Research Article

Trudinger-Moser Embedding on the Hyperbolic Space

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Received 6 November 2013; Accepted 26 December 2013; Published 18 February 2014

Academic Editor: Julio Rossi

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Let $(\mathbb{H}^n, g)$ be the hyperbolic space of dimension n . By our previous work (Theorem 2.3 of (Yang (2012))), for any $0 < \alpha < \alpha_n$, there exists a constant $\tau > 0$ depending only on n and α such that $\sup_{u \in W^{1,n}(\mathbb{H}^n), \|u\|_{1,\tau} \leq 1} \int_{\mathbb{H}^n} (e^{\alpha|u|^{n/(n-1)}} - \sum_{k=0}^{n-2} \frac{\alpha^k |u|^{nk/(n-1)}}{k!}) dv_g < \infty$, where $\alpha_n = n\omega_{n-1}^{1/(n-1)}$, ω_{n-1} is the measure of the unit sphere in $\mathbb{R}^n$, and $\|u\|_{1,\tau} = \|\nabla_g u\|_{L^n(\mathbb{H}^n)} + \tau \|u\|_{L^n(\mathbb{H}^n)}$. In this note we shall improve the above mentioned inequality. Particularly, we show that, for any $0 < \alpha < \alpha_n$ and any $\tau > 0$, the above mentioned inequality holds with the definition of $\|u\|_{1,\tau}$ replaced by $(\int_{\mathbb{H}^n} (|\nabla_g u|^n + \tau |u|^n) dv_g)^{1/n}$. We solve this problem by gluing local uniform estimates.

1. Introduction

Let Ω be a bounded smooth domain in $\mathbb{R}^n$. The classical Trudinger-Moser inequality [1–3] says

$$\sup_{u \in W_0^{1,n}(\Omega), \|u\|_{W_0^{1,n}(\Omega)} \leq 1} \int_{\Omega} e^{\alpha_n |u|^{n/(n-1)}} dx \leq C |\Omega| \quad (1)$$

for some constant C depending only on n , where $W_0^{1,n}(\Omega)$ is the usual Sobolev space and $|\Omega|$ denotes the Lebesgue measure of Ω . In the case where Ω is an unbounded domain of $\mathbb{R}^n$, the above integral is infinite, but it was shown by Cao [4], Panda [5], and do Ó [6] that for any $\tau > 0$ and any $\alpha < \alpha_n$ there holds

$$\sup_{u \in W^{1,n}(\mathbb{R}^n), \int_{\mathbb{R}^n} (|\nabla u|^n + \tau |u|^n) dx \leq 1} \int_{\mathbb{R}^n} \left(e^{\alpha |u|^{n/(n-1)}} - \sum_{k=0}^{n-2} \frac{\alpha^k |u|^{nk/(n-1)}}{k!} \right) dx < \infty. \quad (2)$$

Later Ruf [7], Li and Ruf [8], and Adimurthi and Yang [9] obtained (2) in the critical case $\alpha = \alpha_n$.

The study of Trudinger-Moser inequalities on compact Riemannian manifolds can be traced back to Aubin [10],

Cherrier [11, 12], and Fontana [13]. A particular case is as follows. Let (M, g) be an n -dimensional compact Riemannian manifold without boundary. Then there holds

$$\sup_{\int_M |\nabla_g u|^n dv_g \leq 1, \int_M u dv_g = 0} \int_M e^{\alpha_n |u|^{n/(n-1)}} dv_g < \infty. \quad (3)$$

In view of (2), it is natural to consider extension of (3) on complete noncompact Riemannian manifolds. In [14] we obtained the following results. Let (M, g) be a complete noncompact Riemannian manifold. If the Trudinger-Moser inequality holds on it, then there holds $\inf_{x \in M} \text{vol}_g(B_1(x)) > 0$. If the Ricci curvature has lower bound, say $\text{Ric}_g(M) \geq -K$, the injectivity radius has a positive lower bound i_0 then for any $\alpha < \alpha_n$ there exists a constant $\tau > 0$ depending only on α, n, K , and i_0 such that

$$\sup_{\left(\int_M |\nabla u|^n dv_g \right)^{1/n} + \tau \left(\int_M |u|^n dv_g \right)^{1/n} \leq 1} \int_M \left(e^{\alpha |u|^{n/(n-1)}} - \sum_{k=0}^{n-2} \frac{\alpha^k |u|^{nk/(n-1)}}{k!} \right) dv_g < \infty. \quad (4)$$

Since τ depends on α , (4) is weaker than (2) when (M, g) is replaced by $\mathbb{R}^n$. Moreover, the condition that $\text{Ric}_g(M)$ has

lower bound is not necessary for the validity of the Trudinger-Moser inequality.

In this note, we will continue to study (4) in whole $\mathbb{H}^n$ by gluing local uniform estimates. Particularly, we have the following.

Theorem 1. *Let $(\mathbb{H}^n, g)$ be an n -dimensional hyperbolic space, $\alpha_n = n\omega_{n-1}^{1/(n-1)}$, where ω_{n-1} is the measure of the unit sphere in $\mathbb{R}^n$. Then for any $\alpha < \alpha_n$, any $\tau > 0$, and any $u \in W^{1,n}(\mathbb{H}^n)$ satisfying $\int_{\mathbb{H}^n} (|\nabla_g u|^n + \tau|u|^n) dv_g \leq 1$, there exists some constant β depending only on n and τ such that*

$$\int_{\mathbb{H}^n} \left(e^{\alpha|u|^{n/(n-1)}} - \sum_{k=0}^{n-2} \frac{\alpha^k |u|^{nk/(n-1)}}{k!} \right) dv_g \leq \beta. \quad (5)$$

The proof of Theorem 1 is based on local uniform estimates (Lemma 2 below). This idea comes from [14] and can also be used in other cases [15, 16].

We remark that critical case of (5) was studied by Adimurthi and Tintarev [17], Mancini and Sandeep [18], and Mancini et al. ([19]) via different methods.

The remaining part of this note is organized as follows. In Section 2 we derive local uniform Trudinger-Moser inequalities; in Section 3, Theorem 1 is proved.

2. Local Estimates

To get (5), we need the following uniform local estimates which is an analogy of ([15], Lemma 4.1) or ([16], Lemma 1), and it is of its own interest.

Lemma 2. *For any $p \in \mathbb{H}^n$, any $R > 0$, and any $u \in W_0^{1,n}(B_R(p))$ with $\int_{B_R(p)} |\nabla_g u|^n dv_g \leq 1$, there exists some constant C_n depending only on n such that*

$$\begin{aligned} & \int_{B_R(p)} \left(e^{\alpha_n |u|^{n/(n-1)}} - \sum_{k=0}^{n-2} \frac{\alpha_n^k |u|^{nk/(n-1)}}{k!} \right) dv_g \\ & \leq C_n (\sinh R)^n \int_{B_R(p)} |\nabla_g u|^n dv_g, \end{aligned} \quad (6)$$

where $B_R(p)$ denotes the geodesic ball of $(\mathbb{H}^n, g)$ which is centered at p with radius R .

Proof. It is well known (see, e.g., [20], II.5, Theorem 1) that there exists a homomorphism $\varphi : D = \{x \in \mathbb{R}^n : |x| < 1\}$ such that $\varphi(p) = 0$, that in these coordinates the Riemannian metric g can be represented by

$$g(x) = \frac{4}{(1 - |x|^2)^2} g_0(x), \quad (7)$$

where $g_0(x) = \sum_{i=1}^n (dx^i)^2$ is the standard Euclidean metric on $\mathbb{R}^n$, and that

$$\varphi(B_R(p)) = \mathbb{B}_{\tanh R/2}(0), \quad (8)$$

where $\mathbb{B}_r(0) \subset \mathbb{R}^n$ denotes a ball centered at 0 with radius r . Moreover, the corresponding polar coordinates $(r, \theta) \in [0, \infty) \times \mathbb{S}^{n-1}$ read

$$g = dr^2 + (\sinh r)^2 d\theta^2, \quad (9)$$

where $d\theta^2$ is the standard metric on $\mathbb{S}^{n-1}$.

Denote $f = 2/(1 - |x|^2)$; then $g = f^2 g_0$, $|\nabla_g u| = f^{-1} |\nabla_{g_0}(u \circ \varphi^{-1})|$, and $dv_g = f^n dv_{g_0}$. Calculating directly, we have

$$\int_{B_R(0)} |\nabla_g u|^n dv_g = \int_{\mathbb{B}_{\tanh R/2}(0)} |\nabla_{g_0}(u \circ \varphi^{-1})|^n dv_{g_0}. \quad (10)$$

Since $u \in W_0^{1,n}(B_R(p))$, we have $u \circ \varphi^{-1} \in W_0^{1,n}(\mathbb{B}_{\tanh R/2}(0))$. Noting that $\int_{B_R(p)} |\nabla_g u|^n dv_g \leq 1$, we have by (10)

$$\int_{\mathbb{B}_{\tanh R/2}(0)} |\nabla_{g_0}(u \circ \varphi^{-1})|^n dv_{g_0} \leq 1. \quad (11)$$

The standard Trudinger-Moser inequality (1) implies

$$\begin{aligned} & \int_{\mathbb{B}_{\tanh R/2}(0)} \left(e^{\alpha_n |u \circ \varphi^{-1}|^{n/(n-1)}} - \sum_{k=0}^{n-2} \frac{\alpha_n^k |u \circ \varphi^{-1}|^{nk/(n-1)}}{k!} \right) dv_{g_0} \\ & = \int_{\mathbb{B}_{\tanh R/2}(0)} \sum_{k=n-1}^{\infty} \frac{\alpha_n^k |u \circ \varphi^{-1}|^{nk/(n-1)}}{k!} dv_{g_0} \\ & \leq \int_{\mathbb{B}_{\tanh R/2}(0)} \sum_{k=n-1}^{\infty} \frac{\alpha_n^k (|u \circ \varphi^{-1}| / \|\nabla_{g_0}(u \circ \varphi^{-1})\|_{L^n})^{nk/(n-1)}}{k!} dv_{g_0} \\ & \quad \times \int_{\mathbb{B}_{\tanh R/2}(0)} |\nabla_{g_0}(u \circ \varphi^{-1})|^n dv_{g_0} \\ & \leq C_n \left(\tanh \frac{R}{2} \right)^n \int_{\mathbb{B}_{\tanh R/2}(0)} |\nabla_{g_0}(u \circ \varphi^{-1})|^n dv_{g_0}, \end{aligned} \quad (12)$$

where C_n is a constant depending only on n . This together with (10) immediately leads to

$$\begin{aligned} & \int_{B_R(p)} \left(e^{\alpha_n |u|^{n/(n-1)}} - \sum_{k=0}^{n-2} \frac{\alpha_n^k |u|^{nk/(n-1)}}{k!} \right) dv_g \\ & = \int_{\mathbb{B}_{\tanh R/2}(0)} \left(e^{\alpha_n |u \circ \varphi^{-1}|^{n/(n-1)}} - \sum_{k=0}^{n-2} \frac{\alpha_n^k |u \circ \varphi^{-1}|^{nk/(n-1)}}{k!} \right) f^n dv_{g_0} \\ & \leq C_n \left(\frac{2 \tanh R/2}{1 - (\tanh R/2)^2} \right)^n \int_{\mathbb{B}_{\tanh R/2}(0)} |\nabla_{g_0}(u \circ \varphi^{-1})|^n dv_{g_0} \\ & = C_n (\sinh R)^n \int_{B_R(p)} |\nabla_g u|^n dv_g. \end{aligned} \quad (13)$$

This is exactly (6) and thus ends the proof of the lemma. $\square$

As a corollary of Lemma 2, the following estimates can be compared with (1).

Corollary 3. For any $p \in \mathbb{H}^n$, any $R > 0$, and any $u \in W_0^{1,n}(B_R(p))$ with $\int_{B_R(p)} |\nabla_g u|^n dv_g \leq 1$, there exists some constant C depending only on n such that

$$\frac{1}{\text{Vol}_g(B_R(p))} \int_{B_R(p)} e^{\alpha_n |u|^{n/(n-1)}} dv_g \leq C \frac{\sinh R}{R}. \quad (14)$$

Proof. Since

$$\lim_{R \rightarrow 0^+} \frac{\text{Vol}_g(B_R(p))}{R(\sinh R)^{n-1}} = \lim_{R \rightarrow \infty} \frac{\text{Vol}_g(B_R(p))}{R(\sinh R)^{n-1}} = 1, \quad (15)$$

it follows from (13) that there exists some constant C depending only on n such that

$$\begin{aligned} \frac{1}{\text{Vol}_g(B_R(p))} \int_{B_R(p)} \left(e^{\alpha_n |u|^{n/(n-1)}} - \sum_{k=0}^{n-2} \frac{\alpha_n^k |u|^{nk/(n-1)}}{k!} \right) dv_g \\ \leq C \frac{\sinh R}{R}. \end{aligned} \quad (16)$$

In particular,

$$\int_{B_R(p)} |u|^n dv_g \leq C \frac{\sinh R}{R} \text{Vol}_g(B_R(p)). \quad (17)$$

Here and in the sequel we often denote various constants by the same C ; the reader can easily distinguish them from the context. Noting that for any $q, 0 \leq q \leq n$,

$$\int_{B_R(p)} |u|^q dv_g \leq \text{Vol}_g(B_R(p)) + \int_{B_R(p)} |u|^n dv_g, \quad (18)$$

we conclude

$$\int_{B_R(p)} \sum_{k=0}^{n-2} \frac{\alpha_n^k |u|^{nk/(n-1)}}{k!} dv_g \leq C \frac{\sinh R}{R} \text{Vol}_g(B_R(p)). \quad (19)$$

Combining (16) and (19), we obtain (14). $\square$

3. Proof of Theorem 1

In this section, we will prove Theorem 1 by gluing local estimates (6).

Proof of Theorem 1. Let R be a positive real number which will be determined later. By ([21], Lemma 1.6) we can find a sequence of points $\{x_i\}_{i=1}^\infty \subset \mathbb{H}^n$ such that $\cup_{i=1}^\infty B_{R/2}(x_i) = \mathbb{H}^n$, that $B_{R/4}(x_i) \cap B_{R/4}(x_j) = \emptyset$ for any $i \neq j$, and that for any $x \in \mathbb{H}^n$, x belongs to at most N balls $B_R(x_i)$, where N depends only on n . Let ϕ_i be the cut-off function satisfies the following conditions: (i) $\phi_i \in C_0^\infty(B_R(x_i))$; (ii) $0 \leq \phi_i \leq 1$ on $B_R(x_i)$ and $\phi_i \equiv 1$ on $B_{R/2}(x_i)$; (iii) $|\nabla_g \phi_i(x)| \leq 4/R$. Let $\tau > 0$ be fixed. For any $u \in W^{1,n}(\mathbb{H}^n)$ satisfying

$$\int_{\mathbb{H}^n} (|\nabla_g u|^n + \tau |u|^n) dv_g \leq 1, \quad (20)$$

we have $\phi_i u \in W_0^{1,n}(B_R(x_i))$. For any $\epsilon > 0$, using an elementary inequality $ab \leq \epsilon a^2 + (1/(4\epsilon))b^2$, we find some constant C depending only on n and ϵ such that

$$\begin{aligned} \int_{B_R(x_i)} |\nabla_g(\phi_i u)|^n dv_g \\ \leq (1 + \epsilon) \int_{B_R(x_i)} \phi_i^n |\nabla_g u|^n dv_g + C \int_{B_R(x_i)} |\nabla_g \phi_i|^n |u|^n dv_g \\ \leq (1 + \epsilon) \int_{B_R(x_i)} |\nabla_g u|^n dv_g + \frac{4^n C}{R^n} \int_{B_R(x_i)} |u|^n dv_g \\ \leq (1 + \epsilon) \int_{B_R(x_i)} (|\nabla_g u|^n + \tau |u|^n) dv_g, \end{aligned} \quad (21)$$

where in the last inequality we choose a sufficiently large R to make sure $4^n C/R^n \leq (1 + \epsilon)\tau$. Let $\alpha_\epsilon = \alpha_n/(1 + \epsilon)^{1/(n-1)}$ and $\widetilde{\phi}_i u = \phi_i u/(1 + \epsilon)^{1/n}$. Noting that $\widetilde{\phi}_i u \in W_0^{1,n}(B_R(x_i))$, we have by (21) and Lemma 2

$$\begin{aligned} \int_{B_{R/2}(x_i)} \left(e^{\alpha_\epsilon |u|^{n/(n-1)}} - \sum_{k=0}^{n-2} \frac{\alpha_\epsilon^k |u|^{nk/(n-1)}}{k!} \right) dv_g \\ \leq \int_{B_R(x_i)} \left(e^{\alpha_\epsilon |\phi_i u|^{n/(n-1)}} - \sum_{k=0}^{n-2} \frac{\alpha_\epsilon^k |\phi_i u|^{nk/(n-1)}}{k!} \right) dv_g \\ = \int_{B_R(x_i)} \left(e^{\alpha_n |\widetilde{\phi}_i u|^{n/(n-1)}} - \sum_{k=0}^{n-2} \frac{\alpha_n^k |\widetilde{\phi}_i u|^{nk/(n-1)}}{k!} \right) dv_g \quad (22) \\ \leq C_n (\sinh R)^n \int_{B_R(x_i)} |\nabla_g(\widetilde{\phi}_i u)|^n dv_g \\ \leq C (\sinh R)^n \int_{B_R(x_i)} (|\nabla_g u|^n + \tau |u|^n) dv_g, \end{aligned}$$

where C is a constant depending only on n and τ . By the choice of $\{x_i\}_{i=1}^\infty$ and (22), we have

$$\begin{aligned} \int_{\mathbb{H}^n} \left(e^{\alpha_\epsilon |u|^{n/(n-1)}} - \sum_{k=0}^{n-2} \frac{\alpha_\epsilon^k |u|^{nk/(n-1)}}{k!} \right) dv_g \\ \leq \int_{\cup_{i=1}^\infty B_{R/2}(x_i)} \left(e^{\alpha_\epsilon |u|^{n/(n-1)}} - \sum_{k=0}^{n-2} \frac{\alpha_\epsilon^k |u|^{nk/(n-1)}}{k!} \right) dv_g \\ \leq \sum_{i=1}^\infty \int_{B_{R/2}(x_i)} \left(e^{\alpha_\epsilon |u|^{n/(n-1)}} - \sum_{k=0}^{n-2} \frac{\alpha_\epsilon^k |u|^{nk/(n-1)}}{k!} \right) dv_g \\ \leq \sum_{i=1}^\infty C (\sinh R)^n \int_{B_R(x_i)} (|\nabla_g u|^n + \tau |u|^n) dv_g \\ \leq CN (\sinh R)^n \int_{\mathbb{H}^n} (|\nabla_g u|^n + \tau |u|^n) dv_g \\ \leq CN (\sinh R)^n \end{aligned} \quad (23)$$

for some constant C depending only on n and τ . For any $\alpha < \alpha_n$, we can choose $\epsilon > 0$ sufficiently small such that $\alpha < \alpha_\epsilon$. This ends the proof of Theorem 1. $\square$

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

Acknowledgment

This work is supported by the NSFC 11171347. The authors thank the referee for pointing out some grammar mistakes and the reference [19].

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