Research Article

The Convergence of Double-Indexed Weighted Sums of Martingale Differences and Its Application

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We investigate the complete moment convergence of double-indexed weighted sums of martingale differences. Then it is easy to obtain the Marcinkiewicz-Zygmund-type strong law of large numbers of double-indexed weighted sums of martingale differences. Moreover, the convergence of double-indexed weighted sums of martingale differences is presented in mean square. On the other hand, we give the application to study the convergence of the state observers of linear-time-invariant systems and present the convergence with probability one and in mean square.

1. Introduction

Hsu and Robbins [1] introduced the concept of complete convergence; that is, a sequence of random variables $\{X_n, n \geq 1\}$ is said to *converge completely* to a constant C if $\sum_{n=1}^{\infty} P(|X_n - C| \geq \varepsilon) < \infty$ for all $\varepsilon > 0$. By Borel-Cantelli lemma, it follows that $X_n \rightarrow C$ almost surely (a.s.). The converse is true if $\{X_n, n \geq 1\}$ is independent. But the converse cannot always be true for the dependent case. Hsu and Robbins [1] obtained that the sequence of arithmetic means of independent and identically distributed (i.i.d.) random variables converges completely to the expected value if the variance of the summands is finite. Erdős [2] proved the converse. The result of Hsu-Robbins-Erdős is a fundamental theorem in probability theory, and it has been generalized and extended in several directions by many authors. Baum and Katz [3] gave the following generalization to establish a rate of convergence in the sense of Marcinkiewicz-Zygmund-type strong law of large numbers.

Theorem 1. Let $\alpha > 1/2$, $\alpha p > 1$, and $\{X_n, n \geq 1\}$ be a sequence of i.i.d. random variables. Assume that $EX_1 = 0$ if $\alpha \leq 1$. Then the following statements are equivalent:

- (i) $E|X_1|^p < \infty$;
- (ii) $\sum_{n=1}^{\infty} n^{\alpha p - 2} P(\max_{1 \leq k \leq n} |\sum_{i=1}^k X_i| > \varepsilon n^\alpha) < \infty$ for all $\varepsilon > 0$.

Many authors have extended Theorem 1 to the martingale differences. For example, Yu [4] obtained the complete convergence for weighted sums of martingale differences; Ghosal and Chandra [5] gave the complete convergence of martingale arrays; Stoica [6, 7] investigated the Baum-Katz-Nagaev-type results for martingale differences and the rate of convergence in the strong law of large numbers for martingale differences; Wang et al. [8] also studied the complete convergence and complete moment convergence for martingale differences, which generalized some results of Stoica [6, 7]; Yang et al. [9] obtained the complete convergence for the moving average process of martingale differences and so forth. For other works about convergence analysis, one can refer to Gut [10], Chen et al. [11], Sung [12–14], Sung and Volodin [15], Hu et al. [16], and the references therein.

In this paper, we study the moment complete convergence of double-indexed weighted sums of martingale differences. Then it is easy to obtain the Marcinkiewicz-Zygmund-type strong law of large numbers of double-indexed weighted sums of martingale differences. Moreover, the convergence of double-indexed weighted sums of martingale differences is presented in mean square. For the details, see Theorem 5, Corollary 6, and Theorem 7 in Section 2. On the other hand, we give the applications of Corollary 6 and Theorem 7 to study the convergence of the state observers of linear-time-invariant systems and present their convergence with

probability one and in mean square, respectively (see Theorems 11 and 12 in Section 3).

Recall that the sequence $\{X_n, n \geq 1\}$ is stochastically dominated by a nonnegative random variable X if $\sup_{n \geq 1} P(|X_n| > t) \leq KP(X > t)$ for some positive constant K and for all $t \geq 0$.

Throughout the paper, let $\mathcal{F}_0 = \{\emptyset, \Omega\}$, $\mathbb{1}(B)$ be the indicator function of set B , $x^+ = x\mathbb{1}(x \geq 0)$, and let $K, K_1, K_2, \dots$ denote some positive constants not depending on n , which may be different in various places.

The following lemmas are useful for the proofs of the main results.

Lemma 2 (cf. Hall and Heyde [17, Theorem 2.11]). *If $\{X_i, \mathcal{F}_i, 1 \leq i \leq n\}$ are martingale differences and $p > 0$, then there exists a constant K depending only on p such that*

$$\begin{aligned} & E \left(\max_{1 \leq k \leq n} \left| \sum_{i=1}^k X_i \right|^p \right) \\ & \leq K \left\{ E \left(\sum_{i=1}^n E(X_i^2 | \mathcal{F}_{i-1}) \right)^{p/2} + E \left(\max_{1 \leq i \leq n} |X_i|^p \right) \right\}, \\ & n \geq 1. \end{aligned} \quad (1)$$

Lemma 3 (cf. Sung [12, Lemma 2.4]). *Let $\{X_n, n \geq 1\}$ and $\{Y_n, n \geq 1\}$ be sequences of random variables. Then for any $n \geq 1, q > 1, \varepsilon > 0$, and $a > 0$, one has*

$$\begin{aligned} & E \left(\max_{1 \leq j \leq n} \left| \sum_{i=1}^j (X_i + Y_i) \right| - \varepsilon a \right)^+ \\ & \leq \left(\frac{1}{\varepsilon^q} + \frac{1}{q-1} \right) \frac{1}{a^{q-1}} E \left(\max_{1 \leq j \leq n} \left| \sum_{i=1}^j X_i \right|^q \right) \\ & + E \left(\max_{1 \leq j \leq n} \left| \sum_{i=1}^j Y_i \right| \right). \end{aligned} \quad (2)$$

Lemma 4 (cf. Wang et al. [8, Lemma 2.2]). *Let $\{X_n, n \geq 1\}$ be a sequence of random variables stochastically dominated by a nonnegative random variable X . Then for any $n \geq 1, a > 0$, and $b > 0$, the following two statements hold:*

$$\begin{aligned} & E \left[|X_n|^a \mathbb{1}(|X_n| \leq b) \right] \\ & \leq K_1 \{ E[X^a \mathbb{1}(X \leq b)] + b^a P(X > b) \}, \end{aligned} \quad (3)$$

$$E \left[|X_n|^a \mathbb{1}(|X_n| > b) \right] \leq K_2 E[X^a \mathbb{1}(X > b)].$$

Consequently, $E|X_n|^a \leq K_3 EX^a$. Here K_1, K_2 , and K_3 are positive constants.

2. The Convergence of Double-Indexed Weighted Sums of Martingale Differences

First, we give the complete moment convergence of double-indexed weighted sums of martingale differences.

Theorem 5. *Let $\alpha > 1/2, p \geq 2$, and $\{X_n, \mathcal{F}_n, n \geq 1\}$ be martingale differences stochastically dominated by a nonnegative random variable X with $EX^p < \infty$. Let $\{a_{ni}, 1 \leq i \leq n, n \geq 1\}$ be a triangular array of real numbers. For some $q > 2(\alpha p - 1)/(2\alpha - 1)$, we assume that $E[\sup_{n \geq 1} E(X_n^2 | \mathcal{F}_{n-1})]^{q/2} < \infty$ and*

$$\sum_{i=1}^n |a_{ni}|^q = O(n). \quad (4)$$

Then for every $\varepsilon > 0$,

$$\sum_{n=1}^{\infty} n^{\alpha p - 2 - \alpha} E \left(\max_{1 \leq k \leq n} \left| \sum_{i=1}^k a_{ni} X_i \right| - \varepsilon n^{\alpha} \right)^+ < \infty. \quad (5)$$

Taking $p = 2l$ and $\alpha = 2/p$ for $1 \leq l < 2$ in Theorem 5, we have the following result.

Corollary 6. *Let $1 \leq l < 2, \{X_n, \mathcal{F}_n, n \geq 1\}$ be martingale differences stochastically dominated by a nonnegative random variable X with $EX^{2l} < \infty$. Let $\{a_{ni}, 1 \leq i \leq n, n \geq 1\}$ be a triangular array of real numbers. For some $q > 2l/(2-l)$, one assumes that $E[\sup_{n \geq 1} E(X_n^2 | \mathcal{F}_{n-1})]^{q/2} < \infty$ and (4) holds true. Then for every $\varepsilon > 0$,*

$$\sum_{n=1}^{\infty} n^{-1/l} E \left(\max_{1 \leq k \leq n} \left| \sum_{i=1}^k a_{ni} X_i \right| - \varepsilon n^{1/l} \right)^+ < \infty. \quad (6)$$

In particular, one has

$$\lim_{n \rightarrow \infty} \frac{1}{n^{1/l}} \sum_{i=1}^n a_{ni} X_i = 0, \quad a.s. \quad (7)$$

Next, we investigate the convergence in mean square.

Theorem 7. *Let $r > 1/2$ and $\{X_n, \mathcal{F}_n, n \geq 1\}$ be martingale differences stochastically dominated by a nonnegative random variable X with $EX^2 < \infty$. Let $\{a_{ni}, 1 \leq i \leq n, n \geq 1\}$ be a triangular array of real numbers and*

$$\sum_{i=1}^n a_{ni}^2 = O(n). \quad (8)$$

Then, one has

$$n^{2r-1} E \left(\frac{1}{n^r} \sum_{i=1}^n a_{ni} X_i \right)^2 \leq K, \quad n \geq 1, \quad (9)$$

where K is a positive constant.

Remark 8. Wang et al. [8] obtained the complete convergence and complete moment convergence for nonweighted martingale differences, which generalized some results of Stoica [6, 7]. In this paper, we study the complete moment convergence of double-indexed weighted sums of martingale differences. So we extend the results of Wang et al. [8] and Stoica [6, 7] to the case of double-indexed weighted sums of martingale differences. On the other hand, we give the applications of

Corollary 6 and Theorem 7 to study the convergence of the state observers of linear-time-invariant systems and present the convergence with probability one and in mean square, respectively (see Theorems 11 and 12 in Section 3).

Proof of Theorem 5. Let $X_{ni} = X_i \mathbb{1}(|X_i| \leq n^\alpha)$, $1 \leq i \leq n$. It can be found that $a_{ni}X_i = a_{ni}X_i \mathbb{1}(|X_i| > n^\alpha) + [a_{ni}X_{ni} - a_{ni}E(X_{ni} | \mathcal{F}_{i-1})] + a_{ni}E(X_{ni} | \mathcal{F}_{i-1})$, $1 \leq i \leq n$.

By Lemma 3 with $a = n^\alpha$, for any $q > 1$, we obtain that

$$\begin{aligned} & \sum_{n=1}^{\infty} n^{\alpha p-2-\alpha} E \left(\max_{1 \leq k \leq n} \left| \sum_{i=1}^k a_{ni} X_i \right| - \varepsilon n^\alpha \right)^+ \\ & \leq K_1 \sum_{n=1}^{\infty} n^{\alpha p-2-q\alpha} \\ & \quad \times E \left(\max_{1 \leq k \leq n} \left| \sum_{i=1}^k [a_{ni} X_{ni} - a_{ni} E(X_{ni} | \mathcal{F}_{i-1})] \right|^q \right) \quad (10) \\ & \quad + \sum_{n=1}^{\infty} n^{\alpha p-2-\alpha} E \left(\max_{1 \leq k \leq n} \left| \sum_{i=1}^k a_{ni} X_i \mathbb{1}(|X_i| > n^\alpha) \right| \right) \\ & \quad + \sum_{n=1}^{\infty} n^{\alpha p-2-\alpha} E \left(\max_{1 \leq k \leq n} \left| \sum_{i=1}^k a_{ni} E(X_{ni} | \mathcal{F}_{i-1}) \right| \right) \\ & := H_1 + H_2 + H_3. \end{aligned}$$

For $p \geq 2$, it is easy to see that $q > 2(\alpha p - 1)/(2\alpha - 1) \geq 2$. Consequently, for any $1 \leq s \leq 2$, we get by Hölder's inequality and (4) that

$$\sum_{i=1}^n |a_{ni}|^s \leq \left(\sum_{i=1}^n |a_{ni}|^q \right)^{s/q} \left(\sum_{i=1}^n 1 \right)^{1-s/q} = O(n). \quad (11)$$

So, it can be checked by Markov's inequality, Lemma 4, (11), and $EX^p < \infty$ ($p \geq 2$) that

$$\begin{aligned} H_2 & \leq \sum_{n=1}^{\infty} n^{\alpha p-2-\alpha} \sum_{i=1}^n |a_{ni}| E[|X_i| \mathbb{1}(|X_i| > n^\alpha)] \\ & \leq K_1 \sum_{n=1}^{\infty} n^{\alpha p-1-\alpha} E[X \mathbb{1}(X > n^\alpha)] \\ & = K_1 \sum_{n=1}^{\infty} n^{\alpha p-1-\alpha} \sum_{m=n}^{\infty} E[X \mathbb{1}(m^\alpha < X \leq (m+1)^\alpha)] \quad (12) \\ & = K_1 \sum_{m=1}^{\infty} E[X \mathbb{1}(m^\alpha < X \leq (m+1)^\alpha)] \sum_{n=1}^m n^{\alpha p-1-\alpha} \\ & \leq K_2 \sum_{m=1}^{\infty} m^{\alpha p-\alpha} E[X \mathbb{1}(m^\alpha < X \leq (m+1)^\alpha)] \\ & \leq K_2 EX^p < \infty. \end{aligned}$$

Since $\{X_i, \mathcal{F}_i, 1 \leq i \leq n\}$ are martingale differences, by the martingale property and the proof of (12), one has that

$$\begin{aligned} H_3 & = \sum_{n=1}^{\infty} n^{\alpha p-2-\alpha} E \left(\max_{1 \leq k \leq n} \left| \sum_{i=1}^k a_{ni} E[X_i \mathbb{1}(|X_i| \leq n^\alpha) | \mathcal{F}_{i-1}] \right| \right) \\ & = \sum_{n=1}^{\infty} n^{\alpha p-2-\alpha} E \left(\max_{1 \leq k \leq n} \left| \sum_{i=1}^k a_{ni} E[X_i \mathbb{1}(|X_i| > n^\alpha) | \mathcal{F}_{i-1}] \right| \right) \\ & \leq K_1 \sum_{n=1}^{\infty} n^{\alpha p-2-\alpha} \sum_{i=1}^n |a_{ni}| E[|X_i| \mathbb{1}(|X_i| > n^\alpha)] \\ & \leq K_2 \sum_{n=1}^{\infty} n^{\alpha p-1-\alpha} E[X \mathbb{1}(X > n^\alpha)] \leq K_3 EX^p < \infty. \end{aligned} \quad (13)$$

Next, we turn to prove $H_1 < \infty$ under conditions of Theorem 5. It can be seen that

$$\{[a_{ni}X_{ni} - a_{ni}E(X_{ni} | \mathcal{F}_{i-1})], \mathcal{F}_i, 1 \leq i \leq n\} \quad (14)$$

are also martingale differences. So, by Markov's inequality, (10), and Lemma 2 with $p = q$, it can be found that

$$\begin{aligned} H_1 & = K_1 \sum_{n=1}^{\infty} n^{\alpha p-2-q\alpha} E \left(\max_{1 \leq k \leq n} \left| \sum_{i=1}^k [a_{ni}X_{ni} - a_{ni}E(X_{ni} | \mathcal{F}_{i-1})] \right|^q \right) \\ & \leq K_2 \sum_{n=1}^{\infty} n^{\alpha p-2-q\alpha} E \left(\sum_{i=1}^n E \{ [a_{ni}X_{ni} - a_{ni}E(X_{ni} | \mathcal{F}_{i-1})]^2 | \mathcal{F}_{i-1} \} \right)^{q/2} \\ & \quad + K_3 \sum_{n=1}^{\infty} n^{\alpha p-2-q\alpha} \sum_{i=1}^n E |a_{ni}X_{ni} - a_{ni}E(X_{ni} | \mathcal{F}_{i-1})|^q \\ & =: K_2 H_{11} + K_3 H_{12}. \end{aligned} \quad (15)$$

Obviously, it follows that

$$\begin{aligned} & E \{ [a_{ni}X_{ni} - E(a_{ni}X_{ni} | \mathcal{F}_{i-1})]^2 | \mathcal{F}_{i-1} \} \\ & = E [a_{ni}^2 X_i^2 \mathbb{1}(|X_i| \leq n^\alpha) | \mathcal{F}_{i-1}] \\ & \quad - [E(a_{ni}X_i \mathbb{1}(|X_i| \leq n^\alpha) | \mathcal{F}_{i-1})]^2 \quad (16) \\ & \leq a_{ni}^2 E [X_i^2 \mathbb{1}(|X_i| \leq n^\alpha) | \mathcal{F}_{i-1}] \\ & \leq a_{ni}^2 E (X_i^2 | \mathcal{F}_{i-1}), \quad \text{a.s., } 1 \leq i \leq n. \end{aligned}$$

Combining (11) with $E[\sup_{i \geq 1} E(X_i^2 | \mathcal{F}_{i-1})]^{q/2} < \infty$, we obtain that

$$\begin{aligned} H_{11} &\leq \sum_{n=1}^{\infty} n^{\alpha p - 2 - q\alpha} \left(\sum_{i=1}^n a_{ni}^2 \right)^{q/2} E \left(\sup_{i \geq 1} E(X_i^2 | \mathcal{F}_{i-1}) \right)^{q/2} \\ &\leq K_4 \sum_{n=1}^{\infty} n^{\alpha p - 2 - q\alpha + q/2} < \infty, \end{aligned} \quad (17)$$

following from the fact that $q > 2(\alpha p - 1)/(2\alpha - 1)$. Meanwhile, by C_r inequality, Lemma 4, and (4),

$$\begin{aligned} H_{12} &\leq K_5 \sum_{n=1}^{\infty} n^{\alpha p - 2 - q\alpha} \sum_{i=1}^n |a_{ni}|^q E[|X_i|^q \mathbb{1}(|X_i| \leq n^\alpha)] \\ &\leq K_6 \sum_{n=1}^{\infty} n^{\alpha p - 1 - q\alpha} E[X^q \mathbb{1}(X \leq n^\alpha)] \\ &\quad + K_7 \sum_{n=1}^{\infty} n^{\alpha p - 1} P(X > n^\alpha) \\ &\leq K_6 \sum_{n=1}^{\infty} n^{\alpha p - 1 - q\alpha} E[X^q \mathbb{1}(X \leq n^\alpha)] \\ &\quad + K_7 \sum_{n=1}^{\infty} n^{\alpha p - 1 - \alpha} E[X \mathbb{1}(X > n^\alpha)] \\ &=: K_6 H_{11}^* + K_7 H_{12}^*. \end{aligned} \quad (18)$$

By the conditions $p \geq 2$ and $\alpha > 1/2$, we have that $2(\alpha p - 1)/(2\alpha - 1) - p \geq 0$, which implies $q > p$. So, we obtain by $EX^p < \infty$ that

$$\begin{aligned} H_{11}^* &= \sum_{n=1}^{\infty} n^{\alpha p - 1 - q\alpha} \sum_{i=1}^n E[X^q \mathbb{1}((i-1)^\alpha < X \leq i^\alpha)] \\ &= \sum_{i=1}^{\infty} E[X^q \mathbb{1}((i-1)^\alpha < X \leq i^\alpha)] \sum_{n=i}^{\infty} n^{\alpha p - 1 - q\alpha} \\ &\leq K_8 \sum_{i=1}^{\infty} E[X^p X^{q-p} \mathbb{1}((i-1)^\alpha < X \leq i^\alpha)] i^{\alpha p - q\alpha} \\ &\leq K_8 EX^p < \infty. \end{aligned} \quad (19)$$

By the proof of (12), one has that

$$H_{12}^* = \sum_{n=1}^{\infty} n^{\alpha p - 1 - \alpha} E[X \mathbb{1}(X > n^\alpha)] \leq K_9 EX^p < \infty. \quad (20)$$

Thus, by (15)–(20), we have that $H_1 < \infty$. So, it completes the proof of (5). $\square$

Proof of Corollary 6. If $p = 2l$ and $\alpha = 2/p$, then one has $\alpha p = 2$. So as an application of Theorem 5, one gets (6) immediately. On the other hand, it can be seen that

$$\begin{aligned} &\sum_{n=1}^{\infty} n^{\alpha p - 2 - \alpha} E \left(\max_{1 \leq k \leq n} \left| \sum_{i=1}^k a_{ni} X_i \right| - \varepsilon n^\alpha \right)^+ \\ &= \sum_{n=1}^{\infty} n^{\alpha p - 2 - \alpha} \int_0^\infty P \left(\max_{1 \leq k \leq n} \left| \sum_{i=1}^k a_{ni} X_i \right| - \varepsilon n^\alpha > t \right) dt \\ &\geq \sum_{n=1}^{\infty} n^{\alpha p - 2 - \alpha} \int_0^{\varepsilon n^\alpha} P \left(\max_{1 \leq k \leq n} \left| \sum_{i=1}^k a_{ni} X_i \right| - \varepsilon n^\alpha > t \right) dt \\ &\geq \varepsilon \sum_{n=1}^{\infty} n^{\alpha p - 2} P \left(\max_{1 \leq k \leq n} \left| \sum_{i=1}^k a_{ni} X_i \right| > 2\varepsilon n^\alpha \right). \end{aligned} \quad (21)$$

So by (5) and (21) with $\alpha p = 2$, we have for every $\varepsilon > 0$ that

$$\sum_{n=1}^{\infty} P \left(\max_{1 \leq k \leq n} \left| \sum_{i=1}^k a_{ni} X_i \right| > \varepsilon n^{1/l} \right) < \infty. \quad (22)$$

It follows from Borel-Cantelli lemma that

$$\lim_{n \rightarrow \infty} \frac{1}{n^{1/l}} \sum_{i=1}^n a_{ni} X_i = 0, \quad \text{a.s.} \quad (23)$$

So, (7) holds. $\square$

Proof of Theorem 7. Since $\{a_{ni} X_i, \mathcal{F}_i, 1 \leq i \leq n\}$ are martingale differences, it can be found by Lemmas 2 and 4 and (8) that

$$\begin{aligned} E \left(\frac{1}{n^r} \sum_{i=1}^n a_{ni} X_i \right)^2 &= \frac{1}{n^{2r}} E \left(\sum_{i=1}^n a_{ni} X_i \right)^2 \leq \frac{K_1}{n^{2r}} \sum_{i=1}^n a_{ni}^2 EX_i^2 \\ &\leq \frac{K_2}{n^{2r}} EX^2 \sum_{i=1}^n a_{ni}^2 \leq \frac{K_3}{n^{2r-1}}, \quad n \geq 1. \end{aligned} \quad (24)$$

Consequently, (9) holds true. $\square$

3. Applications to the Convergence of the State Observers of Linear-Time-Invariant Systems

In this section, we give the applications of Corollary 6 and Theorem 7 to study the convergence of the state observers of linear-time-invariant systems.

For $t \geq 0$, consider an MISO (multi-input-single-output) linear-time-invariant system

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t), \\ y(t) &= Cx(t), \end{aligned} \quad (25)$$

where $A \in R^{m_0 \times m_0}$, $B \in R^{m_0 \times m_1}$, and $C \in R^{1 \times m_0}$ are known system matrices, and for $t \geq 0$, $u(t) \in R^{m_1}$ is the control input,

$x(t) \in R^{m_0}$ is the state, and $y(t) \in R$ is the system output. The initial state $x(0)$ is unknown. We are interested in estimation of $x(t)$, from some limited observations on $y(t)$.

In our setup, the output $y(t)$ is only measured at a sequence of sampling time instants $\{t_i\}$ with measured values $\gamma(t_i)$, and noise d_i

$$\gamma(t_i) = y(t_i) - d_i. \quad (26)$$

We would like to estimate the state $x(t)$ from information on $u(t)$, $\{t_i\}$, and $\{\gamma(t_i)\}$. In practical systems, the irregular sampling sequences $\{\gamma(t_i)\}$ can be generated by different means such as randomized sampling, event-triggered sampling, and signal quantization.

It is obvious that state estimation will not be possible if the system is not observable. Also, in this paper, d_k is assumed to be martingale difference. We give the following assumption.

Assumption 9. The system (25) is observable; that is, the observability matrix

$$W_o' = \left[C', (CA)', \dots, (CA^{m_0-1})' \right] \quad (27)$$

has full rank.

For both $t > t_0$ and $t < t_0$, the solution to system (25) can be expressed as

$$x(t) = e^{A(t-t_0)} x(t_0) + \int_{t_0}^t e^{A(t-\tau)} Bu(\tau) d\tau. \quad (28)$$

Suppose that $\{t_i, 1 \leq i \leq n\}$ is a sequence of sampling times. For $t_i \leq t_n$, we have

$$\gamma(t_i) + d_i = y(t_i) = Ce^{A(t_i-t_n)} x(t_n) + C \int_{t_n}^{t_i} e^{A(t_i-\tau)} Bu(\tau) d\tau. \quad (29)$$

Since the second term is known, it will be denoted by $v(t_i, t_n) = C \int_{t_n}^{t_i} e^{A(t_i-\tau)} Bu(\tau) d\tau$. This leads to the observations

$$Ce^{A(t_i-t_n)} x(t_n) = \gamma(t_i) - v(t_i, t_n) + d_i, \quad 1 \leq i \leq n. \quad (30)$$

Define

$$\Phi_n = \begin{bmatrix} Ce^{A(t_1-t_n)} \\ \vdots \\ Ce^{A(t_{n-1}-t_n)} \\ C \end{bmatrix}, \quad \Gamma_n = \begin{bmatrix} \gamma(t_1) \\ \vdots \\ \gamma(t_{n-1}) \\ \gamma(t_n) \end{bmatrix}, \quad (31)$$

$$V_n = \begin{bmatrix} v(t_1, t_n) \\ \vdots \\ v(t_{n-1}, t_n) \\ 0 \end{bmatrix}, \quad D_n = \begin{bmatrix} d_1 \\ \vdots \\ d_{n-1} \\ d_n \end{bmatrix}.$$

Then, (30) can be written as

$$\Phi_n x(t_n) = \Gamma_n - V_n + D_n. \quad (32)$$

Suppose that Φ_n is full rank, which will be established later. Then, a least-squares estimate of $x(t_n)$ is given by

$$\hat{x}(t_n) = (\Phi_n' \Phi_n)^{-1} \Phi_n' (\Gamma_n - V_n). \quad (33)$$

Here, G' denotes the transpose of G . From (32) and (33), the estimation error for $x(t_n)$ at sampling time t_n is

$$e(t_n) = \hat{x}(t_n) - x(t_n) = (\Phi_n' \Phi_n)^{-1} \Phi_n' D_n \\ = \left(\frac{1}{n^r} \Phi_n' \Phi_n \right)^{-1} \frac{1}{n^r} \Phi_n' D_n \quad (34)$$

for some $1/2 < r < 1$. For convergence analysis, one must consider a typical entry in $(1/n^r) \Phi_n' D_n$. By the Cayley Hamilton theorem (see Ogata [18]), the matrix exponential can be expressed by a polynomial function of A of order at most $m_0 - 1$,

$$e^{At} = \alpha_1(t) I + \dots + \alpha_{m_0}(t) A^{m_0-1}, \quad (35)$$

where the time functions $\alpha_i(t)$ can be derived by the Lagrange-Hermite interpolation method (see Ogata [18]). This implies that

$$Ce^{A(t_i-t_n)} = [\alpha_1(t_i-t_n), \dots, \alpha_{m_0}(t_i-t_n)] \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{m_0-1} \end{bmatrix} \\ = \varphi'(t_i-t_n) W_o, \quad (36)$$

where $\varphi'(t_i-t_n) = [\alpha_1(t_i-t_n), \dots, \alpha_{m_0}(t_i-t_n)]$ and W_o is the observability matrix.

Denote

$$\Psi_n = \begin{bmatrix} \varphi'(t_1-t_n) \\ \vdots \\ \varphi'(0) \end{bmatrix}. \quad (37)$$

Then

$$\Phi_n = \Psi_n W_o, \quad (38)$$

which implies that

$$\frac{1}{n^r} \Phi_n' \Phi_n = W_o' \frac{1}{n^r} \Psi_n' \Psi_n W_o, \\ \frac{1}{n^r} \Phi_n' D_n = \frac{1}{n^r} W_o' \Psi_n' D_n. \quad (39)$$

As a result, for any $r > 0$, one has

$$e(t_n) = \left(\frac{1}{n^r} \Phi_n' \Phi_n \right)^{-1} \frac{1}{n^r} \Phi_n' D_n = W_o^{-1} \left(\frac{1}{n^r} \Psi_n' \Psi_n \right)^{-1} \frac{1}{n^r} \Psi_n' D_n. \quad (40)$$

Under Assumption 9, W_o^{-1} exists. Convergence results will be established by the following two sufficient conditions: $(1/n^r) \Psi_n' \Psi_n \rightarrow 0$ and $(1/n^r) \Psi_n' \Psi_n \geq \beta I$, for some $\beta > 0$. So we need the following persistent excitation (PE) condition, which was used by Wang et al. [19] and Thanh et al. [20].

Assumption 10. For some $1/2 < r < 1$,

$$\beta = \inf_{n \geq 1} \sigma_{\min} \left(\frac{1}{n^r} \Psi_n' \Psi_n \right) > 0, \quad (41)$$

where $\sigma_{\min}(H)$ is the small eigenvalue of H for a suitable symmetric H .

We can investigate the convergence of double-indexed summations of random variables form

$$\frac{1}{n^r} \sum_{i=1}^n a_{ni} d_i \quad (42)$$

for some $1/2 < r < 1$. Here, $\{a_{ni}\}$ is a triangular array of real numbers and $\{d_i\}$ is a sequence of martingale differences. It can be seen that (42) is a special case of (7) in Corollary 6. The j th component of $(1/n^r) \Psi_n' D_n$ takes the form

$$\frac{1}{n^r} \sum_{i=1}^n \alpha_j(t_i - t_n) d_i, \quad (43)$$

where $\{\alpha_j(t_i - t_n)\}$ is a triangular array of real numbers. The convergence analysis of (43) for $e(t_n)$ is a special case of (42) or (7) in Corollary 6.

Recently, Wang et al. [19] investigated the convergence analysis of the state observers of linear-time-invariant systems under ρ^* -mixing sampling. Thanh et al. [20] studied the convergence analysis of double-indexed and randomly weighted sums of ρ^* -mixing sequence and gave its application to state observers. For more related works, one can refer to [18–23] and the references therein.

As an application of Corollary 6 to the observers and state estimation, we obtain the following theorem.

Theorem 11. Let Assumptions 9 and 10 hold. Let $1/2 < r < 1$ and $\{d_n, \mathcal{F}_n, n \geq 1\}$ be martingale differences stochastically dominated by a nonnegative random variable d with $E d^{2/r} < \infty$. Suppose that for any $q > 2/(2r - 1)$, one has $E[\sup_{n \geq 1} E(d_n^2 | \mathcal{F}_{n-1})]^{q/2} < \infty$ and

$$\sum_{i=1}^n |\alpha_j(t_i - t_n)|^q = O(n), \quad (44)$$

where $1 \leq j \leq m_0$. Then

$$\frac{1}{n^r} \|\Psi_n' D_n\| \longrightarrow 0, \quad a.s. \quad (45)$$

Consequently,

$$e(t_n) \longrightarrow 0, \quad a.s. \quad (46)$$

As an application to Theorem 7, we get the following result.

Theorem 12. Let $1/2 < r < 1$ and Assumptions 9 and 10 hold. Assume that $\{d_n, \mathcal{F}_n, n \geq 1\}$ are martingales differences stochastically dominated by a nonnegative random variable d with $E d^2 < \infty$. For $1 \leq j \leq m_0$, it is supposed that

$$\sum_{i=1}^n \alpha_j^2(t_i - t_n) = O(n). \quad (47)$$

Then

$$\zeta = \sup_{n \geq 1} n^{2r-1} E e'(t_n) e(t_n) < \infty. \quad (48)$$

Remark 13. If we assume that, for each $1 \leq i \leq n$, $\{\varphi(t_i - t_n)\}$ is uniformly bounded, then we can find that condition (44) holds for any q . On the other hand, similar to Theorems 11 and 12, Wang et al. [19] also obtained the convergence of the state observers with probability one and in mean square under ρ^* -mixing sampling (see Theorems 4 and 5 of Wang et al. [19]). So Theorems 11 and 12 generalize the results of Wang et al. [19] to the case of martingale differences.

Proof of Theorem 11. It can be seen that

$$\frac{1}{n^r} \Psi_n' D_n = \begin{bmatrix} \frac{1}{n^r} \sum_{i=1}^n \alpha_1(t_i - t_n) d_i \\ \vdots \\ \frac{1}{n^r} \sum_{i=1}^n \alpha_{m_0}(t_i - t_n) d_i \end{bmatrix}. \quad (49)$$

To prove (45), it suffices to look at the j th component

$$\frac{1}{n^r} \sum_{i=1}^n \alpha_j(t_i - t_n) d_i \quad (50)$$

of

$$\frac{1}{n^r} \Psi_n' D_n. \quad (51)$$

For any $q > 2/(2r - 1)$, by $E[\sup_{n \geq 1} E(d_n^2 | \mathcal{F}_{n-1})]^{q/2} < \infty$ and (44), we can obtain (45) from Corollary 6 with $l = 1/r$, $a_{ni} = \alpha_j(t_i - t_n)$ in (43), and $X_n = d_n$.

On the other hand, by Assumption 9, W_0^{-1} exists, and by (41) in Assumption 10, $((1/n^r) \Psi_n' \Psi_n)^{-1}$ exists and

$$\sigma_{\max} \left(\left(\frac{1}{n^r} \Psi_n' \Psi_n \right)^{-1} \right) \leq \frac{1}{\beta}, \quad (52)$$

where $\sigma_{\max}(\cdot)$ is the largest eigenvalue. Together with

$$e(t_n) = W_0^{-1} \left(\frac{1}{n^r} \Psi_n' \Psi_n \right)^{-1} \frac{1}{n^r} \Psi_n' D_n \quad (53)$$

and (45), it follows (46). $\square$

Proof of Theorem 12. For $1 \leq j \leq m_0$, by (47), (8) holds. Applying Theorem 7 with $a_{ni} = \alpha_j(t_i - t_n)$, $X_n = d_n$, and $1/2 < r < 1$, we obtain that for a typical term

$$\frac{1}{n^r} \sum_{i=1}^n \alpha_j(t_i - t_n) d_i \quad (54)$$

in (49),

$$n^{2r-1} E \left(\frac{1}{n^r} \sum_{i=1}^n \alpha_j(t_i - t_n) d_i \right)^2 \leq K_1, \quad n \geq 1. \quad (55)$$

Together with (49), (53), and (55), we obtain that

$$n^{2r-1} E e'(t_n) e(t_n) \leq m_0 K_2 < \infty, \quad (56)$$

where K_2 is a positive constant. Lastly, by (56), (48) holds true. $\square$

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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References

- [1] P. L. Hsu and H. Robbins, "Complete convergence and the law of large numbers," *Proceedings of the National Academy of Sciences of the United States of America*, vol. 33, no. 2, pp. 25–31, 1947.
- [2] P. Erdős, "On a Theorem of Hsu and Robbins," *Annals of Mathematical Statistics*, vol. 20, no. 2, pp. 286–291, 1949.
- [3] L. E. Baum and M. Katz, "Convergence rates in the law of large numbers," *Transactions of the American Mathematical Society*, vol. 120, no. 1, pp. 108–123, 1965.
- [4] K. F. Yu, "Complete convergence of weighted sums of martingale differences," *Journal of Theoretical Probability*, vol. 3, no. 2, pp. 339–347, 1990.
- [5] S. Ghosal and T. K. Chandra, "Complete convergence of martingale arrays," *Journal of Theoretical Probability*, vol. 11, no. 3, pp. 621–631, 1998.
- [6] G. Stoica, "Baum-Katz-Nagaev type results for martingales," *Journal of Mathematical Analysis and Applications*, vol. 336, no. 2, pp. 1489–1492, 2007.
- [7] G. Stoica, "A note on the rate of convergence in the strong law of large numbers for martingales," *Journal of Mathematical Analysis and Applications*, vol. 381, no. 2, pp. 910–913, 2011.
- [8] X. J. Wang, S. H. Hu, W. Z. Yang, and X. H. Wang, "Convergence rates in the strong law of large numbers for martingale difference sequences," *Abstract and Applied Analysis*, vol. 2012, Article ID 572493, 13 pages, 2012.
- [9] W. Z. Yang, S. H. Hu, and X. J. Wang, "Complete convergence for moving average process of martingale differences," *Discrete Dynamics in Nature and Society*, vol. 2012, Article ID 128492, 16 pages, 2012.
- [10] A. Gut, *Probability: A Graduate Course*, Springer Science+Business Media, 2005.
- [11] P. Chen, T.-C. Hu, and A. Volodin, "Limiting behaviour of moving average processes under φ -mixing assumption," *Statistics and Probability Letters*, vol. 79, no. 1, pp. 105–111, 2009.
- [12] S. H. Sung, "Moment inequalities and complete moment convergence," *Journal of Inequalities and Applications*, vol. 2009, Article ID 271265, 14 pages, 2009.
- [13] S. H. Sung, "Complete convergence for weighted sums of ρ^* -mixing random variables," *Discrete Dynamics in Nature and Society*, vol. 2010, Article ID 630608, 13 pages, 2010.
- [14] S. H. Sung, "Convergence of moving average processes for dependent random variables," *Communications in Statistics*, vol. 40, no. 13, pp. 2366–2376, 2011.
- [15] S. H. Sung and A. Volodin, "A note on the rate of complete convergence for weighted sums of arrays of Banach space valued random elements," *Stochastic Analysis and Applications*, vol. 29, no. 2, pp. 282–291, 2011.
- [16] T.-C. Hu, A. Rosalsky, and A. Volodin, "A Complete Convergence Theorem for Row Sums from Arrays of Rowwise Independent Random Elements in Rademacher Type p Banach Spaces," *Stochastic Analysis and Applications*, vol. 30, no. 2, pp. 343–353, 2012.
- [17] P. Hall and C. C. Heyde, *Martingale Limit Theory and Its Application*, Academic Press, New York, NY, USA, 1980.
- [18] K. Ogata, *Modern Control Engineering*, Prentice-Hall, Englewood Cliffs, NJ, USA, 4th edition, 2002.
- [19] L. Y. Wang, C. Li, G. G. Yin, L. Guo, and C.-Z. Xu, "State observability and observers of linear-time-invariant systems under irregular sampling and sensor limitations," *IEEE Transactions on Automatic Control*, vol. 56, no. 11, pp. 2639–2654, 2011.
- [20] L. V. Thanh, G. G. Yin, and L. Y. Wang, "State observers with random sampling times and convergence analysis of double-indexed and randomly weighted sums of mixing processes," *SIAM Journal on Control and Optimization*, vol. 49, no. 1, pp. 106–124, 2011.
- [21] D. W. Huang and L. Guo, "Estimation of nonstationary ARMAX models based on the Hannan-Rissanen method," *The Annals of Statistics*, vol. 18, no. 4, pp. 1729–1756, 1990.
- [22] L. Y. Wang, G. Yin, J. F. Zhang, and Y. L. Zhao, *System Identification with Quantized Observations*, Birkhäuser, Boston, Mass, USA, 2010.
- [23] F. Wu, G. Yin, and L. Y. Wang, "Moment exponential stability of random delay systems with two-time-scale Markovian switching," *Nonlinear Analysis: Real World Applications*, vol. 13, no. 6, pp. 2476–2490, 2012.