ &= (\gamma_{2,(d),0} + \gamma_{2,(d),1} t + \gamma_{2,(d),2} t^2 + \cdots), \end{aligned} \quad (88)$$

which leads to a recursive relationship

$$\sum_{i=0}^k (k-i+1) \gamma_{2,(d),i}^* = \gamma_{2,(d),k}. \quad (89)$$

Tables 1 and 2 are a few examples of the explicit and implicit integration coefficients.

6. Problem Tested

The problems shown in Table 3 are used to test the performance of the method.

7. Numerical Result

Tables 4, 5, 6, 7, and 8 give the numerical results for problems given in the previous section. The results for the 2PBBD are compared with those of 2PBDD and 1PBD according to Omar [5] and Suleiman et al. [8], respectively. Also given are graphs, where $\text{Log}_{10}(\text{MAXE})$ is plotted against $\text{Log}_{10}(H)$ and $\text{Log}_{10}(\text{Time})$. The following notations are used in the tables:

H : step size,

2PBBD: 2-point block backward difference method,

2PBDD: 2-point block divided difference method,

TABLE 6: Numerical result for Problem 3.

H	Method	NS	$\text{Log}_{10}\text{MAXE}$	Time
10^{-2}	2PBBD	2513	-6.17611	24465
	2PBDD	2513	-5.87541	24221
	1PBD	5026	-7.07922	28522
10^{-3}	2PBBD	25133	-9.17248	152099
	2PBDD	25133	-8.87081	159389
	1PBD	50265	-10.01264	139073
10^{-4}	2PBBD	251328	-10.04346	1418931
	2PBDD	251328	-10.0434	1366837
	1PBD	502655	-9.26354	1219971
10^{-5}	2PBBD	2513274	-8.80560	14315026
	2PBDD	2513274	-8.80560	13192255
	1PBD	5026548	-8.32281	11525418
10^{-6}	2PBBD	25132742	-7.99962	141938590
	2PBDD	25132742	-7.99962	130102816
	1PBD	50265482	-7.45728	106896913
10^{-7}	2PBBD	251327412	-6.87690	1412455700
	2PBDD	251327412	-6.87690	1300828490
	1PBD	502654824	-6.44359	1063913703

TABLE 7: Numerical result for Problem 4.

H	Method	NS	$\text{Log}_{10}\text{MAXE}$	Time
10^{-2}	2PBBD	2513	-5.29159	23909
	2PBDD	2513	-5.69715	26684
	1PBD	5026	-6.58585	31772
10^{-3}	2PBBD	25133	-7.74946	121769
	2PBDD	25133	-8.68299	131297
	1PBD	50265	-9.58464	121536
10^{-4}	2PBBD	251328	-9.39201	1102741
	2PBDD	251328	-10.04970	1068239
	1PBD	502655	-9.26367	1046930
10^{-5}	2PBBD	2513274	-8.80463	10004344
	2PBDD	2513274	-8.80548	9407995
	1PBD	5026548	-8.32278	10087395
10^{-6}	2PBBD	25132742	-7.99900	99147571
	2PBDD	25132742	-7.99900	91521335
	1PBD	50265482	-7.45728	93207026
10^{-7}	2PBBD	251327412	-6.87692	988919951
	2PBDD	251327412	-6.87692	911824508
	1PBD	502654824	-6.44359	922126253

1PBD: 1-point backward difference method,
 NS: total number of steps,
 MAXE: maximum error,
 TIME: total execution times (in microsecond).

Two sets of scaled graphs were plotted, namely, (i) $\text{Log}_{10}(\text{MAXE})$ against $\text{Log}_{10}(H)$ and (ii) $\text{Log}_{10}(\text{MAXE})$ against $\text{Log}_{10}(\text{TIME})$. For a particular abscissa, the lowest value of the ordinate is considered to be the more efficient at the abscissa considered. Hence, for the first set of graphs,

TABLE 8: Numerical result for Problem 5.

H	Method	NS	$\text{Log}_{10} \text{MAXE}$	Time
10^{-2}	2PBBD	2450	-3.65914	25345
	2PBDD	2450	-3.37200	26172
	1PBD	4900	-5.08070	27139
10^{-3}	2PBBD	24500	-6.62483	106625
	2PBDD	24500	-6.32538	116848
	1PBD	49000	-8.06583	108650
10^{-4}	2PBBD	245000	-9.64707	1021740
	2PBDD	245000	-9.30862	996446
	1PBD	490000	-10.55259	888813
10^{-5}	2PBBD	2450000	-10.06216	9797882
	2PBDD	2450000	-10.06228	8683164
	1PBD	4900000	-9.42795	8211650
10^{-6}	2PBBD	24500000	-8.99818	97582403
	2PBDD	24500000	-8.99818	85958740
	1PBD	49000000	-8.73189	80890328
10^{-7}	2PBBD	245000000	-8.10589	978631746
	2PBDD	245000000	-8.10589	844019109
	1PBD	490000000	-7.70882	790057441

FIGURE 2: Graph of $\text{Log}_{10}(\text{MAXE})$ plotted against $\text{Log}_{10}(H)$ for Problem 1. FIGURE 3: Graph of $\text{Log}_{10}(\text{MAXE})$ plotted against $\text{Log}_{10}(\text{TIME})$ for Problem 1.

that is, $\text{Log}_{10}(\text{MAXE})$ against $\text{Log}_{10}(H)$, the method 2PBBD is better when $\text{Log}_{10}(H) < -5$, and loses out for value of $\text{Log}_{10}(H) > -5$. For the second set of graphs, as the time

increases, the 2PBBD is the method of choice since it is lowest for all five sets of problems (see Figures 2, 3, 4, 5, 6, 7, 8, 9, 10, and 11). It gives us the impression of stability, where the errors

FIGURE 4: Graph of $\text{Log}_{10}(\text{MAXE})$ plotted against $\text{Log}_{10}(H)$ for Problem 2.

FIGURE 6: Graph of $\text{Log}_{10}(\text{MAXE})$ plotted against $\text{Log}_{10}(H)$ for Problem 3.

FIGURE 5: Graph of $\text{Log}_{10}(\text{MAXE})$ plotted against $\text{Log}_{10}(\text{TIME})$ for Problem 2.

FIGURE 7: Graph of $\text{Log}_{10}(\text{MAXE})$ plotted against $\text{Log}_{10}(\text{TIME})$ for Problem 3.

grow most slowly compared with the other methods, 2PBDD and 1PBD.

8. Conclusion

Of the 3 methods, 2PBBD is therefore preferred as a general code and should be included as a collection of methods, as

FIGURE 8: Graph of $\text{Log}_{10}(\text{MAXE})$ plotted against $\text{Log}_{10}(H)$ for Problem 4.

FIGURE 10: Graph of $\text{Log}_{10}(\text{MAXE})$ plotted against $\text{Log}_{10}(H)$ for Problem 5.

FIGURE 9: Graph of $\text{Log}_{10}(\text{MAXE})$ plotted against $\text{Log}_{10}(\text{TIME})$ for Problem 4.

a code for parallelization purposes, as an assembly of codes to be tested and studied, and as a code for solving ODEs.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

FIGURE 11: Graph of $\text{Log}_{10}(\text{MAXE})$ plotted against $\text{Log}_{10}(\text{TIME})$ for Problem 5.

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