Research Article

Some Properties of Distances and Best Proximity Points of Cyclic Proximal Contractions in Metric Spaces

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Abstract

This paper presents some results concerning the properties of distances and existence and uniqueness of best proximity points of \( p \)-cyclic proximal, weak proximal contractions, and some of their generalizations for the non-self-mapping \( T : \bigcup_{i \in p} A_i \to \bigcup_{i \in p} B_i \) (\( p \geq 2 \)), where \( A_i \) and \( B_i \), for all \( i \in p \) = \{1, 2, … , \( p \)\}, are nonempty subsets of \( X \) which satisfy \( T(A_i) \subseteq B_i \), \( \forall i \in \overline{p} \), such that \( (X, d) \) is a metric space. The boundedness and the convergence of the sequences of distances in the domains and in their respective image sets of the cyclic proximal and weak cyclic proximal non-self-mapping, and of some of their generalizations are investigated. The existence and uniqueness of the best proximity points and the properties of convergence of the iterates to such points are also addressed.

1. Introduction

The characterization and study of existence and uniqueness of best proximity points is an important tool in fixed point theory concerning cyclic nonexpansive mappings including the problems of (strict) contractions, asymptotic contractions, contractive, weak-contractive mappings, and cyclic mappings and also in related problems of proximal contractions, weak proximal contractions, and approximation results and methods [1–15]. The application of the theory of fixed points in stability issues of dynamic systems, [16–21] has been proved to be a very useful tool. See, for instance, [22–26] and references therein. Some best approximation problems in semiconvex and locally convex structures and Hyers-Ulam type stability in multivalued functions and in additive-quadratic functional equations are investigated in [27–30] and some of the references therein. Recent trends concerning best proximity points and related problems are dealt with in [31–35] and some references therein. In particular, the problem of best proximity points of two mappings in a cyclic disposal is investigated in [31] under a nonlinear contractive condition. In [32], several results are obtained for proximal and weak proximal contractions of several types as well as for generalized proximal nonexpansive mappings. A modified Suzuki \( \alpha - \psi \) proximal contraction is proposed and discussed in [33] and "ad hoc" best proximity and fixed point results are obtained. Generalizations of proximal contractions of first and second kinds are given in [34, 35] for non-self-mappings and related optimal approximate solution theorems are obtained.

This paper is devoted to formulating and proving some results being concerned with the boundedness and convergence properties of distances and the convergence of the built iterated sequences to unique existing best proximity points of \( p \)-cyclic proximal and weak proximal contractions of the form \( T : \bigcup_{i \in \overline{p}} A_i \to \bigcup_{i \in \overline{p}} B_i \) (\( p \geq 2 \)) where \( A_i \) and \( B_i \), for all \( i \in \overline{p} \) = \{1, 2, … , \( p \)\}, are nonempty subsets of \( X \) which satisfy \( T(A_i) \subseteq B_i \), for all \( i \in \overline{p} \), with \( (X, d) \) being a metric space. In the most general case, all the \( A_i \) and \( B_i \) pairs of subsets, for all \( i \in \overline{p} \), are assumed to be pairwise disjoint.
The results are also extended to a class of generalized $p$-cyclic proximal and weak proximal contractions in the sense that the contractiveness constraints are referred to finite sets of consecutive iterations rather than to each iteration. The boundedness and convergence of the sequences of distances in the domains and image sets of the cyclic proximal and weak cyclic proximal non-self-mappings are investigated. The existence and uniqueness of the best proximity points and their allocation as limit points, or limit cycles of best proximity points, are also addressed. These last properties are achieved if the metric space is complete under approximative compactness’ assumptions of the image subsets of the cyclic mapping with respect to the domain subsets.

2. $p$-Cyclic Proximal Contractions, Extensions, Boundedness, and Convergence of Distances

Consider the metric space $(X, d)$ and subsets $A_i$ and $B_i$ of $X$ for $i \in \mathbb{P}$, where $\mathbb{P} = \{1, 2, \ldots, p\}$ with $p \geq 2$. Consider also a non-self-mapping $T : \bigcup_{i \in \mathbb{P}} A_i \to \bigcup_{i \in \mathbb{P}} B_i$, satisfying $T(A_i) \subseteq B_i$, for all $i \in \mathbb{P}$. Assume that $D_i = d(A_{i+1}, B_i)$, $D_A = d(A_1, A_{1+1})$, and $D_B = d(B_1, B_{i+1})$, for all $i \in \mathbb{P}$ by assuming also that $A_{n+p} = A_1$, and $B_{n+p} = B_1$, for all $i \in \mathbb{P}$, for all $n \in Z_0$. If the pair $(a_i, a_{i+1}) \in A_i \times A_{i+1}$ satisfies $d(a_i, a_{i+1}) = D_1$ for any $i \in \mathbb{P}$, then $A_i \subseteq A_{i+1}$ and $A_{i+1}$ are best proximity points in $A_i$ and $B_i$ with respect to $T : \bigcup_{i \in \mathbb{P}} A_i \to \bigcup_{i \in \mathbb{P}} B_i$.

In the following, the fact that the best proximity points are best proximity points with respect to the mapping is not mentioned explicitly.

$A_{0i} \subseteq A_i$ is the set of best proximity points of $A_i$ and $B_{0i} \subseteq B_i$ is the set of best proximity points of $B_i$. Through the paper, it is assumed that $A_{0i} \neq \emptyset$ and $B_{0i} \neq \emptyset$, for all $i \in \mathbb{P}$. An important remark is that the above statement can be considered for the particular case that $A_i = A_{1+1}$ which is well known in the context of $p$-cyclic self-mappings $T : \bigcup_{i \in \mathbb{P}} A_i \to \bigcup_{i \in \mathbb{P}} B_i$ with $T(A_i) \subseteq B_i$ for all $i \in \mathbb{P}$. However, the proposed statement is more general in the sense of the following illustrative example.

Example 1. Consider a metric space $(X, d)$ and $T : A_1 \cup A_2 \to B_1 \cup B_2$ such that $A_i, B_i \subseteq X$ are nonempty with $(A_1 \cup A_2) \subseteq B_1 \cup B_2$, $D_1 = d(A_{i+1}, B_i)$, $D_A = d(A_1, A_{i+1}) = d(A_0, A_{i+1})$, and $D_B = d(B_{i+1}, B_i)$ for $i = 1, 2$. Assume also that $A_0 = \{x_1, x_2\} \subseteq A_1$, $B_0 = \{x_1, x_2\} \subseteq B_1$, $A_{02} = \{x_3\} \subseteq A_2$, and $B_2 = \{x_3\} \subseteq B_2$. Then, we can formulate the following simple $2$-cyclic proximal-type problem. Fix $x_1 \in A_0$ as a best proximity point of $A_1$ and then compute $x_2 \in A_{02}$ and $x_3 \in A_{01}$, best proximity points of $A_2$ and $A_1$, such that

\[
\begin{align*}
d(x_2, x_1) &= d(A_2, B_1) = d(A_{02}, B_0) = D_1 \quad (\text{2-cyclic proximal constraint, first step}); \\
d(x_3, x_2) &= d(A_1, B_2) = d(A_{01}, B_0) = D_2 \quad (\text{2-cyclic proximal constraint, second step}); \\
d(A_1, A_2) &= d(A_0, A_2) = d(x_1, x_2) = d(x_2, x_3) = D_A \quad (\text{2-cyclic best proximity constraints}); \\
d(Tx_2, Tx_1) &= d(Tx_3, Tx_2) = D_B = d(B_1, B_2) = d(B_{01}, B_{02}) \quad (\text{2-cyclic associate best proximity constraints for the images}).
\end{align*}
\]

Note that there are four potentially distinct constraints related to $D_1, D_2, D_A$, and $D_B$ which can be distinct so that the problem is more general than the simple case of $D = d(A_i, A_{i+1})$ for $i = 1, 2$ for the $2$-cyclic self-mapping $T : A_1 \cup A_2 \to A_1 \cup A_2$. A variant proximal-type problem arises if $A_1 = B_1$ and $A_2 = B_2$ and the best proximity points are taken as follows:

$x_1 \in A_{01}, Tx_1 \in A_{02}, x_2 \in A_{01}, Tx_2 \in A_{02}, x_3 \in A_{01},$ and then $d(x_2, Tx_1) = d(x_3, Tx_2) = D_A = d(A_1, A_2)$.

The following definitions will be then used through the paper.

Definition 2. $T : \bigcup_{i \in \mathbb{P}} A_i \to \bigcup_{i \in \mathbb{P}} B_i$ is said to be a $p$-cyclic proximal contraction with respect to its domain (CPD$_p$) if there are real constants $\alpha_i \in [0, 1)$, for all $i \in \mathbb{P}$, such that any two sequences $\{x_{n+p}\} \subseteq A_{i+j}$ and $\{\overline{x}_{n+p}\} \subseteq A_{i+j}$, for all $i \in \mathbb{P}$, satisfy the constraints

\[
\begin{align*}
d(x_{n+p+i}, \overline{x}_{n+p+i}) &\leq \alpha_i d(x_{n+p+i}, \overline{x}_{n+p+i}) + (1 - \alpha_i) D_A, \\
&\forall i \in \mathbb{P} \quad \forall n \in Z_0, \\
&\forall i \in \mathbb{P} \quad \forall n \in Z_0,
\end{align*}
\]

provided that $x_0, \overline{x}_0 \in A_j$, for any given $j \in \mathbb{P}$ with $A_{i+j} = A_{i+j}$ if $i < p - j$ and that $d(x_{n+p+i}, T_{n+p+i}) = d(\overline{x}_{n+p+i}, T_{n+p+i}) = d(A_{i+1}, B_i) = D_i$, for all $i \in \mathbb{P}$.

Definition 3. $T : \bigcup_{i \in \mathbb{P}} A_i \to \bigcup_{i \in \mathbb{P}} B_i$ is said to be a weak $p$-cyclic proximal contraction with respect to its domain (WCPD$_p$) if there are $p$ real constants $\alpha_i \geq 0$, for all $i \in \mathbb{P}$, subject to $\alpha = \prod_{i \in \mathbb{P}} |\alpha_i| \in [0, 1)$, such that any two sequences $\{x_{n+p}\} \subseteq A_{i+j}$ and $\{\overline{x}_{n+p}\} \subseteq A_{i+j}$, for all $i \in \mathbb{P}$, satisfy the constraints (1) and (2) provided that $x_0, \overline{x}_0 \in A_j$ for any given $j \in \mathbb{P}$ and that $d(x_{n+p+i}, T_{n+p+i}) = d(\overline{x}_{n+p+i}, T_{n+p+i}) = D_i$, for all $i \in \mathbb{P}$.

Definition 4. $T : \bigcup_{i \in \mathbb{P}} A_i \to \bigcup_{i \in \mathbb{P}} B_i$ is said to be a generalized $p$-cyclic proximal contraction with respect to its domain (GCPD$_p$) if there are $p$ bounded real functions $\alpha_i : A_i \to R_{\overline{p}}$, for all $i \in \mathbb{P}$, such that any two sequences $\{x_{n+p}\} \subseteq A_{i+j}$ and $\{\overline{x}_{n+p}\} \subseteq A_{i+j}$, for all $i \in \mathbb{P}$, satisfy the constraints (1) and (2) with the replacements $\alpha_i \to \sup_{x \in A_i} \alpha_i(x)$, for all $i \in \mathbb{P}$, provided that $x_0, \overline{x}_0 \in A_j$ for any given $j \in \mathbb{P}$ and that $d(x_{n+p+i}, T_{n+p+i}) = d(\overline{x}_{n+p+i}, T_{n+p+i}) = D_i$, for all $i \in \mathbb{P}$.

Definition 5. $T : \bigcup_{i \in \mathbb{P}} A_i \to \bigcup_{i \in \mathbb{P}} B_i$ is said to be a generalized weak $p$-cyclic proximal contraction with respect to its domain (GWCPD$_p$) if there are $p$ bounded real functions $\alpha_i : A_i \to R_{\overline{p}}$, for all $i \in \mathbb{P}$, and a strictly increasing sequence of integers $\{n_k\}$, subject to $n_0 \leq \overline{N}_0 <
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\[ \lim_{k \to \infty} \nu_k = +\infty, \quad \alpha(n_k, n_{k+1}) = \min_{x \in A_i} \alpha(x, n_k, n_{k+1}) \leq \nu < +\infty, \quad \text{and} \quad \alpha = \sup_{k \in \mathbb{Z}_0^+} \alpha(n_k, n_{k+1}) \in [0, 1), \]

\[ \alpha(n_k, n_{k+1}) = \min_{x \in A_i} \alpha(x, n_k, n_{k+1}), \quad \forall k \in \mathbb{Z}_0^+, \]

such that any two sequences \( \{x_{n+k}\} \subseteq A_{i+j} \) and \( \{x_{n+k}\} \subseteq A_{i+j} \) for all \( k \in \mathbb{P} \), provided that \( x_0, x_0 \in A_j \) for any given \( j \in \mathbb{P} \), satisfy the constraints

\[ d \left( x_{n+k}, x_{n+k+1}, x_{n+k+2} \right) \leq \alpha(n_k, n_{k+1}) + (1 - \alpha(n_k, n_{k+1})) D_A, \]

\[ \forall k \in \mathbb{Z}_0^+, \]

and the constraints (1) and (2) provided that \( x_0, x_0 \in A_j \) for any given \( j \in \mathbb{P} \) and that \( d(x_{n+k+1}, x_{n+k+2}) = D_A, \) for all \( k \in \mathbb{P} \).

The following assertions are obvious without proof from Definitions 2–5.

**Assertion I.** If \( T : \bigcup_{i \in \mathbb{P}} A_i \to \bigcup_{i \in \mathbb{P}} B_i \) is CPD, then it is WCPD.

**Assertion II.** \( T : \bigcup_{i \in \mathbb{P}} A_i \to \bigcup_{i \in \mathbb{P}} B_i \) is WCPD, then it is GCPD.

**Assertion III.** \( T : \bigcup_{i \in \mathbb{P}} A_i \to \bigcup_{i \in \mathbb{P}} B_i \) is GCPD, then it is GWCPD.

Note that the converse implications of those in Assertions 1 are not true in general. The relevant distances satisfy the following convergence and boundedness result.

**Lemma 6.** Consider a metric space \((X, d)\) with subsets \( A_i, B_i \subset X \) and a \( p \)-cyclic mapping \( T : \bigcup_{i \in \mathbb{P}} A_i \to \bigcup_{i \in \mathbb{P}} B_i \) which is GWCPD, subject to \( D_A = d(A_{i+1}, B_i), D_A = d(A_{i+1}, B_{i+1}), \) and \( D_B = d(B_i, B_{i+1}) \), with \( T(A_{i+1}) \subseteq B_{i+1} \) for all \( i \in \mathbb{P} \) such that \( A_{i+1} \subseteq A_i \) and \( B_{i+1} \subseteq B_i \) are nonempty, for all \( i \in \mathbb{P} \).

Consider also any sequences \( \{x_{n+k}\}, \{y_{n+k}\} \subseteq \bigcup_{i \in \mathbb{P}} A_i \) which satisfy \( d(x_{n+k+1}, x_{n+k+2}) = d(x_{n+k+1}, x_{n+k+2}) = D_A \), for all \( i \in \mathbb{P} \). Then, the following properties hold.

1. The sequences of distances \( \{d(x_{n+k}, x_{n+k+1})\} \to D_A \), \( \{d(x_{n+k}, x_{n+k+1})\} \to 0 \) and they are bounded for any given initial points \( x_0, x_0 \in A_j \) for any given \( j \in \mathbb{P} \).

2. If, furthermore, \( T : \bigcup_{i \in \mathbb{P}} A_i \to \bigcup_{i \in \mathbb{P}} B_i \) is continuous in \( cl(T(A_0)) \), for all \( i \in \mathbb{P} \), then \( d(T(x_0), T(x_0)) \to 0 \) and \( d(T(x_0), T(x_0)) \to D_B \) and both sequences of distances are bounded for any given initial points \( x_0, x_0 \in A_j \) for any given \( j \in \mathbb{P} \).

3. If the sets of best proximity points \( A_{i+1} \) and \( B_{i+1} \) for all \( i \in \mathbb{P} \), are bounded, then the sequences \( \{d(x_{n+k}, x_{n+k+1})\}, \{d(x_{n+k}, x_{n+k+1})\}, \{d(T(x_0), T(x_0))\}, \{d(T(x_0), T(x_0))\} \) are uniformly bounded for any initial points \( x_0, x_0 \in A_i \) for some \( j \in \mathbb{P} \).

4. If the sets of best proximity points \( A_{i+1} \) and \( B_{i+1} \) for all \( i \in \mathbb{P} \), are bounded, then the sequences \( \{d(x_{n+k}, x_{n+k+1})\}, \{d(x_{n+k}, x_{n+k+1})\}, \{d(T(x_0), T(x_0))\}, \{d(T(x_0), T(x_0))\} \) are uniformly bounded for any initial points \( x_0, x_0 \in A_i \) for any arbitrary given \( j \in \mathbb{P} \).

5. The corresponding image sequences \( \{T(x_{n+k})\} \subseteq cl(B_{i+1}) \), for all \( i \in \mathbb{P} \), are also Cauchy sequences if \( T(A_0) \subseteq B_0 \) for all \( i \in \mathbb{P} \) and \( T : \bigcup_{i \in \mathbb{P}} A_i \to \bigcup_{i \in \mathbb{P}} B_i \) is continuous in \( cl(T(A_0)) \), for all \( i \in \mathbb{P} \).

Proof. Note that, for any \( i \in \mathbb{P} \), \( \emptyset \neq A_i \subseteq A_i \).
for all $i \in (1, \max(n_{k+1} - n_k)) \cap Z_{q_{1i}}$, since $\overline{a} \in [0, 1)$, 
limsup_{k \to \infty} (n_{k+1} - n_k) \leq \overline{a} < +\infty$ and $\alpha_i \in B(A; \mathbb{R}_0)$ with 
max_{j \in \mathbb{Z}} (sup_{x \in \overline{x}} \alpha_i(x)) \leq \overline{a} \alpha_{k+1}.$

Thus, $\exists \lim_{k \to \infty} d(x_{n_{k+1}+1}, x_{n_{k+1}+p-1}) = \lim_{k \to \infty} d(x_{n_{k+1}+1}, x_{n_{k+1}+p-1}) = D_A$, for all 
i \in (1, \max(n_{k+1} - n_k)) \cap Z_{q_{1i}}$ from (9a) and the distance subsequences 
$d(x_{n_{k+1}+1}, x_{n_{k+1}+p-1})$, for all i \in (1, \max(n_{k+1} - n_k)) \cap Z_{q_{1i}}$ is bounded from (9b) for any given initial points 
x_{0}, \overline{x}_{0} \in \bigcup_{j \in \mathbb{Z}} A_j. Also, one gets from (6), subject to (3), that 

\[ \exists \lim_{k \to \infty} d(x_{n_{k+1}+1}, x_{n_{k+1}+p-1}) = \lim_{k \to \infty} d\left(x_{n_{k+1}+1}, x_{n_{k+1}, p+1}\right) = 0, \]

for all i \in (1, \max(n_{k+1} - n_k)) \cap Z_{q_{1i}}. Those results also imply that the sequences of distances 
$d(x_{n_{k+1}, p+1}) \rightarrow D_A$ or $d(x_{n_{k+1}, p+1}) \rightarrow 0$. It is now proved by contradiction that 
$d(Tx_{n_{k+1}}, T\overline{x}_{n_{k+1}}) \rightarrow D_B$ and $d(Tx_{n_{k+1}}, T\overline{x}_{n_{k+1}}) \rightarrow 0$. Assume that, for each given $x_{0} \in A_{0,j} \subset \bigcup_{j \in \overline{p}} A_j$ for some $j \in \overline{p}$ and any $\epsilon \in \mathbb{R}_+$, there are some $\delta \in \mathbb{R}_+$, some $i \in \overline{p}$, some 
k_{0} = k_{0}(\epsilon, x_{0}) \in Z_{q_{1i}}$, sequences of integers $n_{k+1} \in Z_{0,i}$, and sequences of best proximity points $x_{n_{k+1}+1} \in cl(A_{0,j+1})$ and 
$\overline{T}x_{n_{k+1}+1} \in cl(B_{0,j+1})$ for $k > k_{0}$, such that $d(Tx_{n_{k+1}}, T\overline{x}_{n_{k+1}})$ does not converge to zero so that it has some subsequence which does not converge either:

\[ d(A_{0,j+1}) = d(A_{0,j}, A_{0,j+1}) = D_A, \]

\[ d(T(A_{0,j}), T(A_{0,j+1})) = d(B_{0,j}, B_{0,j+1}) = D_B, \quad \forall i \in \overline{p}, \]

\[ d(x_{n_{k+1}+1}, x_{n_{k+1}+p-1}) < D_A + \epsilon, \]

\[ d(Tx_{n_{k+1}+1}, Tx_{n_{k+1}+p-1}) \geq \delta + D_B, \quad \forall k > k_{0}, \]

\[ d(x_{n_{k+1}+1}, x_{n_{k+1}+p-1}) < \epsilon, \quad d(Tx_{n_{k+1}+1}, Tx_{n_{k+1}+p-1}) \geq \delta, \quad \forall k > k_{0}, \]
$$Tx_{n+1} \in clB_{0+j} \subseteq clB_{j+1},$$

$$Tx_{n+p+i} \in clB_{0+j-1} \subseteq clB_{j-1},$$

$$\forall j \in \overline{p}$$

(11)

since $T(A_0) \subseteq B_0$, for all $i \in \overline{p}$. This implies that $\{x_{n+i}^{p+i} - x_{n+1}^{p+1}\} \rightarrow 0$ while $\{Tx_{n+p+i} - Tx_{n+1}^{p+1}\}$ does not converge to zero.

Since $\{x_{n+p+i}^{p+i} - x_{n+i}^{p+i}\} \rightarrow 0$, $\{x_{n+i}^{p+i}\} \subseteq \text{cl}(A_{0+j})$, $\{Tx_{n+p+i}^{p+i}\} \subseteq \text{cl}(T(A_{0+j})) \subseteq \text{cl}(B_{0+j})$, and $T : \bigcup_i \mathcal{P} A_i \rightarrow \bigcup_i \mathcal{P} B_i$ is continuous in $\text{cl}(A_{0+j})$, for all $j \in \overline{p}$ and any given $i \in \overline{p}$, then $\{Tx_{n+p+i} - Tx_{n+p+i}^{p+i}\} \rightarrow 0$. Also, $\{x_{n+i}^{p+i} - 1\} \subseteq \text{cl}(A_{0+j-1})$, $\{Tx_{n+p+i}^{p+i} - 1\} \subseteq \text{cl}(T(A_{0+j-1})) \subseteq \text{cl}(B_{0+j-1})$, for all $i \in \overline{p}$ and any given $j \in \overline{p}$; then $\{Tx_{n+p+i} - Tx_{n+p+i}^{p+i}\} \rightarrow D_B, \{Tx_{n+p+i-1}^{p+i} - Tx_{n+p+i}^{p+i}\} \rightarrow D_B$. Thus, $\{d(Tx_n, TX_n)\} \rightarrow 0$ and $\{d(Tx_n, TnX_n)\} \rightarrow 0$ if $x_0, x_0 \in A_j \in \bigcup_i \mathcal{P} A_i$, for any given $j \in \overline{p}$.

On the other hand, $\{d(Tx_n, TX_n)\} \subseteq \bigcup_i \mathcal{P} A_i$ and $\{d(TpX_n, TnX_n)\}$ are bounded, since $\{d(x_0, TX_n)\}$ is bounded from (6) because $T : \bigcup_i \mathcal{P} A_i \rightarrow \bigcup_i \mathcal{P} B_i$ is GCWCPD, and since one has for some positive real constant $M = M(d(x_0, x_0))$ that

$$d(Tx_n, TX_n) \leq d(Tx_n, x_n) + d(x_n, TX_n) + d(x_n, x_n)$$

$$\leq 2 \max_{i \leq p} D_i + M, \quad \forall n \in \mathbb{Z}+,$$

$$d(Tx_n, TnX_n)$$

$$\leq d(Tx_n, x_{n+1}) + d(x_{n+1}, TnX_n) + d(x_{n+1}, x_{n+1})$$

$$\leq 2 \max_{i \leq p} D_i + M, \quad \forall n \in \mathbb{Z}+,$$

(12)

If the sets of best proximity points $A_0 i$ for all $i \in \overline{p}$ are bounded, then the sequences of distances $\{d(x_n, TX_n)\}, \{d(x_n, x_n)\}, \{d(Tx_n, TX_n)\}$, and $\{d(Tx_n, TnX_n)\}$ are uniformly bounded for any initial best proximity points $x_0, x_0 \in A_0, \subseteq \bigcup_i \mathcal{P} A_i$ where $i$ follows by taking $M = M \sup_{i \leq p} \mathcal{P} \mathcal{L}_{A,i} d(x_0, x_0))$. Property (i) has been fully proved.

To prove Property (ii), take any sequences $x_{n+1}^{p+i} \subseteq A_{0+i}$ and $x_{n+1}^{p+i} \subseteq A_{0+i}$ for all $i \in \overline{p}$, for given initial points $x_0, x_0 \in A_j \subseteq \bigcup_i \mathcal{P} A_i$ for some $j \in \overline{p}$. Note, from (6) for $\{x_{n+1}^{p+i}\} = \{x_{n+1}^{p+i}\}$, for all $i \in \overline{p}$, that

$$d(x_{n+1}^{p+i}, x_{n+1}^{p+i})$$

$$\leq H_d(x_{n+1}^{p+i}, x_{n+1}^{p+i}) \leq h_{0\alpha}d(x_{n+1}^{p+i}, x_{n+1}^{p+i}),$$

$$\forall k \in \mathbb{Z}_+, \quad \forall i \in \overline{p},$$

(13)
\[ \bigcup_{i \in \mathcal{P}} A_i \text{ for any given } j \in \mathcal{P}. \] The corresponding image sequences \( \{T x_{n_p+i}\} \subseteq \mathcal{C} T (A_{n_p+i}) \) for all \( i \in \mathcal{P} \), are also convergent, then Cauchy sequences \( \text{iff } T (A_{n_p+i}) \subseteq B_{n_p+i} \), for all \( i \in \mathcal{P} \), and \( T : \bigcup_{i \in \mathcal{P}} A_i \to \bigcup_{i \in \mathcal{P}} B_i \) is continuous in \( \mathcal{C} T (A_{n_p+i}) \), for all \( i \in \mathcal{P} \).

**Definition 8.** \( T : \bigcup_{i \in \mathcal{P}} A_i \to \bigcup_{i \in \mathcal{P}} B_i \) is said to be a \( p \)-cyclic proximal contraction with respect to its image (CP\( p \)) if there are real constants \( \beta_i \in [0, 1), \) for all \( i \in \mathcal{P} \), such that any two sequences \( \{T x_{n_p+i}\} \subseteq T (A_{n_p+i}) \subseteq B_{n_p+i} \) and \( \{\mathcal{C} T (A_{n_p+i})\} \subseteq T (A_{n_p+i}) \subseteq B_{n_p+i} \), for all \( i \in \mathcal{P} \), being point-to-point images of sequences \( \{x_{n_p+i}\} \subseteq A_{n_p+i} \) and \( \{\mathcal{C} T (A_{n_p+i})\} \subseteq A_{n_p+i} \) for any given \( j \in \mathcal{P} \) which satisfies \( d(x_{n_p+i+1}, T x_{n_p+i}) = d(x_{n_p+i+1}, T \mathcal{C} T (A_{n_p+i})) = D_{i} \), for all \( i \in \mathcal{P} \), where \( B_{i+j} = B_{i+j-p} \) for \( i > j \), and for all \( i \in \mathcal{P} \), such that the initial points \( T x_{0}, T \mathcal{X} \subseteq T (A_{i}) \subseteq B_{i} \) are the images of points \( x_{0}, \mathcal{X} \subseteq A_{i} \), for any given \( j \in \mathcal{P} \), satisfy the constraints

\[
\begin{align*}
d(T x_{n_p+i+1}, T \mathcal{X}_{n_p+i}) & \leq \sum_{j=1}^{i} \left[ \sup_{x \in A_{j}} \beta_j T (x) \right] d(T x_{n_p+i}, T \mathcal{X}_{n_p+i}) \quad \forall \mathcal{P}, \forall n \in \mathcal{P}. \tag{16} \\
& \leq \beta d(T x_{n_p+i}, T \mathcal{X}_{n_p+i}) + (1 - \beta) D_{i}, \quad \forall \mathcal{P}, \forall n \in \mathcal{P}. \tag{17}
\end{align*}
\]

**Definition 9.** \( T : \bigcup_{i \in \mathcal{P}} A_i \to \bigcup_{i \in \mathcal{P}} B_i \) is said to be a weak \( p \)-cyclic proximal contraction with respect to its image (WC\( p \)) if there are \( p \) real constants \( \beta_i \geq 0, \) for all \( i \in \mathcal{P} \), subject to \( \beta = \prod_{i=0}^{p-1} \beta_i \in [0, 1), \) such that any two sequences \( \{T x_{n_p+i}\} \subseteq T (A_{n_p+i}) \subseteq B_{n_p+i} \) and \( T \mathcal{X}_{n_p+i} \subseteq T (A_{n_p+i}) \subseteq B_{n_p+i} \), for all \( i \in \mathcal{P} \), being point-to-point images of sequences \( \{x_{n_p+i}\} \subseteq A_{n_p+i} \) and \( \{\mathcal{C} T (A_{n_p+i})\} \subseteq A_{n_p+i} \) for any given \( j \in \mathcal{P} \) such that the initial points \( T x_{0}, T \mathcal{X} \subseteq T (A_{i}) \subseteq B_{i} \) are the images of points \( x_{0}, \mathcal{X} \subseteq A_{i} \), for any given \( j \in \mathcal{P} \), satisfy constraints (16) and (17).

**Definition 10.** \( T : \bigcup_{i \in \mathcal{P}} A_i \to \bigcup_{i \in \mathcal{P}} B_i \) is said to be a generalized \( p \)-cyclic proximal contraction with respect to its image (GC\( p \)) if there are \( p \) bounded real functions \( \beta_i : A_i \to \mathbb{R}_{+}, \) for all \( i \in \mathcal{P} \), such that any sequences \( \{T x_{n_p+i}\} \subseteq T (A_{n_p+i}) \subseteq B_{n_p+i} \) and \( \{\mathcal{X}_{n_p+i}\} \subseteq T (A_{n_p+i}) \subseteq B_{n_p+i} \), for all \( i \in \mathcal{P} \), being point-to-point images of sequences \( \{x_{n_p+i}\} \subseteq A_{n_p+i} \) and \( \{\mathcal{X}_{n_p+i}\} \subseteq A_{n_p+i} \) for any given \( j \in \mathcal{P} \), such that the initial points \( T x_{0}, T \mathcal{X} \subseteq T (A_{i}) \subseteq B_{i} \) are the images of points \( x_{0}, \mathcal{X} \subseteq A_{i} \), for any given \( j \in \mathcal{P} \), satisfy the constraints (16) and (17) with the replacements \( \beta_i \to \sup_{x \in A_i} \beta_i (T x) \), for all \( i \in \mathcal{P} \).

**Definition 11.** \( T : \bigcup_{i \in \mathcal{P}} A_i \to \bigcup_{i \in \mathcal{P}} B_i \) is said to be a generalized weak \( p \)-cyclic proximal contraction with respect to its image (GW\( p \)) if there are \( p \) bounded real functions \( \beta_i : A_i \to \mathbb{R}_{+}, \) for all \( i \in \mathcal{P} \), and a strictly increasing sequence of integers \( \{n_k\} \), subject to \( n_0 \leq \mathcal{N} 0 < +\infty \), \( \lim_{k \to \infty} n_k \to +\infty \), \( \lim_{k \to \infty} n_k = n_k \leq \infty < +\infty \), \( \mathcal{X} \) and \( \beta \) be \( \sup_{x \in A_i} \beta (n_k, n_{k+1}) \in (0, 1) \), where

\[ \beta (n_k, n_{k+1}) = \prod_{j=1}^{n_k} \left[ \sup_{x \in A_i} \beta_j (T x_{j+p}) \right], \quad \forall k \in \mathcal{Z}_{n_k}, \tag{18} \]

\[ \leq \beta (n_k, n_{k+1}) \prod \left[ \sup_{x \in A_i} \beta_j (T x_{j+p}) \right] (1 - \beta (n_k, n_{k+1})) D_{i}, \quad \forall k \in \mathcal{Z}_{n_k}. \tag{19} \]

\[ \leq \beta (n_k, n_{k+1}) \prod \left[ \sup_{x \in A_i} \beta_j (T x_{j+p}) \right] (1 - \beta (n_k, n_{k+1})) D_{i}, \quad \forall k \in \mathcal{Z}_{n_k}. \tag{20} \]

\[ \leq \beta (n_k, n_{k+1}) \prod \left[ \sup_{x \in A_i} \beta_j (T x_{j+p}) \right] (1 - \beta (n_k, n_{k+1})) D_{i}, \quad \forall k \in \mathcal{Z}_{n_k}. \tag{21} \]

The following assertions are obvious without proof from Definitions 8–11 and are a parallel result to Assertions 1.

**Assertions 2.** If \( T : \bigcup_{i \in \mathcal{P}} A_i \to \bigcup_{i \in \mathcal{P}} B_i \) is CP\( p \), then it is WC\( p \).

If \( T : \bigcup_{i \in \mathcal{P}} A_i \to \bigcup_{i \in \mathcal{P}} B_i \) is WC\( p \), then it is GC\( p \).

If \( T : \bigcup_{i \in \mathcal{P}} A_i \to \bigcup_{i \in \mathcal{P}} B_i \) is GC\( p \), then it is GW\( p \).

Note that the converse implications of those in Assertions 1 are not true in general.

The relevant distances satisfy the following convergence and boundedness result which is a counterpart of Lemmas 6 and 7. Its proof is close to that of Lemma 6 and Assertions 2 by using (16) and (17) for Definition 11 and their variants for Definitions 8–10.
Lemma 12. Assume that \( T : \bigcup_{i \in \mathbb{P}} A_i \rightarrow \bigcup_{i \in \mathbb{P}} B_i \) is either \( GWCPP \) or \( GCPP \) or \( WCPP \) or \( CPP \) under the assumptions of Lemma 6, and consider any sequences \( \{x_{n_{j,i}}\}, \{x_{n_{j,i}}\} \subseteq \bigcup_{i \in \mathbb{P}} A_i \) which satisfy \( d(x_{n_{j,i+1}}, T x_{n_{j,i+1}}) = d(x_{n_{j,i+1}}, T x_{n_{j,i+1}}) = D_i \), for all \( i \in \mathbb{P} \). Then, the following properties hold.

(i) The sequences of distances \( \{d(x_n, x_m)\} \rightarrow D_A \), \( \{d(x_n, x_m)\} \rightarrow 0 \) and they are bounded for any given initial points \( x_0, x_0 \in A_j \subset \bigcup_{i \in \mathbb{P}} A_i \), for some \( i \in \mathbb{P} \).

(ii) If, furthermore, \( T(A_{0,i}) \subseteq B_{0,i} \), for all \( i \in \mathbb{P} \), and \( T : \bigcup_{i \in \mathbb{P}} A_i \rightarrow \bigcup_{i \in \mathbb{P}} B_i \) is continuous in \( clT(A_{0,i}) \), for all \( i \in \mathbb{P} \), then \( d(T(x_n), T(x_m)) \rightarrow 0 \) and \( d(T(x_n), T(x_{n+1})) \rightarrow D_B \) and it is bounded any given initial points \( x_0, x_0 \in A_{0,j} \subset \bigcup_{i \in \mathbb{P}} A_i \) for some \( j \in \mathbb{P} \).

If the sets of best proximity points \( A_{0,i} \) and \( B_{0,i} \), for all \( i \in \mathbb{P} \), are Cauchy sequences for any given initial point \( x_0 \in A_{0,j} \subset \bigcup_{i \in \mathbb{P}} A_i \), for any given \( i \in \mathbb{P} \). The corresponding image sequences \( \{T(x_n)\} \subseteq \bigcup_{i \in \mathbb{P}} B_{0,i} \), for all \( i \in \mathbb{P} \), are also convergent; then Cauchy sequences if \( T(A_{0,i}) \subseteq B_{0,i} \), for all \( i \in \mathbb{P} \), and \( T : \bigcup_{i \in \mathbb{P}} A_i \rightarrow \bigcup_{i \in \mathbb{P}} B_i \) are continuous in \( clT(A_{0,i}) \), for all \( i \in \mathbb{P} \).

Remark 13. The result \( d(T(x_n), T x_{n+1}) \rightarrow D_B \) of Lemma 12, as well as Lemma 12(ii), obtained under the assumption that \( T : \bigcup_{i \in \mathbb{P}} A_i \rightarrow \bigcup_{i \in \mathbb{P}} B_i \) is continuous in \( clT(A_{0,i}) \), and also holds without such a continuity assumption if the contractive conditions (16) and (17) become modified to the right limits as follows:

\[
d(T(x_{n_{j,i+1}}), T x_{n_{j,i+1}}) \leq \beta_i d(T x_{n_{j,i}}, T x_{n_{j,i+1}}) + (1 - \beta_i) D_B, \quad \forall i \in \mathbb{P}, \quad \forall n \in \mathbb{Z}_{0,i},
\]

\[
d(T(x_{n_{j,i+1}}), T x_{n_{j,i+1}}) \leq \beta d(T x_{n_{j,i}}, T x_{n_{j,i}}), \quad \forall i \in \mathbb{P}, \quad \forall n \in \mathbb{Z}_{0,i},
\]

provided that any discontinuity points in \( \bigcup_{i \in \mathbb{P}} clT(A_{0,i}) \), if any, are of first-class finite-jump type under right best proximity constraints

\[
d(x_{n_{j,i+1}}, T x_{n_{j,i+1}}) = d(x_{n_{j,i+1}}, T x_{n_{j,i+1}}) = D_i, \quad \forall i \in \mathbb{P}.
\]

In the same way, the result \( d(T(x_n), T x_{n+1}) \rightarrow D_B \) of Lemmas 6 and 7, as well as their properties (ii) obtained under the assumption that \( T : \bigcup_{i \in \mathbb{P}} A_i \rightarrow \bigcup_{i \in \mathbb{P}} B_i \) is continuous in \( clT(A_{0,i}) \) and also holds under finite-jump discontinuities in \( clT(A_{0,i}) \) for sequences \( \{x_n\}, \{x_{n+1}\}, \{T x_n\}, \) and \( \{T x_{n+1}\} \) satisfying the contractive proximal conditions (1) and (2) if Definition 2, or their counterparts of Definitions 3–5 for right values \( T x_n \) and \( T x_{n+1} \) under right best proximity constraints (24).}

### 3. Best Proximity Points and Related Convergence Results

We first recall the subsequent useful definition [2–4, 7] as follows.

**Definition 14.** Let \( A \) and \( B \) be two nonempty subsets of a metric space \( (X, d) \) and let \( d(y, A) = \inf \{d(y, x) : x \in A \} \) for \( y \in X \). \( A \) is said to be approximately compact with respect to \( B \) if each sequence \( \{x_n\} \subseteq A \) satisfying \( d(y, x_n) \rightarrow d(y, A) \) for some \( y \in B \) has a convergent subsequence.

Note that if the sets of best proximity points \( A_0 \subseteq A \) and \( B_0 \subseteq B \) are nonempty if Definition 14 holds, then \( A \) is approximately compact with respect to \( B \) if every sequence \( \{x_n\} \subseteq A \) such that \( d(y, x_n) \rightarrow d \) for some \( y \in B_0 \) has a convergent subsequence \( \{x_n\} \subseteq \{x_n\} \) since \( D = d(y, A) = d(B_0, A) = d(B_0, A_0) \). Note that every set is approximately compact with respect to itself and that every compact set is approximately compact with respect to any nonempty subset of a metric space. Also, if \( B \) is compact and \( A \) is approximately compact with respect to \( B \), each sequence \( \{x_n\} \subseteq A \) has a convergent sequence. If \( A \) and \( B \) are nonempty and closed and \( A \) is approximately compact with respect to \( B \), then \( B_0 \) is closed. See, for instance, [2–4, 7]. A result on existence and uniqueness of best proximity points follows for \( p \)-cyclic proximal contraction fulfilling Definitions 2–5 under Lemmas 6 and 7 follows.

**Theorem 15.** Consider a complete metric space \( (X, d) \) with nonempty closed subsets \( A_i, B_i \subseteq X \) and a \( p \)-cyclic mapping \( T : \bigcup_{i \in \mathbb{P}} A_i \rightarrow \bigcup_{i \in \mathbb{P}} B_i \) being either \( GWCPP \), or \( GCPP \), or \( WCPP \), or \( CPP \), subject to set distances \( D_i = d(A_{i+1}, B_i) \), \( D_A = d(A_i, A_{i+1}) \), and \( D_B = d(B_i, B_{i+1}) \), for all \( i \in \mathbb{P} \) such that \( A_0 \) is nonempty and \( B_0 \) is approximately compact with respect to \( A_1 \) and \( T(A_0) \subseteq B_0 \), for all \( i \in \mathbb{P} \). The following properties hold.

(i) \( T : \bigcup_{i \in \mathbb{P}} A_i \rightarrow \bigcup_{i \in \mathbb{P}} B_i \) has a unique best proximity point \( x^*_i \) at \( A_i \), such that \( d(x^*_i, x^*_i) = D_A \), for all \( i \in \mathbb{P} \), and all the sequences \( \{x_n\} \subseteq \bigcup_{i \in \mathbb{P}} A_i \) converge to a unique limit cycle \( \{x^*_1, x^*_2, \ldots, x^*_p\} \).

(ii) Furthermore, if \( T : \bigcup_{i \in \mathbb{P}} A_i \rightarrow \bigcup_{i \in \mathbb{P}} B_i \) either is continuous, respectively, has eventual finite-jump discontinuity points, then \( x^*_i = T x^*_i \), for all \( i \in \mathbb{P} \), respectively, \( x^*_i = T x^*_i \), for all \( i \in \mathbb{P} \), are unique best proximity points such that \( d(T x^*_i, T x^*_i) = D_B \), for all \( i \in \mathbb{P} \), respectively, \( d(T x^*_i, T x^*_i) = D_B \), for all \( i \in \mathbb{P} \), and all the sequences \( \{x_n\} \subseteq \bigcup_{i \in \mathbb{P}} A_i \) converge to a unique limit cycle \( \{T x^*_1, T x^*_2, \ldots, T x^*_p\} \).

**Proof.** Since \( A_0 \) is nonempty and \( T(A_0) \subseteq B_0 \), for all \( i \in \mathbb{P} \), then \( T(A_0) \) and \( B_0 \), are nonempty, for all \( i \in \mathbb{P} \). Also, \( A_{0,i} \) is closed since \( B_i \) is approximately compact with respect to...
A_i. Consider any sequences \( \{x_{n+1}, x_{n+2}, \ldots, x_{n+p}\} \subseteq \bigcup_{i \in p} A_i \) which satisfy
\[
d(x_{n+1}, T x_{n+1}, T x_{n+1}) = D_i, \quad \forall i \in p.
\]
One gets, from Lemma 6(i), if the mapping \( T : \bigcup_{i \in p} A_i \rightarrow \bigcup_{i \in p} B_i \) is GWPDP and, from Lemma 7(ii), if the mapping \( T : \bigcup_{i \in p} A_i \rightarrow \bigcup_{i \in p} B_i \) is either GCPDP or WCPDP or CPDP, that, since \( T(A_0) \subseteq B_0 \), for all \( i \in p \),
\[
\left\{ d(\overline{x}_{p(n+1)+j}, x_{p(n+1)+j}) \right\} \rightarrow 0,
\]
\[
\left\{ d(\overline{x}_{p(n+1)+j}, x_{p(n+1)+j}) \right\} \rightarrow 0,
\]
(25)

One gets, from Lemma 6(i), if the mapping \( T : \bigcup_{i \in p} A_i \rightarrow \bigcup_{i \in p} B_i \) is GWPDP and, from Lemma 7(ii), if the mapping \( T : \bigcup_{i \in p} A_i \rightarrow \bigcup_{i \in p} B_i \) is either GCPDP or WCPDP or CPDP, that, since \( T(A_0) \subseteq B_0 \), for all \( i \in p \),
(30)

(5) Since \( B_i \) is closed and approximately compact with respect to \( A_i \), for all \( i \in p \), one gets from (28) that a subsequence of \( \{T x_{n+1}\} \) is convergent for each \( i \in p \); say \( \{T x_{n+1}\} \rightarrow z^*_i \in T(A_i) \subseteq B_i \), for all \( i \in p \). Since \( \{x_{n+1+i}\} \rightarrow x^*_i \), all its subsequences converge to the same limit so that \( \{x_{n+1+i}\} \rightarrow x^*_i \) and then \( z^*_i = T x^*_i \) is unique, since each \( x^*_i \) is unique, within each \( B_i \), for all \( i \in p \) and, again, from (28), \( D_i = d(x^*_i+1, T x^*_i) \), for all \( i \in p \) if \( T : \bigcup_{i \in p} A_i \rightarrow \bigcup_{i \in p} B_i \) is continuous at \( x^*_i \) and \( D_i = d(x^*_i, T x^*_i) \) if \( T : \bigcup_{i \in p} A_i \rightarrow \bigcup_{i \in p} B_i \) has a finite-jump discontinuity at \( x^*_i \), then \( z^*_i = T x^*_i \) (see Remark 13). The result has been proved.

A further result on the existence and uniqueness of best proximity points follows for \( p \)-cyclic proximal contractions subject to Definitions 8–11 under Lemma 12 and whose proof is very close to that of Theorem 15.

**Theorem 16.** Consider a complete metric space \((X,d)\) with nonempty subsets \(A_i, B_i \subseteq X\) and an \( p \)-cyclic mapping \( T : \bigcup_{i \in p} A_i \rightarrow \bigcup_{i \in p} B_i \) being either GWCPDP or GCPIp or WCPDP or CPIp, subject to set distances \( D_i = d(A_i, B_i) \), \( D_A = d(A_0, A_1) \), and \( D_B = d(B_0, B_1) \), for all \( i \in p \), such that \( A_0 \) is nonempty and closed and \( B_0 \) is nonempty and \( A_i \) is approximately compact with respect to \( B_i \) and \( T(A_0) \subseteq B_0 \), for all \( i \in p \). The following properties hold.

(i) \( T : \bigcup_{i \in p} A_i \rightarrow \bigcup_{i \in p} B_i \) has a unique best proximity point \( x^*_i \) at \( A_i \) such that \( d(x^*_i, x^*_i) = D_A \), for all \( i \in p \), and all the sequences \( \{x_n\} \subseteq \bigcup_{i \in p} A_i \) converge to a unique limit cycle \( \{x^*_1, x^*_2, \ldots, x^*_p\} \).
(26)

(ii) Furthermore, if \( T : \bigcup_{i \in p} A_i \rightarrow \bigcup_{i \in p} B_i \) either is continuous, respectively, has eventual finite-jump discontinuity points, then \( z^*_i = T x^*_i \), for all \( i \in p \), respectively, \( z^*_i = (T x^*_i)^j \), for all \( i \in p \), are unique best proximity points such that \( d(T x^*_i, T x^*_i) = D_B \), for all \( i \in p \), respectively, \( d(T x^*_i, T x^*_i) = D_B \), for all \( i \in p \), and all the sequences \( \{x_n\} \subseteq \bigcup_{i \in p} A_i \) converge to a unique limit cycle \( \{x^*_1, x^*_2, \ldots, x^*_p\} \).

Proof. Since \( T(A_0) \subseteq B_0 \), for all \( i \in p \), then \( T(A_0) \) and \( B_0 \) are nonempty, since \( A_0 \) is nonempty, for all \( i \in p \). \( B_0 \) is, furthermore, closed since \( A_i \) is approximately compact with respect to \( B_i \). Thus, \( A_0 \), \( T(A_0) \), and \( B_0 \) are nonempty and closed, for all \( i \in p \). Consider any sequences \( \{x_{n+1}, x_{n+2}, \ldots, x_{n+p}\} \subseteq \bigcup_{i \in p} A_i \) which satisfy

(30)
\[ \{ d(TX_{p(n+1)+\ell}, TX_{p(n+\ell+1)}) \} \to 0, \]
\[ \{ d(TX_{p(n+1)+\ell+1}, TX_{p(n+\ell+1)+1}) \} \to 0, \]
\[ \{ d(TX_{p(n+\ell+1)}, TX_{p(n+\ell+1)+1}) \} \to 0, \]
\[ d(y_{n+j}, B_{n+i}) = d(y_{n+j}, B_{0+i}), \]
\[ d(z_{p(n+\ell+1)}^{1}, TX_{p(n+\ell+1)+1}) \to 0, \] for some \( j, \ell \in \overline{p} \) and some nonnegative integer \( \delta < p \). But then \( d(TX_{p(n+1)+\ell}, TX_{p(n+\ell+1)}) \) does not converge to zero so that \( x_i^{\ast} = x_i \in A_0 \) is unique, for all \( i \in \overline{p} \). The distance convergence properties are independent of the fact that for the initial condition \( \ell \) is as equal or distinct as \( j \), as discussed in Lemma 6. If and another sequence. Since each \( z_i^* \) is unique, within each \( A_i \), for all \( i \in \overline{p} \) and, again, from (33), \( D_i = d(x_i^{*}, T x_i^{*}) \), for all \( i \in \overline{p} \) if \( T : \bigcup_{i \in \overline{p}} A_i \to \bigcup_{i \in \overline{p}} B_i \) is continuous at \( x_i^{*} \) and \( D_i = d(x_i^{*}, T x_i^{*}) \) if \( T : \bigcup_{i \in \overline{p}} A_i \to \bigcup_{i \in \overline{p}} B_i \) has a finite-jump discontinuity at \( x_i^{*} \), then \( z_i^{*} = T x_i^{*} \) (see Remark 13). The result has been proved.

**Example 17.** Consider a 2-cyclic proximal contraction with respect to its domain: \( T : A_1 \cup A_2 \to B_1 \cup B_2 \), where \( A_1 \) and \( B_1 \) for \( i = 1, 2 \) are nonempty closed subsets of \( R \). Take any sequences \( \{x_n\} \subset A_1, \{y_n\} \subset A_2 \) being subsequences of \( \{z_n\} \subset A_1 \cup A_2 \) defined either by \( z_n = x_{n0} \) for some \( n \in Z_0 \), and subject to the constraints below under the Euclidean metric \( d(x, y) = |x - y| \), for all \( x, y \in R \) for some contractive real constant \( \alpha \in [0, 1) \) such that \( \alpha a_1 = a_1 = \alpha \) and \( (R, d) \) is a complete metric space and also a Banach space. Assume that \( A_1 = [a_1, \bar{a}_1] \) and \( A_2 = [a_2, \bar{a}_2] \) with \( \bar{a}_1 < 0 \) and \( a_2 > 0 \) and \( B_1 = [b_1, \bar{a}_1] \) and \( B_2 = [b_2, \bar{a}_2] \) with \( \bar{a}_2 < 0 \) and \( a_2 = 0 \) and \( b_2 < b_1 \) and \( \bar{a}_2 > \bar{a}_1 \), so that \( T(A_1) \subset B_1, T(A_2) \subset B_2 \) with

\[ D_A = d(A_1, A_2) = |a_2| + a_1, \]
\[ D_B = d(B_1, B_2) = |\bar{a}_1| + b_2, \]
\[ D_1 = d(A_2, B_1) = a_1 + |\bar{a}_1|, \]
\[ D_2 = d(A_1, B_2) = |\bar{a}_1| + b_2, \]
and \( T : A_1 \cup A_2 \to B_1 \cup B_2 \) is a CPD (Definition 2 with \( p = 2 \)) if the subsequent constraints hold for all \( n \in Z_0 \):

\[ y_{n+1} + |TX_n| = D_1, \quad Ty_n + |x_{n+1}| = D_2, \]
\[ |x_{n+2} - x_n| \leq |Ty_n - Ty_{n+1}| \leq \alpha |x_{n+1} - x_n|, \]
\[ |y_{n+1} - y_n| = |Ty_n - Ty_{n+1}| \leq \alpha |y_{n+1} - y_n|, \]
\[ |y_{n+1} - x_{n+1}| \leq |Ty_n - Ty_{n+1}| \leq \alpha |y_{n+1} - y_n|, \]
\[ |y_{n+1} - x_n| \leq |Ty_n - Ty_{n+1}| \leq \alpha |y_{n+1} - y_n|, \]
\[ |y_{n+1} - y_n| \leq |Ty_n - Ty_{n+1}| \leq \alpha |y_{n+1} - y_n|, \]
\[ |y_{n+1} - x_n| \leq |Ty_n - Ty_{n+1}| \leq \alpha |y_{n+1} - y_n|, \]
\[ |y_{n+1} - x_n| \leq |Ty_n - Ty_{n+1}| \leq \alpha |y_{n+1} - y_n|, \]
\[ |y_{n+1} - x_n| \leq |Ty_n - Ty_{n+1}| \leq \alpha |y_{n+1} - y_n|, \]
In particular, (37)–(40a) and (40b) are satisfied if, for all \( n \in \mathbb{Z}_0^+ \),
\[
Ty_n \geq D_2 - \alpha y_n - \frac{1 - \alpha}{2} D_A, \quad (41)
\]
\[
Tx_n \leq -D_1 - \alpha x_n + \frac{1 - \alpha}{2} D_A, \quad (42)
\]
(equivalently, \( D_1 - \alpha |x_n| - \frac{1 - \alpha}{2} D_A \geq |Tx_n| \)),
\[
y_{n+1} \leq \alpha |x_n| - \frac{1 - \alpha}{2} D_A, \quad (43)
\]
\[
y_{n+1} \leq \alpha y_n + \frac{1 - \alpha}{2} D_A.
\]
Parallel results for the case when \( T : A_1 \cup A_2 \rightarrow B_1 \cup B_2 \) is WCPD, GCPCD, or GWCPD (Definitions 3–5 with \( p = 2 \)) can be discussed in the same way with the appropriate extensions for the contractive constant or function. It follows that
\[
|dT(y_n, y_{n+1})| \rightarrow 0, \quad |d(x_n, x_{n+1})| \rightarrow 0, \quad |d(z_n, z_{n+1})| \rightarrow D_A, \quad |y_n| \rightarrow a_2, \quad |x_n| \rightarrow \bar{a}_1, \quad |d(Ty_n, Ty_{n+1})| \rightarrow 0, \quad |d(Tx_n, Tx_{n+1})| \rightarrow 0 \quad \text{and, according to (40a), (40b), and (35)-(36),}
\]
\[
|d(Tz_n, Tz_{n+1})| \rightarrow D_B, \quad \text{since } |T(y_n - x_n)| \rightarrow D_B = D_1 + D_2 - D_A, \quad |Ty_n| \rightarrow T\bar{a}_2 = b_2 \quad \text{and } |Tx_n| \rightarrow T\bar{a}_1 = b_1.
\]

Example 18. Consider Example 17 in the case that \( T : A_1 \cup A_2 \rightarrow B_1 \cup B_2 \) is CPI, WCP1, GCPI, or GWCP1 (Definitions 8–11 with \( p = 2 \)); (40a) using (37) can be reformulated accordingly. In particular, if it is CPI, then one gets for some real constant \( \beta \in [0, 1) \)
\[
|Tx_{n+1}| + Ty_{n+1}
= Ty_{n+1} - Tx_{n+1} = D_1 + D_2 - |x_{n+2}| - y_{n+2}
= D_A + D_B - |x_{n+2}| - y_{n+2}
\leq \beta \left( |x_n| + y_n \right) + (1 - \beta) D_B, \quad \forall n \in \mathbb{Z}_0^+.
\]

Then,
\[
|T(y_n - x_n)| \equiv |dT(Tx_n, Ty_{n+1})| \rightarrow D_B = D_1 + D_2 - D_A, \quad |y_n| \equiv |d(Tz_n, Tz_{n+1})| \rightarrow D_A, \quad |x_n| \equiv |d(Tx_n, Ty_{n+1})| \rightarrow 0,
\]
\[
|d(x_n, x_{n+1})| \rightarrow 0, \quad |Ty_n| \rightarrow T\bar{a}_2 = b_2, \quad |Tx_n| \rightarrow T\bar{a}_1 = b_1, \quad |y_n| \rightarrow a_2, \quad \text{and } |x_n| \rightarrow \bar{a}_1.
\]

Conflict of Interests
The authors declare that there is no conflict of interests regarding the publication of this paper.

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