Research Article

Local Fractional Derivative Boundary Value Problems for Tricomi Equation Arising in Fractal Transonic Flow

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The local fractional decomposition method is applied to obtain the nondifferentiable numerical solutions for the local fractional Tricomi equation arising in fractal transonic flow with the local fractional derivative boundary value conditions.

1. Introduction

The Tricomi equation [1] is the second-order linear partial differential equations of mixed type, which had been applied to describe the theory of plane transonic flow [2–7]. The Tricomi equation was used to describe the differentiable problems for the theory of plane transonic flow. However, for the fractal theory of plane transonic flow with nondifferentiable terms, the Tricomi equation is not applied to describe them. Recently, the local fractional calculus [8] was applied to describe the nondifferentiable problems, such as the fractal heat conduction [8, 9], the damped and dissipative wave equations in fractal strings [10], the local fractional Schrödinger equation [11], the wave equation on Cantor sets [12], the Navier-Stokes equations on Cantor sets [13], and others [14–19]. Recently, the local fractional Tricomi equation arising in fractal transonic flow was suggested in the form [19]

$$\frac{y^\alpha}{\Gamma(1+\alpha)} \frac{\partial^{2\alpha} u(x, y)}{\partial x^{2\alpha}} + \frac{\partial^{2\alpha} u(x, y)}{\partial y^{2\alpha}} = 0, \quad (1)$$

where the quantity $u(x, y)$ is the nondifferentiable function, and the local fractional operator denotes [8]

$$\frac{\partial^\alpha u(x, t)}{\partial t^\alpha} = \frac{\Delta^\alpha (u(x, t) - u(x, t_0))}{(t - t_0)^\alpha}, \quad (2)$$

where

$$\Delta^\alpha (u(x, t) - u(x, t_0)) \cong \Gamma(1+\alpha) [u(x, t) - u(x, t_0)]. \quad (3)$$

The local fractional decomposition method [12] was used to solve the diffusion equation on Cantor time-space. The aim of this paper is to use the local fractional decomposition method to solve the local fractional Tricomi equation arising in fractal transonic flow with the local fractional derivative boundary value conditions. The structure of this paper is as follows. In Section 2, the local fractional integrals and derivatives are introduced. In Section 3, the local fractional decomposition method is suggested. In Section 4, the nondifferentiable numerical solutions for local fractional Tricomi equation with the local fractional derivative boundary value conditions are given. Finally, the conclusions are shown in Section 5.

2. Local Fractional Integrals and Derivatives

In this section, we introduce the basic theory of the local fractional integrals and derivatives [8–19], which are applied in the paper.

Definition 1 (see [8–19]). For $|x - x_0| < \delta$, for $\varepsilon, \delta > 0$ and $\varepsilon \in R$, we give the function $f(x) \in C_\alpha(a, b)$, when

$$|f(x) - f(x_0)| < \varepsilon^\alpha, \quad 0 < \alpha \leq 1, \quad (4)$$

is valid.

Definition 2 (see [8–19]). Let (t_j, t_{j+1}) , $j = 0, \dots, N-1$, $t_0 = a$, and $t_N = b$ with $\Delta t_j = t_{j+1} - t_j$ and $\Delta t = \max\{\Delta t_0, \Delta t_1, \dots\}$, be a partition of the interval $[a, b]$. The local fractional integral of $f(x)$ in the interval $[a, b]$ is defined as

$$\begin{aligned} {}_a I_b^{(\alpha)} f(x) &= \frac{1}{\Gamma(1+\alpha)} \int_a^b f(t) (dt)^\alpha \\ &= \frac{1}{\Gamma(1+\alpha)} \lim_{\Delta t \rightarrow 0} \sum_{j=0}^{N-1} f(t_j) (\Delta t_j)^\alpha. \end{aligned} \quad (5)$$

As the inverse operator of (6), local fractional derivative of $f(x)$ of the order α in the interval (a, b) is presented as [8–19]

$$\frac{d^\alpha f(x_0)}{dx^\alpha} = D_x^{(\alpha)} f(x_0) = \frac{\Delta^\alpha (f(x) - f(x_0))}{(x - x_0)^\alpha}, \quad (6)$$

where

$$\Delta^\alpha (f(x) - f(x_0)) \cong \Gamma(1+\alpha) [f(x) - f(x_0)]. \quad (7)$$

The formulas of local fractional derivative and integral, which appear in the paper, are valid [8]:

$$\begin{aligned} \frac{d^\alpha}{dx^\alpha} \frac{x^{n\alpha}}{\Gamma(1+n\alpha)} &= \frac{x^{(n-1)\alpha}}{\Gamma(1+(n-1)\alpha)}, \quad n \in N, \\ \frac{d^\alpha}{dx^\alpha} E_\alpha(x^\alpha) &= E_\alpha(x^\alpha), \\ \frac{d^\alpha}{dx^\alpha} \sin_\alpha(x^\alpha) &= \cos_\alpha(x^\alpha), \\ \frac{d^\alpha}{dx^\alpha} \cos_\alpha(x^\alpha) &= -\sin_\alpha(x^\alpha), \\ {}_0 I_x^{(\alpha)} \frac{x^{(n-1)\alpha}}{\Gamma(1+(n-1)\alpha)} &= \frac{x^{n\alpha}}{\Gamma(1+n\alpha)}, \quad n \in N, \\ {}_0 I_x^{(\alpha)} \cos_\alpha(x^\alpha) &= \sin_\alpha(x^\alpha). \end{aligned} \quad (8)$$

3. Analysis of the Method

In this section, we give the local fractional decomposition method [12]. We consider the following local fractional operator equation in the form

$$L_\alpha^{(2)} u + R_\alpha u = 0, \quad (9)$$

where $L_\alpha^{(2)}$ is linear local fractional operators of the order 2α with respect to x and R_α is the linear local fractional operators of order less than 2α . We write (9) as

$$L_{xx}^{(2\alpha)} u + R_\alpha u = 0, \quad (10)$$

where the 2α -th local fractional differential operator denotes

$$L_\alpha^{(n)} = L_{xx}^{(2\alpha)} = \frac{\partial^{2\alpha}}{\partial x^{2\alpha}}, \quad (11)$$

and the linear local fractional operators of order less than 2α denote

$$R_\alpha = \frac{\Gamma(1+\alpha)}{y^\alpha} \frac{\partial^{2\alpha} u(x, y)}{\partial y^{2\alpha}}. \quad (12)$$

Define the 2α -fold local fractional integral operator

$$L_\alpha^{(-2\alpha)} m(s) = {}_0 I_x^{(\alpha)} {}_0 I_x^{(\alpha)} m(s) \quad (13)$$

so that we obtain the local fractional iterative formula as follows:

$$L_\alpha^{(-2\alpha)} L_{xx}^{(2\alpha)} u + L_\alpha^{(-2\alpha)} L_\alpha^{(-2\alpha)} R_\alpha u = 0, \quad (14)$$

which leads to

$$u(x) = u_0(x) + L_\alpha^{(-2\alpha)} L_\alpha^{(-2\alpha)} R_\alpha u. \quad (15)$$

Therefore, for $n \geq 0$, we obtain the recurrence formula in the form

$$\begin{aligned} u_{n+1}(x) &= L_\alpha^{(-2)} R_\alpha u_n(x), \\ u_0(x) &= r(x). \end{aligned} \quad (16)$$

Finally, the solution of (9) reads

$$u(x) = \lim_{n \rightarrow \infty} \phi_n(x) = \lim_{n \rightarrow \infty} \sum_{n=0}^{\infty} u_n(x). \quad (17)$$

4. The Nondifferentiable Numerical Solutions

In this section, we discuss the nondifferentiable numerical solutions for the local fractional Tricomi equation arising in fractal transonic flow with the local fractional derivative boundary value conditions.

Example 1. We consider the initial-boundary value conditions for the local fractional Tricomi equation in the form [19]

$$u(0, y) = 0, \quad (18)$$

$$u(l, y) = 0, \quad (19)$$

$$u(x, 0) = \frac{x^\alpha}{\Gamma(1+\alpha)}, \quad (20)$$

$$\frac{\partial^\alpha u(x, 0)}{\partial x^\alpha} = \frac{x^\alpha}{\Gamma(1+\alpha)}. \quad (21)$$

Using (20)-(21), we structure the recurrence formula in the form

$$\begin{aligned} u_{n+1}(x, y) &= L_\alpha^{(-2)} \left[\frac{\Gamma(1+\alpha)}{y^\alpha} \frac{\partial^{2\alpha} u_n(x, y)}{\partial y^{2\alpha}} \right], \\ u_0(x, y) &= \frac{x^\alpha}{\Gamma(1+\alpha)} + \frac{x^\alpha}{\Gamma(1+\alpha)} \frac{y^\alpha}{\Gamma(1+\alpha)}. \end{aligned} \quad (22)$$

Hence, for $n = 0$, the first term of (22) reads

$$u_1(x, y) = 0. \quad (23)$$

For $n = 1$ the second term of (22) is given as

$$u_2(x, y) = 0. \quad (24)$$

Hence, we obtain

$$u_0(x, y) = u_1(x, y) = \cdots = u_n(x, y) = 0. \quad (25)$$

Finally, the solution of (9) with the local fractional derivative boundary value conditions (19)–(21) can be written as

$$\begin{aligned} u(x, y) &= \lim_{n \rightarrow \infty} \phi_n(x, y) \\ &= \lim_{n \rightarrow \infty} \sum_{n=0}^{\infty} u_n(x, y) \\ &= \frac{x^\alpha}{\Gamma(1+\alpha)} + \frac{x^\alpha}{\Gamma(1+\alpha)} \frac{y^\alpha}{\Gamma(1+\alpha)} \end{aligned} \quad (26)$$

which is in accordance with the result from the local fractional variational iteration method [19].

Example 2. Let us consider the initial-boundary value conditions for the local fractional Tricomi equation in the form

$$\begin{aligned} u(0, y) &= 0, \\ u(l, y) &= 0, \\ u(x, 0) &= 0, \\ \frac{\partial^\alpha u(x, 0)}{\partial x^\alpha} &= \cos_\alpha(x^\alpha). \end{aligned} \quad (27)$$

In view of (27), we set up the recurrence formula in the form

$$\begin{aligned} u_{n+1}(x, y) &= L_\alpha^{(-2)} \left[\frac{\Gamma(1+\alpha)}{y^\alpha} \frac{\partial^{2\alpha} u_n(x, y)}{\partial y^{2\alpha}} \right], \\ u_0(x, y) &= \cos_\alpha(x^\alpha) \frac{y^\alpha}{\Gamma(1+\alpha)}. \end{aligned} \quad (28)$$

Hence, from (28) we get the following equations:

$$\begin{aligned} u_1(x, y) &= L_\alpha^{(-2)} \left[\frac{\Gamma(1+\alpha)}{y^\alpha} \frac{\partial^{2\alpha} u_0(x, y)}{\partial y^{2\alpha}} \right] \\ &= L_\alpha^{(-2)} \left[\frac{\Gamma(1+\alpha)}{y^\alpha} \frac{\partial^{2\alpha}}{\partial y^{2\alpha}} \left(\cos_\alpha(x^\alpha) \frac{y^\alpha}{\Gamma(1+\alpha)} \right) \right] \\ &= 0, \end{aligned}$$

$$u_2(x, y) = L_\alpha^{(-2)} \left[\frac{\Gamma(1+\alpha)}{y^\alpha} \frac{\partial^{2\alpha} u_1(x, y)}{\partial y^{2\alpha}} \right]$$

$$= L_\alpha^{(-2)} \left[\frac{\Gamma(1+\alpha)}{y^\alpha} \frac{\partial^{2\alpha}}{\partial y^{2\alpha}} (0) \right]$$

$$= 0,$$

$$u_3(x, y) = L_\alpha^{(-2)} \left[\frac{\Gamma(1+\alpha)}{y^\alpha} \frac{\partial^{2\alpha} u_2(x, y)}{\partial y^{2\alpha}} \right]$$

$$= L_\alpha^{(-2)} \left[\frac{\Gamma(1+\alpha)}{y^\alpha} \frac{\partial^{2\alpha}}{\partial y^{2\alpha}} (0) \right]$$

$$= 0,$$

$$u_4(x, y) = L_\alpha^{(-2)} \left[\frac{\Gamma(1+\alpha)}{y^\alpha} \frac{\partial^{2\alpha} u_3(x, y)}{\partial y^{2\alpha}} \right]$$

$$= L_\alpha^{(-2)} \left[\frac{\Gamma(1+\alpha)}{y^\alpha} \frac{\partial^{2\alpha}}{\partial y^{2\alpha}} (0) \right]$$

$$= 0,$$

$\vdots$

$$u_n(x, y) = 0. \quad (29)$$

Finally, we obtain the solution of (9) with the local fractional derivative boundary value conditions (27), namely,

$$\begin{aligned} u(x, y) &= \lim_{n \rightarrow \infty} \phi_n(x, y) \\ &= \lim_{n \rightarrow \infty} \sum_{n=0}^{\infty} u_n(x, y) \end{aligned} \quad (30)$$

$$= \cos_\alpha(x^\alpha) \frac{y^\alpha}{\Gamma(1+\alpha)},$$

whose graph is shown in Figure 1.

Example 3. Let us consider the initial-boundary value conditions for the local fractional Tricomi equation in the form

$$\begin{aligned} u(0, y) &= 0, \\ u(l, y) &= 0, \\ u(x, 0) &= \frac{x^\alpha}{\Gamma(1+\alpha)}, \end{aligned} \quad (31)$$

$$\frac{\partial^\alpha u(x, 0)}{\partial x^\alpha} = \sin_\alpha(x^\alpha).$$

FIGURE 1: The plot of the solution of (9) with the local fractional derivative boundary value conditions (27) when $\alpha = \ln 2 / \ln 3$.

Making use of (31), the recurrence formula can be written as

$$\begin{aligned} u_{n+1}(x, y) &= L_{\alpha}^{(-2)} \left[\frac{\Gamma(1+\alpha)}{y^{\alpha}} \frac{\partial^{2\alpha} u_n(x, y)}{\partial y^{2\alpha}} \right], \\ u_0(x, y) &= \frac{x^{\alpha}}{\Gamma(1+\alpha)} + \sin_{\alpha}(x^{\alpha}) \frac{y^{\alpha}}{\Gamma(1+\alpha)}. \end{aligned} \quad (32)$$

Applying (32) gives the following equations:

$$\begin{aligned} u_1(x, y) &= L_{\alpha}^{(-2)} \left[\frac{\Gamma(1+\alpha)}{y^{\alpha}} \frac{\partial^{2\alpha} u_0(x, y)}{\partial y^{2\alpha}} \right] \\ &= L_{\alpha}^{(-2)} \left[\frac{\Gamma(1+\alpha)}{y^{\alpha}} \frac{\partial^{2\alpha}}{\partial y^{2\alpha}} \left(\frac{x^{\alpha}}{\Gamma(1+\alpha)} + \sin_{\alpha}(x^{\alpha}) \frac{y^{\alpha}}{\Gamma(1+\alpha)} \right) \right] \\ &= 0, \\ u_2(x, y) &= L_{\alpha}^{(-2)} \left[\frac{\Gamma(1+\alpha)}{y^{\alpha}} \frac{\partial^{2\alpha} u_1(x, y)}{\partial y^{2\alpha}} \right] \\ &= L_{\alpha}^{(-2)} \left[\frac{\Gamma(1+\alpha)}{y^{\alpha}} \frac{\partial^{2\alpha}}{\partial y^{2\alpha}} (0) \right] \\ &= 0, \\ u_3(x, y) &= L_{\alpha}^{(-2)} \left[\frac{\Gamma(1+\alpha)}{y^{\alpha}} \frac{\partial^{2\alpha} u_2(x, y)}{\partial y^{2\alpha}} \right] \\ &= L_{\alpha}^{(-2)} \left[\frac{\Gamma(1+\alpha)}{y^{\alpha}} \frac{\partial^{2\alpha}}{\partial y^{2\alpha}} (0) \right] \\ &= 0, \end{aligned}$$

FIGURE 2: The plot of the solution of (9) with the local fractional derivative boundary value conditions (31) when $\alpha = \ln 2 / \ln 3$.

$$\begin{aligned} u_4(x, y) &= L_{\alpha}^{(-2)} \left[\frac{\Gamma(1+\alpha)}{y^{\alpha}} \frac{\partial^{2\alpha} u_3(x, y)}{\partial y^{2\alpha}} \right] \\ &= L_{\alpha}^{(-2)} \left[\frac{\Gamma(1+\alpha)}{y^{\alpha}} \frac{\partial^{2\alpha}}{\partial y^{2\alpha}} (0) \right] \\ &= 0, \\ &\vdots \\ u_n(x, y) &= 0. \end{aligned} \quad (33)$$

Finally, the solution of (9) with the local fractional derivative boundary value conditions (31) reads

$$\begin{aligned} u(x, y) &= \lim_{n \rightarrow \infty} \phi_n(x, y) \\ &= \lim_{n \rightarrow \infty} \sum_{n=0}^{\infty} u_n(x, y) \\ &= \frac{x^{\alpha}}{\Gamma(1+\alpha)} + \sin_{\alpha}(x^{\alpha}) \frac{y^{\alpha}}{\Gamma(1+\alpha)}, \end{aligned} \quad (34)$$

and its graph is shown in Figure 2.

5. Conclusions

In this work we discussed the nondifferentiable numerical solutions for the local fractional Tricomi equation arising in fractal transonic flow with the local fractional derivative boundary value conditions by using the local fractional decomposition method and their plots were also shown in the MatLab software.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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