The Uniqueness of Solution for a Class of Fractional Order Nonlinear Systems with $p$-Laplacian Operator

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We are concerned with the uniqueness of solutions for a class of $p$-Laplacian fractional order nonlinear systems with nonlocal boundary conditions. Based on some properties of the $p$-Laplacian operator, the criterion of uniqueness for solutions is established.

1. Introduction

Fractional order differential systems arise from many branches of applied mathematics and physics, such as gas dynamics, Newtonian fluid mechanics, nuclear physics, and biological process [1–12]. In the recent years, there has a significant development in fractional calculus. For example, by using the contraction mapping principle, ur Rehman and Khan [13] established the existence and uniqueness of positive solutions for the fractional order differential equation with multipoint boundary conditions:

$\mathcal{D}_t^\alpha y(t) = f \left( t, y(t), \mathcal{D}_t^\beta y(t) \right), \quad t \in (0, 1),
\quad y(0) = 0, \quad \mathcal{D}_t^\beta y(1) = \sum_{i=1}^{p-2} a_i \mathcal{D}_t^\mu (\xi_i),
\quad y(0) = 0, \quad \mathcal{D}_t^\mu y(1) = \sum_{i=1}^{p-2} b_j \mathcal{D}_t^\nu y(\xi_j),
(1)$

where $1 < \alpha \leq 2, 1 < \beta < 1, \xi_i \in [0, +\infty)$, and $0 < \xi_i < 1$, with $\sum_{i=1}^{m-2} \xi_i < 1$. In [14], by using the fixed point theorem of mixed monotone operator, Zhang et al. studied the existence and uniqueness of positive solution for the following fractional order differential systems with multipoint boundary conditions:

$-\mathcal{D}_t^\mu x(t) = f \left( t, x(t), \mathcal{D}_t^\beta x(t), y(t) \right),
\quad -\mathcal{D}_t^\beta y(t) = g \left( t, x(t) \right), \quad t \in (0, 1),
\quad \mathcal{D}_t^\beta x(0) = 0, \quad \mathcal{D}_t^\mu y(1) = \sum_{i=1}^{p-2} a_i \mathcal{D}_t^\mu (\xi_i),
\quad y(0) = 0, \quad \mathcal{D}_t^\nu y(1) = \sum_{i=1}^{p-2} b_j \mathcal{D}_t^\nu y(\xi_j),
(2)$

where $1 < \gamma < \alpha \leq 2, 1 < \alpha - \beta < \gamma, 0 < \beta \leq \mu < 1, 0 < \nu < 1$, and $0 < \xi_1 < \xi_2 < \cdots < \xi_{p-2} < 1, a_i, b_j \in [0, +\infty)$ with $\sum_{j=1}^{p-2} a_j \xi_j^{\alpha-\mu-1} < 1$ and $\sum_{j=1}^{p-2} b_j \xi_j^{\gamma-\nu-1} < 1; \mathcal{D}_t$ is the standard Riemann-Liouville derivative. Some interesting results were also obtained by Zhang et al. [1, 2, 5, 7, 9], Goodrich [15–17], and Ahmad and Nieto [18].

On the other hand, the $p$-Laplacian equation

$\left( \varphi_p \left( x'(t) \right) \right)' = f \left( t, x(t), x'(t) \right),
\quad x(0) = 0, \quad x'(1) = a \mathcal{D}_t^\mu x(\xi),
(3)$

where $\varphi_p(s) = |s|^{p-2}s, p > 1$, can describe the turbulent flow in a porous medium; see [19]. Recently, by using Krasnoselskii’s fixed point theorem and the Leggett-Williams theorem, Wang et al. [20] investigated the existence of positive solutions for the nonlinear fractional order differential equation with a $p$-Laplacian operator:

$\mathcal{D}_t^\alpha \left( \varphi_p \left( \mathcal{D}_t^\beta x(t) \right) \right) + f \left( t, x(t) \right) = 0,
\quad x(0) = 0, \quad \mathcal{D}_t^\beta x(0) = 0, \quad x(1) = ax(\xi),
(4)$
where \( 0 < \beta \leq 2, 0 < \alpha \leq 1, 0 \leq a \leq 1, \) and \( 0 < \xi < 1 \). And then, by looking for a more suitable upper and lower solution, Ren and Chen [21] established the existence of positive solutions for four points fractional order boundary value problem:

\[
\mathcal{D}_t^\beta \left( q_p \left( \mathcal{D}_t^\alpha x \right) \right) (t) = f (t, x (t)), \quad t \in (0, 1),
\]

where \( \mathcal{D}_t^\alpha \) and \( \mathcal{D}_t^\beta \) are the standard Riemann–Liouville derivatives, \( p \)-Laplacian operator is defined as \( \varphi_p (s) = |s|^{p-2} s, p > 1 \), and the nonlinearity \( f \) may be singular at both \( t = 0, 1 \) and \( x = 0 \).

Inspired by the above work, in this paper, we study the uniqueness of positive solutions for the following fractional order differential system with \( p \)-Laplacian operator:

\[
\mathcal{D}_t^\alpha \left( q_{p_1} \left( \mathcal{D}_t^\beta x \right) \right) (t) = \lambda f (t, x (t)),
\]

where \( \mathcal{D}_t^\alpha, \mathcal{D}_t^\beta, \mathcal{D}_t^\gamma, \) and \( \mathcal{D}_t^\delta \) are the standard Riemann–Liouville derivatives with \( \alpha, \beta, \gamma, \delta \in (1, 2], \) and \( p > 1 \), and the positive parameters, \( p \)-Laplacian operator is defined as \( \varphi_{p_1} (s) = |s|^{p_1-2} s, p_1 > 1, \) and \( \varphi_{p_2} (s) = |s|^{p_2-2} s, p_2 > 1, \) where \( c_i \in \mathbb{R} \) \( (i = 1, 2, \ldots, n) \) and \( n \) is the smallest integer greater than or equal to \( \beta \).

The main results of this paper are based on the following property of \( p \)-Laplacian operator, which is easy to be proved.

**Lemma 3.** (1) If \( q \geq 2 \) and \( |x|, |y| \leq M \), then

\[
\left| \varphi_{q} (x) - \varphi_{q} (y) \right| \leq (q - 1) M^{q-2} |x - y|.
\]

(2) If \( 1 < q < 2, x y > 0, \) and \( |x|, |y| \geq m > 0 \), then

\[
\left| \varphi_{q} (x) - \varphi_{q} (y) \right| \leq (q - 1) m^{q-2} |x - y|.
\]

Applying Definitions 1 and 2 and Property 1, we have the following lemma.

**Lemma 4.** Let \( y \in L^1_{\xi}[0, 1], 1 < \alpha, \beta \leq 2, 0 < \xi, \eta < 1, \) and \( 0 \leq a, b \leq 1 \). The fractional order boundary value problem,

\[
\mathcal{D}_t^\alpha \left( q_{p_1} \left( \mathcal{D}_t^\beta x \right) \right) (t) = h (t), \quad t \in (0, 1),
\]

where

\[
K_1 (t, s) = k_1 (t, s) + a k_1 (\xi, s) t^{\alpha-1} \frac{1}{1 - a^{\xi-1}},
\]

\[
K_2 (t, s) = k_2 (t, s) + b k_2 (\eta, s) t^{\beta-1} \frac{1}{1 - b^{\eta-1}},
\]

has the unique solution

\[
x (t) = \int_0^1 K_1 (t, s) q_{p_1} \left( \int_0^s K_2 (s, \tau) h (\tau) d\tau \right) ds.
\]
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$$k_1 (t, s) = \begin{cases} \frac{(t(1-s)^{\alpha-1} - (t-s)^{\alpha-1})}{\Gamma (\alpha)}, & 0 \leq s \leq t \leq 1, \\ \frac{(t(1-s)^{\alpha-1})}{\Gamma (\alpha)}, & 0 \leq t \leq s \leq 1, \end{cases}$$

$$k_2 (t, s) = \begin{cases} \frac{(t(1-s)^{\beta-1} - (t-s)^{\beta-1})}{\Gamma (\beta)}, & 0 \leq s \leq t \leq 1, \\ \frac{(t(1-s)^{\beta-1})}{\Gamma (\beta)}, & 0 \leq t \leq s \leq 1, \end{cases}$$

and $b_1 = b^{\beta-1}$.

Similar to (14), the fractional order boundary value problem,

$$D_t^\delta y (t) = h (t), \quad t \in (0, 1),$$

$$y (0) = 0, \quad y (1) = cy (\zeta),$$

has unique solution

$$y (t) = \int_0^1 K_3 (t, s) \varphi (s, t) h (s) ds,$$  \hspace{1cm} (17)

where

$$K_3 (t, s) = k_3 (t, s) + \frac{ck_3(\xi, s)t^{\delta-1}}{1-\xi^{\delta-1}},$$

$$K_4 (t, s) = k_4 (t, s) + \frac{dk_4(\mu, s)t^{\gamma-1}}{1-d\mu^{\gamma-1}},$$

$$k_3 (t, s) = \begin{cases} \frac{(t(1-s)^{\gamma-1} - (t-s)^{\gamma-1})}{\Gamma (\gamma)}, & 0 \leq t \leq s \leq 1, \\ \frac{(t(1-s)^{\gamma-1})}{\Gamma (\gamma)}, & 0 \leq s \leq t \leq 1, \end{cases}$$

$$k_4 (t, s) = \begin{cases} \frac{(t(1-s)^{\gamma-1} - (t-s)^{\gamma-1})}{\Gamma (\gamma)}, & 0 \leq s \leq t \leq 1, \\ \frac{(t(1-s)^{\gamma-1})}{\Gamma (\gamma)}, & 0 \leq t \leq s \leq 1, \end{cases}$$

and $d_1 = d^{\gamma-1}$.

**Lemma 5.** Let $1 < \alpha, \beta, \gamma, \delta \leq 2$, $0 < \xi, \zeta, \eta, \mu < 1$, and $0 \leq a, b, c, d \leq 1$. The functions $K_i (t, s), i = 1, 2, 3, 4$, are continuous on $[0, 1] \times [0, 1]$ and satisfy

(i) $K_i (t, s) \geq 0$, $i = 1, 2, 3, 4$ for $t, s \in [0, 1]$;

(ii) for $t, s \in [0, 1]$,

$$\sigma_1 (s) t^{\alpha-1} \leq K_2 (t, s) \leq \sigma_2 (s) t^{\alpha-1},$$

$$\sigma_3 (s) t^{\gamma-1} \leq K_4 (t, s) \leq \sigma_4 (s) t^{\gamma-1},$$

where

$$\sigma_1 (s) = \frac{b_1k_2 (\eta, s)}{1-b_1H^{\beta-1}},$$

$$\sigma_3 (s) = \frac{(1-s)^{\beta-1}}{\Gamma (\beta)} + \frac{b_1k_2 (\eta, s)}{1-b_1H^{\beta-1}},$$

$$\sigma_2 (s) = \frac{d_1k_4 (\mu, s)}{1-d_1H^{\gamma-1}},$$

$$\sigma_4 (s) = \frac{(1-s)^{\gamma-1}}{\Gamma (\gamma)} + \frac{d_1k_4 (\mu, s)}{1-d_1H^{\gamma-1}}.$$

(iii) For $t, s \in [0, 1]$,

$$K_1 (t, s) \leq r_1 (s) s^{\alpha-1}, \quad K_3 (t, s) \leq r_2 (s) s^{\gamma-1},$$

where

$$r_1 = \frac{1}{\Gamma (\alpha)} \left[ 1 + \frac{a}{1-\zeta^{\alpha-1}} \right],$$

$$r_2 = \frac{1}{\Gamma (\gamma)} \left[ 1 + \frac{c}{1-\zeta^{\gamma-1}} \right].$$

**Proof.** The proof is obvious; we omit the proof. \hfill $\Box$

The basic space used in this paper is $E = C([0, 1]; \mathbb{R}) \times C([0, 1]; \mathbb{R})$, where $\mathbb{R}$ is a real number set. Obviously, the space $E$ is a Banach space if it is endowed with the norm as follows:

$$\| (u, v) \| := \| u \| + \| v \|, \quad \| u \| = \max_{t \in [0, 1]} | u (t) |,$$

$$\| v \| = \max_{t \in [0, 1]} | v (t) |,$$

for any $(u, v) \in E$. By Lemma 4, $(x, y) \in E$ is a solution of the fractional order system (1) if and only if $(x, y) \in E$ is a solution of the integral equation

$$x (t) = \lambda r_1 \int_0^t K_1 (t, s) \varphi (s, t) f (s, y (s)) ds,$$

$$y (t) = \rho r_2 \int_0^t K_3 (t, s) \varphi (s, t) g (s, x (s)) ds,$$

$$t \in [0, 1],$$

$$t \in [0, 1].$$

We define an operator $T : E \to E$ by

$$T (x, y) (t) = (F (x, y), G (x, y)),$$  \hspace{1cm} (25)

where

$$F (x, y) = \lambda r_1 \int_0^t K_1 (t, s) \varphi (s, t) f (s, y (s)) ds,$$

$$G (x, y) = \rho r_2 \int_0^t K_3 (t, s) \varphi (s, t) g (s, x (s)) ds.$$
\[ G(x, y) = \rho \int_0^1 K_3(t, s) \varphi(q_2) \left( \int_0^1 K_4(s, \tau) g(s, x(\tau)) d\tau \right) ds. \] (26)

It is easy to see that \((x, y)\) is the solution of the boundary value problem (6) if and only if \((x, y)\) is the fixed point of \(T\). As \(f, g \in C([0, 1] \times \mathbb{R}, \mathbb{R})\), we know that \(T : E \to E\) is a continuous and compact operator.

### 3. Main Results

Now we here introduce a new concept: the \(\Phi\)-contraction mapping.

**Definition 6.** A function \(\psi : (-\infty, +\infty) \to [0, +\infty)\) is called a nonlinear \(\Phi\)-contraction mapping if it is continuous and nondecreasing and satisfies \(\psi(r) \leq r, r > 0\).

**Theorem 7.** Suppose that \(p_1, p_2 > 2\), if there exist nonnegative functions \(a_i(t), i = 1, 2, 3, 4\), such that
\[
0 < \int_0^1 \delta_i(t) a_i(t) dt < +\infty, \quad i = 1, 2, 3, 4, \tag{27}
\]
and the following conditions are satisfied:

\(H_1\) for any \((t, u) \in (0, 1) \times \mathbb{R},\)
\[
f(t, u) \geq a_1(t), \quad g(t, u) \geq a_2(t). \tag{28}
\]

\(H_2\) there exist \(\Phi\)-contraction mappings \(\psi_1, \psi_2\) as
\[
|f(t, u) - f(t, v)| \leq a_3(t) \psi_1(|u - v|),
\]
a.e. \((t, u), (t, v) \in [0, 1] \times \mathbb{R},\)
\[
g(t, u) - g(t, v) \leq a_4(t) \psi_2(|u - v|),
\]
a.e. \((t, u), (t, v) \in [0, 1] \times \mathbb{R}.
\]

Then the fractional order differential system (6) has a unique solution provided that
\[
\Lambda = \lambda^{\alpha}(q_1 - 1) r_1 B(\alpha, (\beta - 1)(q_1 - 2) + 1)
\]
\[
\times \left( \int_0^1 \delta_1(\tau) a_1(\tau) d\tau \right)^{q_2 - 2} \int_0^1 \delta_2(\tau) a_2(\tau) d\tau
\]
\[
+ \rho^{\beta}(q_2 - 1) r_2 B(\delta, (\gamma - 1)(q_2 - 2) + 1)
\]
\[
\times \left( \int_0^1 \delta_3(\tau) a_3(\tau) d\tau \right)^{q_3 - 2} \int_0^1 \delta_4(\tau) a_4(\tau) d\tau < 1. \tag{30}
\]

**Proof.** In the case \(p_1, p_2 > 2\), we have \(1 < q_1, q_2 < 2\). Now we prove that \(T\) is a contraction mapping. By (27)-(28) and Lemma 5, we have
\[
\int_0^1 K_2(s, \tau) f(s, y(\tau)) d\tau \geq s^{\beta - 1} \int_0^1 \delta_1(\tau) a_1(\tau) d\tau,
\]
\[
\int_0^1 K_4(s, \tau) g(s, x(\tau)) d\tau \geq s^{\gamma - 1} \int_0^1 \delta_2(\tau) a_2(\tau) d\tau. \tag{31}
\]

By (12), (28), and (31), for any \((u_1, v_1), (u_2, v_2) \in E\) and for \(t > 0\), we have
\[
\left| \varphi_{q_1} \left( \int_0^1 K_2(s, \tau) f(s, v_1(\tau)) d\tau \right) - \varphi_{q_1} \left( \int_0^1 K_2(s, \tau) f(s, v_2(\tau)) d\tau \right) \right|
\]
\[
\leq (q_1 - 1) s^{\beta - 1} \left( \int_0^1 \delta_1(\tau) a_1(\tau) d\tau \right)^{q_2 - 2}
\]
\[
\times \int_0^1 K_2(s, \tau) |f(\tau, v_1(\tau)) - f(\tau, v_2(\tau))| d\tau
\]
\[
\leq (q_1 - 1) s^{\beta - 1} \left( \int_0^1 \delta_1(\tau) a_1(\tau) d\tau \right)^{q_2 - 2}
\]
\[
\times \int_0^1 \delta_2(\tau) a_2(\tau) d\tau \left\| v_1 - v_2 \right\|
\]
\[
\leq (q_1 - 1) s^{(\beta - 1)(q_2 - 2)} \left( \int_0^1 \delta_1(\tau) a_1(\tau) d\tau \right)^{q_2 - 2}
\]
\[
\times \int_0^1 \delta_2(\tau) a_2(\tau) d\tau \left\| u_1 - u_2 \right\|. \tag{32}
\]

Similarly, we also have
\[
\left| \varphi_{q_2} \left( \int_0^1 K_4(s, \tau) g(s, v_1(\tau)) d\tau \right) - \varphi_{q_2} \left( \int_0^1 K_4(s, \tau) g(s, v_2(\tau)) d\tau \right) \right|
\]
\[
\leq (q_2 - 1) s^{\gamma - 1} \left( \int_0^1 \delta_2(\tau) a_2(\tau) d\tau \right)^{q_3 - 2}
\]
\[
\times \int_0^1 \delta_1(\tau) a_1(\tau) d\tau \left\| u_1 - u_2 \right\|. \tag{33}
\]

So it follows from (14), (17), and (31)-(32) that
\[
\left| F(u_1, v_1)(t) - F(u_2, v_2)(t) \right|
\]
\[
= \left| \lambda^{\alpha} \int_0^1 K_1(t, s) ds \right|
\]
\[
\times \left[ \varphi_{q_1} \left( \int_0^1 K_2(s, \tau) f(s, v_1(\tau)) d\tau \right) - \varphi_{q_1} \left( \int_0^1 K_2(s, \tau) f(s, v_2(\tau)) d\tau \right) \right]
\]
\[
\leq \lambda^{\alpha} r_1 \int_0^1 (1 - s)^{\alpha - 1}
\]
\[
\times \left| \varphi_{q_1} \left( \int_0^1 K_2(s, \tau) f(s, v_1(\tau)) d\tau \right) \right|
\]
\[
\times \left| \varphi_{q_2} \left( \int_0^1 K_4(s, \tau) g(s, v_1(\tau)) d\tau \right) \right|
\]
\[-\varphi_{q_1} \left( \int_0^1 K_2 (s, \tau) f(s, v_2 (\tau)) \, d\tau \right) \, ds \]
\[\leq \lambda^{q_1} (q_1 - 1) \int_0^1 (1 - s)^{q_1 - 1} (s^{(\beta - 1)q_1 - 1}) \, ds \]
\[\times \left( \int_0^1 \delta_1 (r) a_1 (r) \, dr \right)^{q_1 - 2} \]
\[\times \int_0^1 \delta_2 (r) a_2 (r) \, dr \|v_1 - v_2\| \]
\[\leq \lambda^{q_1} (q_1 - 1) r_1 B(\alpha, (\beta - 1) (q_1 - 2) + 1) \]
\[\times \left( \int_0^1 \delta_1 (r) a_1 (r) \, dr \right)^{q_1 - 2} \]
\[\times \int_0^1 \delta_3 (r) a_3 (r) \, dr \|v_1 - v_2\| \]
\[+ \rho^{q_1} (q_2 - 1) r_2 B(\delta, (\gamma - 1) (q_2 - 2) + 1) \]
\[\times \left( \int_0^1 \delta_2 (r) a_2 (r) \, dr \right)^{q_2 - 2} \]
\[\times \int_0^1 \delta_4 (r) a_4 (r) \, dr \|u_1 - u_2\| \]
\[\leq \Lambda (\|v_1 - v_2\| + \|u_1 - u_2\|) \]
\[= \Lambda \|(u_1, v_1) - (u_2, v_2)\|, \quad (35)\]

where

\[\Lambda = \lambda^{q_1} (q_1 - 1) r_1 B(\alpha, (\beta - 1) (q_1 - 2) + 1) \]
\[\times \left( \int_0^1 \delta_1 (r) a_1 (r) \, dr \right)^{q_1 - 2} \int_0^1 \delta_3 (r) a_3 (r) \, dr \]
\[\times \int_0^1 \delta_4 (r) a_4 (r) \, dr \|v_1 - v_2\|, \quad (36)\]

Noticing that \(0 < \Lambda < 1\), we obtain that \(F : C[0, 1] \to C[0, 1]\) is a contraction mapping. By means of the Banach contraction mapping principle, we get that \(T\) has a unique fixed point in \(E\) which implies that the fractional order differential system (6) has a unique solution.

**Theorem 8.** Suppose that \(1 < p_1, p_2 \leq 2\), if there exist nonnegative functions \(b_i(t), i = 1, 2, 3, 4\), such that

\[0 < \int_0^1 \delta_i (t) b_i (t) \, dt < +\infty, \quad i = 1, 2, 3, 4, \quad (37)\]

and the following conditions are satisfied:

\[(H_3)\] for any \((t, w) \in (0, 1) \times \mathbb{R},\]
\[
|f(t, w)| \leq b_1 (t), \quad g(t, w) \leq b_1 (t), \quad (38)\]

\[(H_4)\] there exist \(\mathcal{D}\)-contraction mappings \(\phi_1, \phi_2\) as
\[
|f(t, u) - f(t, v)| \leq b_1 (t) \phi_1 (|u - v|), \quad \text{a.e.} (t, u), (t, v) \in [0, 1] \times \mathbb{R}, \quad (39)\]

\[
|g(t, u) - g(t, v)| \leq b_2 (t) \phi_2 (|u - v|), \quad \text{a.e.} (t, u), (t, v) \in [0, 1] \times \mathbb{R}. \]

Then the fractional order differential system (6) has a unique solution provided that

\[
\overline{\Lambda} = \lambda^{q_1} (q_1 - 1) r_1 B(\alpha, (\beta - 1) (q_1 - 2) + 1) \]
\[\times \left( \int_0^1 \delta_1 (r) b_1 (r) \, dr \right)^{q_1 - 2} \int_0^1 \delta_3 (r) b_1 (r) \, dr \]
\[\times \int_0^1 \delta_4 (r) b_2 (r) \, dr < 1, \quad (40)\]

**Proof.** In the case \(1 < p_1, p_2 \leq 2\), we get \(q_1, q_2 \geq 2\); here we still prove that \(T\) is a contraction mapping if the conditions
of theorem are satisfied. By (37)-(38) and Lemma 5, for any 
\((x,y) \in E\), we have

\[
\begin{align*}
&\int_0^1 K_2(s,\tau)f(s,y(\tau))d\tau \leq s^{\beta-1}\int_0^1 \delta_1(\tau)b_1(\tau)d\tau, \\
&\int_0^1 K_4(s,\tau)g(s,x(\tau))d\tau \leq s^{\gamma-1}\int_0^1 \delta_4(\tau)b_4(\tau)d\tau.
\end{align*}
\]

(41)

By (11), (39), and (41), for any \((u_1, v_1), (u_2, v_2) \in E\) and for 
\(t > 0\), we have

\[
\begin{align*}
&\left| \varphi_{q_1}\left( \int_0^1 K_2(s,\tau)f(s,v_1(\tau))d\tau \right) \\
&- \varphi_{q_1}\left( \int_0^1 K_2(s,\tau)f(s,v_2(\tau))d\tau \right) \right| \\
&\leq (q_1 - 1) \left( s^{\beta-1}\int_0^1 \delta_3(\tau)b_3(\tau)d\tau \right)^{q_1-2} \\
&\times \int_0^1 K_2(s,\tau)\left( f(\tau,v_1(\tau)) - f(\tau,v_2(\tau)) \right)d\tau \\
&\leq (q_1 - 1) \left( s^{\beta-1}\int_0^1 \delta_3(\tau)b_3(\tau)d\tau \right)^{q_1-2} \\
&\times \int_0^1 \delta_3(\tau)b_1(\tau)d\tau \left\| v_1 - v_2 \right\|.
\end{align*}
\]

(42)

Similarly, we also have

\[
\begin{align*}
&\left| \varphi_{q_2}\left( \int_0^1 K_4(s,\tau)g(s,u_1(\tau))d\tau \right) \\
&- \varphi_{q_2}\left( \int_0^1 K_4(s,\tau)g(s,u_2(\tau))d\tau \right) \right| \\
&\leq (q_2 - 1) \left( s^{\gamma-1}\int_0^1 \delta_4(\tau)b_4(\tau)d\tau \right)^{q_2-2} \\
&\times \int_0^1 \delta_4(\tau)b_2(\tau)d\tau \left\| u_1 - u_2 \right\|.
\end{align*}
\]

(43)

So it follows from (14), (17), and (42)-(43) that

\[
\begin{align*}
&\left| F(u_1, v_1)(t) - F(u_2, v_2)(t) \right| \\
= &\left| \lambda^{q_1}\int_0^1 K_1(t,\tau)d\tau \\
&\times \left[ \varphi_{q_1}\left( \int_0^1 K_2(s,\tau)f(s,v_1(\tau))d\tau \right) \\
&- \varphi_{q_1}\left( \int_0^1 K_2(s,\tau)f(s,v_2(\tau))d\tau \right) \right] ds \right|
\end{align*}
\]

\[
\leq \lambda^{q_1}r_1 \int_0^1 (1-s)^{\eta-1} \\
\times \left[ \varphi_{q_1}\left( \int_0^1 K_2(s,\tau)f(s,v_1(\tau))d\tau \right) \\
- \varphi_{q_1}\left( \int_0^1 K_2(s,\tau)f(s,v_2(\tau))d\tau \right) \right] ds
\]

\[
\leq \lambda^{q_1}r_1 (q_1 - 1) \int_0^1 (1-s)^{\eta-1}s^{(\beta-1)(q_1-2)}ds \\
\times \left( \int_0^1 \delta_3(\tau)b_3(\tau)d\tau \right)^{q_1-2} \\
\times \int_0^1 \delta_3(\tau)b_1(\tau)d\tau \left\| v_1 - v_2 \right\| \\
\leq \lambda^{q_1} (q_1 - 1) r_1 B(\alpha, (\beta - 1) (q_1 - 2) + 1) \\
\times \left( \int_0^1 \delta_3(\tau)b_3(\tau)d\tau \right)^{q_1-2} \\
\times \int_0^1 \delta_3(\tau)b_1(\tau)d\tau \left\| v_1 - v_2 \right\|,
\]

\[
\left| G(u_1, v_1)(t) - G(u_2, v_2)(t) \right|
\]

\[
= \rho^{q_2}\int_0^1 K_3(t,\tau)d\tau \\
\times \left[ \varphi_{q_2}\left( \int_0^1 K_4(s,\tau)g(s,u_1(\tau))d\tau \right) \\
- \varphi_{q_2}\left( \int_0^1 K_4(s,\tau)g(s,u_2(\tau))d\tau \right) \right] ds
\]

\[
\leq \rho^{q_2}r_2 B(\delta, (\gamma - 1) (q_2 - 2) + 1) \\
\times \left( \int_0^1 \delta_4(\tau)b_4(\tau)d\tau \right)^{q_2-2} \\
\times \int_0^1 \delta_4(\tau)b_2(\tau)d\tau \left\| u_1 - u_2 \right\|.
\]

(44)

Hence

\[
\left| T(u_1, v_1) - T(u_2, v_2) \right|
\]

\[
= \left| (F(u_1, v_1) - F(u_2, v_2), G(u_1, v_1) - G(u_2, v_2)) \right|
\]

\[
\leq \left\| F(u_1, v_1) - F(u_2, v_2) \right\| \\
+ \left\| G(u_1, v_1) - G(u_2, v_2) \right\|
\]

\[
\leq \lambda^{q_1} (q_1 - 1) r_1 B(\alpha, (\beta - 1) (q_1 - 2) + 1) \\
\times \left( \int_0^1 \delta_3(\tau)b_3(\tau)d\tau \right)^{q_1-2} \\
\times \int_0^1 \delta_3(\tau)b_1(\tau)d\tau \left\| v_1 - v_2 \right\|
\]

\[
\leq \lambda^{q_1} (q_1 - 1) r_1 B(\alpha, (\beta - 1) (q_1 - 2) + 1) \\
\times \left( \int_0^1 \delta_3(\tau)b_3(\tau)d\tau \right)^{q_1-2} \\
\times \int_0^1 \delta_3(\tau)b_1(\tau)d\tau \left\| v_1 - v_2 \right\|
\]
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\[ \rho_3 (q_2 - 1) r_2 B (\delta, (\gamma - 1) (q_2 - 2) + 1) \]
\[ \times \left( \int_0^1 \delta_4 (r) b_4 (r) dr \right)^{q_2 - 2} \]
\[ \times \int_0^1 \delta_4 (r) b_4 (r) dr \| u_1 - u_2 \| \]
\[ \leq \Lambda (\| v_1 - v_2 \| + \| u_1 - u_2 \|) = \Lambda (\| u_1, v_1 \| - (u_2, v_2)) \],
\]
(45)

where
\[ \Lambda = \lambda^q (q_1 - 1) r_1 B (\alpha, (\beta - 1) (q_1 - 2) + 1) \]
\[ \times \left( \int_0^1 \delta_3 (r) b_3 (r) dr \right)^{q_1 - 2} \]
\[ \times \int_0^1 \delta_3 (r) b_3 (r) dr \| u_1 - u_2 \| \]
\[ + \rho_3 (q_2 - 1) r_2 B (\delta, (\gamma - 1) (q_2 - 2) + 1) \]
\[ \times \left( \int_0^1 \delta_4 (r) b_4 (r) dr \right)^{q_2 - 2} \]
\[ \times \int_0^1 \delta_4 (r) b_4 (r) dr < 1. \]
\]
(50)

**Corollary 9.** Suppose that \( p_1 > 2 \) and \( 1 < p_2 \leq 2 \) if there exist nonnegative functions \( a_i (t) \), \( i = 1, 2, 3, 4 \), such that
\[ 0 < \int_0^1 \delta_1 (t) a_1 (t) dt < +\infty, \quad i = 1, 2, 3, 4, \]
and the following conditions are satisfied:
\begin{enumerate}
\item [(H_1)] for any \((t, w) \in (0, 1) \times \mathbb{R} \),
\[ f (t, w) \geq a_1 (t), \quad |g (t, w)| \leq a_2 (t), \]
\item [(H_2)] there exist \( \mathscr{D} \)-contraction mappings \( \psi_1, \psi_2 \) as
\[ |f (t, u) - f (t, v)| \leq a_3 (t) \psi_1 (|u - v|), \]
\[ a.e. \ (t, u), (t, v) \in [0, 1] \times \mathbb{R}, \]
\[ |g (t, u) - g (t, v)| \leq a_4 (t) \psi_2 (|u - v|), \]
\[ a.e. \ (t, u), (t, v) \in [0, 1] \times \mathbb{R}. \]
\end{enumerate}

Then the fractional order differential system (6) has a unique solution provided that
\[ \Lambda_1 = \lambda^q (q_1 - 1) r_1 B (\alpha, (\beta - 1) (q_1 - 2) + 1) \]
\[ \times \left( \int_0^1 \delta_1 (r) a_1 (r) dr \right)^{q_1 - 2} \]
\[ \times \int_0^1 \delta_1 (r) a_1 (r) dr < 1. \]
\]
(54)

**Corollary 10.** Suppose that \( p_1 > 2 \) and \( 1 < p_1 \leq 2 \) if there exist nonnegative functions \( a_i (t), \) \( i = 1, 2, 3, 4 \), such that
\[ 0 < \int_0^1 \delta_i (t) a_i (t) dt < +\infty, \quad i = 1, 2, 3, 4, \]
and the following conditions are satisfied:
\begin{enumerate}
\item [(H_1)] for any \((t, w) \in (0, 1) \times \mathbb{R} \),
\[ |f (t, w)| \leq a_1 (t), \quad g (t, w) \geq a_2 (t), \]
\item [(H_2)] there exist \( \mathscr{D} \)-contraction mappings \( \psi_1, \psi_2 \) as
\[ |f (t, u) - f (t, v)| \leq a_3 (t) \psi_1 (|u - v|), \]
\[ a.e. \ (t, u), (t, v) \in [0, 1] \times \mathbb{R}, \]
\[ |g (t, u) - g (t, v)| \leq a_4 (t) \psi_2 (|u - v|), \]
\[ a.e. \ (t, u), (t, v) \in [0, 1] \times \mathbb{R}. \]
\end{enumerate}

Then the fractional order differential system (6) has a unique solution provided that
\[ \Lambda_2 = \lambda^q (q_1 - 1) r_1 B (\alpha, (\beta - 1) (q_1 - 2) + 1) \]
\[ \times \left( \int_0^1 \delta_2 (r) a_2 (r) dr \right)^{q_2 - 2} \]
\[ \times \int_0^1 \delta_2 (r) a_2 (r) dr < 1. \]
\]
(50)

**Conflict of Interests**

The authors declare that there is no conflict of interests regarding the publication of this paper.

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**References**


