Research Article

Landau-Type Theorems for Certain Biharmonic Mappings

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Abstract

Let \( F(\bar{z}) = |\bar{z}|^2 \frac{f(z) + h(z)}{f(z)} \) be a biharmonic mapping of the unit disk \( D \), where \( g \) and \( h \) are harmonic in \( D \). In this paper, the Landau-type theorems for biharmonic mappings of the form \( L(F) \) are provided. Here \( L \) represents the linear complex operator \( L(\bar{z} \frac{\partial}{\partial z} - \frac{\partial}{\partial \bar{z}}) \) defined on the class of complex-valued \( C^1 \) functions in the plane. The results, presented in this paper, improve the related results of earlier authors.

1. Introduction

Suppose that \( f(z) = u(x, y) + iv(x, y), z = x + iy \) is a four times continuously differentiable complex-valued function in a domain \( D \subset C \). If \( f \) satisfies the biharmonic equation \( \Delta(\Delta f) = 0 \), then we call that \( f \) is biharmonic, where \( \Delta \) represents the Laplacian operator:

\[
\Delta = 4 \frac{\partial^2}{\partial z \partial \bar{z}} = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}.
\]

Biharmonic functions arise in many physical situations, particularly in fluid dynamics and elasticity problems, and have many important applications in engineering (see [1] for details). It is known that a mapping \( F \) is biharmonic in a simply connected domain \( D \) if and only if \( F \) has the following representation:

\[
F(z) = |z|^2 g(z) + h(z),
\]

where \( g(z) \) and \( h(z) \) are complex-valued harmonic functions in \( D \) [1]. Also, it is known that \( g(z) \) and \( h(z) \) can be expressed as

\[
g(z) = g_1(z) + \overline{g_2(z)}, \quad z \in D,
\]

\[
h(z) = h_1(z) + \overline{h_2(z)}, \quad z \in D,
\]

where \( g_1, g_2, k_1, \) and \( k_2 \) are analytic in \( D \) [2, 3].

For a continuously differentiable mapping \( f \) in \( D \), we define

\[
\Lambda_f(z) = \max_{0 \leq \theta \leq 2\pi} \left| f_z(z) + e^{-2i\theta} f_{\bar{z}}(z) \right| = \left| f_z(z) \right| + \left| f_{\bar{z}}(z) \right|,
\]

\[
\lambda_f(z) = \min_{0 \leq \theta \leq 2\pi} \left| f_z(z) + e^{-2i\theta} f_{\bar{z}}(z) \right| = \left| f_z(z) \right| - \left| f_{\bar{z}}(z) \right|.
\]

We use \( J_f \) to denote the Jacobian of \( f \)

\[
J_f(z) = \left| f_z(z) \right|^2 - \left| f_{\bar{z}}(z) \right|^2.
\]

Then \( J_f = \lambda_f / \Lambda_f \) if \( J_f \geq 0 \).

In [4], the authors considered the following differential operator \( L \) defined on the class of complex-valued \( C^1 \) functions:

\[
L = \bar{z} \frac{\partial}{\partial z} - z \frac{\partial}{\partial \bar{z}}.
\]

Evidently, \( L \) is a complex linear operator and satisfies the usual product rule:

\[
L(af + bg) = aL(f) + bL(g),
\]

\[
L(fg) = fL(g) + gL(f),
\]

where \( a \) and \( b \) are complex constants; \( f \) and \( g \) are \( C^1 \) functions. In addition, the operator \( L \) possesses a number of
interesting properties. For instance, it is easy to see that the operator $L$ preserves both harmonicity and biharmonicity. Many other basic properties are stated in [4].

Landau’s theorem states that if $f$ is an analytic function on the unit disk $D$ with $f(0) = f'(0) = 1 = 0$ and $|f(z)| < M$ for $z \in D$, then $f$ is univalent in the disk $D_{r_0} = \{z \in \mathbb{C} : |z| < r_0\}$ with $r_0 = 1/(M + \sqrt{M^2 - 1})$, and $f(D_{r_0})$ contains a disk $D_{r_0}$ with $R_0 = M r_0^2$. This result is sharp, with the extremal function $f(z) = Mz(1 - Mz)/(M - z)$. Recently, many authors considered the Landau-type theorems for harmonic mappings [5–9] and biharmonic mappings [1, 4, 10–13]. Chen et al. [10] obtained the Landau-type theorems for biharmonic mappings of the form $L(f)$ as follows.

**Theorem A** (see [10]). Let $F(z) = |z|^2 g(z) + h(z)$ be a biharmonic mapping of the unit disk $D = \{z \in \mathbb{C} : |z| < 1\}$ such that $F(0) = h(0) = 0$ and $J_0(0) = 1$, where $g(z)$ and $h(z)$ are harmonic in $D$. Assume that both $|g(z)|$ and $|h(z)|$ are bounded by $M$. Then there is a constant $\rho_1$ ($0 < \rho_1 < 1$) such that $L(F)$ is univalent in $D_{\rho_1}$, where $\rho_1$ satisfies the following equation:

$$
\frac{\pi}{4M} - \frac{6M p_1^2}{(1 - \rho_1)^2} - \frac{4M p_1^3}{(1 - \rho_1)^3} = 0,
$$

where $m_1 \approx 6.059$ is the minimum value of the function

$$
\frac{2 - x^2 + (4/\pi) \arctan x}{x(1 - x^2)}, \quad 0 < x < 1.
$$

The minimum is attained at $x_0 = 0.588$. Moreover, the range $L(F)(D_{\rho_1})$ contains a schlicht disk $D_{\rho_1}$, where

$$
R_1 = \rho_1 \left[ \frac{\pi}{4M} - \frac{2M p_1^2}{(1 - \rho_1)^2} - \frac{16M p_1}{(1 - \rho_1)} \arctan p_1 \right].
$$

**Theorem B** (see [10]). Let $F(z) = |z|^2 g(z) + h(z)$ be a biharmonic mapping in $D$ such that $g(0) = 0$, $J_0(0) = 1$, and $|g(z)| < M$, where $g(z)$ is harmonic in $D$. Then there is a constant $\rho_2$ ($0 < \rho_2 < 1$) such that $L(F)$ is univalent in $D_{\rho_2}$, where $\rho_2$ satisfies the following equation:

$$
\frac{\pi}{4M} - \frac{48M}{\pi^2 m_1} \arctan p_2 - \frac{2M p_2^3}{(1 - \rho_2)^3} = 0,
$$

where $m_1$ is defined as in Theorem A. Moreover, $L(F)(D_{\rho_2})$ contains a disk $D_{\rho_2}$ with

$$
R_2 = \rho_2 \left[ \frac{\pi}{4M} - \frac{16M}{\pi^2 m_1} \arctan p_1 \right].
$$

However, these results are not sharp. The main object of this paper is to improve Theorems A and B. We get three versions of Landau-type theorems for biharmonic mappings of the form $L(F)$, where $F$ belongs to the class of biharmonic mappings, and Theorems II and 14 improve Theorems A and B. In order to establish our main results, we need to recall the following lemmas.

**Lemma 1** (see [6, 14]). Suppose that $f(z)$ is a harmonic mapping of the unit disk $D$ such that $|f(z)| \leq M$ for all $D$. Then

$$
\Lambda_f(z) = \frac{4M}{\pi (1 - |z|^2)}, \quad z \in D.
$$

The inequality is sharp.

**Lemma 2** (see [9, 12, 15]). Suppose that $f(z) = h(z) + \overline{g(z)}$ is a harmonic mapping of the unit disk $D$ such that $|f(z)| \leq M$ for all $z \in D$ with $h(z) = \sum_{n=0}^{\infty} a_n z^n$ and $g(z) = \sum_{n=1}^{\infty} b_n z^n$. Then $|a_n| \leq M$ and for any $n \geq 1$

$$
|a_n| + |b_n| \leq \frac{4M}{\pi},
$$

These estimates are sharp.

**Lemma 3** (see [8, 11]). Suppose that $f$ is a harmonic mapping of $D$ with $f(0) = \lambda_J(0) = 1$. If $\Lambda_f(z) \leq \Lambda_f(0)$ for $z \in D$; then

$$
|a_n| + |b_n| \leq \frac{\Lambda^2 - 1}{n \Lambda}, \quad n = 2, 3, \ldots
$$

These estimates are sharp.

**Lemma 4** (see [11]). Suppose that $f(z) = h(z) + \overline{g(z)}$ is a harmonic mapping of the unit disk $D$ such that $|f(z)| \leq M$ for all $z \in D$ with $h(z) = \sum_{n=0}^{\infty} a_n z^n$ and $g(z) = \sum_{n=1}^{\infty} b_n z^n$. If $|f(0)| = 1$; then $\lambda_f(0) \geq \Lambda_0(M)$, where $M_0 = \pi/2 \sqrt{2^2 - 16} = 1.1296$ and

$$
\Lambda_0(M) = \frac{\sqrt{M^2 - 1 + \sqrt{M^2 + 1}}}{4M}, \quad 1 \leq M \leq M_0.
$$

**Lemma 5** (see [13]). Suppose that $f(z) = h(z) + \overline{g(z)}$ is a harmonic mapping of the unit disk $D$ with $h(z) = \sum_{n=0}^{\infty} a_n z^n$ and $g(z) = \sum_{n=1}^{\infty} b_n z^n$. If $f$ satisfies $|f(z)| \leq M$ for all $z \in D$ and $|f(0)| = 1$, then

$$
\left( \sum_{n=2}^{\infty} (|a_n| + |b_n|)^2 \right)^{1/2} \leq \sqrt{\Lambda^4 - 1} \cdot \lambda_f(0).
$$

**Lemma 6.** Suppose that $M > 0$, $\Lambda \geq 0$. Then the equation

$$
\varphi(r) = 1 - \frac{12M r^2}{\pi (1 - r^2)} - \frac{8M r^3}{\pi (1 - r^3)} - \frac{\Lambda^2 - 1}{\Lambda} \cdot \frac{2r - r^2}{(1 - r)^2} = 0
$$

has a unique root in $(0, 1)$.

**Proof.** It is easy to prove that the function $\varphi$ is continuous and strictly decreasing on $(0, 1)$, $\varphi(0) = 1 > 0$, and $\lim_{r \to 1^-} \varphi(r) = -\infty$. Hence, the assertion follows from the mean value theorem. This completes the proof.
Lemma 7. Suppose that $M_1 > 0$, $M_2 \geq 1$, and $\lambda_0(M_2)$ is defined by (16). Then the equation
\[ \frac{12 M r^2}{\pi (1 - r^2)} - \frac{8 M r^3}{\pi (1 - r^3)} - \lambda_0(M_2) \sqrt{M_2^2 - 1} \cdot \left[ \frac{2 r \sqrt{4 r^2 + r^4 + 1} + r \sqrt{4 r^4 - 3 r^2 + 4}}{(1 - r^2)^{3/2}} \right] = 0 \] has a unique root in $(0, 1)$.

Lemma 8. Let $M \geq 1$. Then the equation
\[ 1 - \sqrt{M^4 - 1} \cdot \left[ \frac{3 r \sqrt{4 r^2 - 3 r^2 + 4} + 2 r \sqrt{4 r^4 + r^4 + 1}}{(1 - r^2)^{3/2}} \right] = 0 \]
has a unique root in $(0, 1)$.

Lemma 9. For any $z_1 \neq z_2$ in $D_r$ ($0 < r < 1$), we have
\[ \int_0^1 |t z_1 + (1 - t) z_2|^2 dt \geq \frac{|z_1|^3 + |z_2|^3}{3(|z_1| + |z_2|)} > 0. \] (21)

2. Main Results

We first establish a new version of the Landau-type theorem for biharmonic mappings on the unit disk $D$ as follows.

Theorem 10. Let $F(z) = |z|^2 g(z) + h(z)$ be a biharmonic mapping of the unit disk $D$, with $M_F(0) = \lambda_F(0) - 1 = 0$, $|g(z)| \leq M$, and $A_F(z) \leq A$ for $z \in D$, where $M > 0$, $A > 1$. Then $L(F)$ is univalent in the disk $D_{r_0}$, where $r_0$ is the unique root in $(0, 1)$ of the equation
\[ 1 - \frac{12 M r^2}{\pi (1 - r^2)} - \frac{8 M r^3}{\pi (1 - r^3)} - \frac{\Lambda^2 - 1}{\Lambda} \cdot \frac{2 r - r^2}{(1 - r)^2} = 0, \] (22)
and $L(F)(D_{r_0})$ contains a schlicht disk $D_{\sigma_0}$, where
\[ \sigma_0 = r_0 \left[ 1 - \frac{\Lambda^2 - 1}{\Lambda} \cdot \frac{r_0}{1 - r_0} - \frac{4 M r_0^2}{\pi (1 - r_0)^2} \right]. \] (23)

Proof. Let $F(z) = |z|^2 g(z) + h(z)$ satisfy the hypothesis of Theorem 10, where
\[ g(z) = g_1(z) + \bar{g}(z) = \sum_{n=0}^\infty a_n z^n + \sum_{n=0}^\infty b_n \bar{z}^n, \]
\[ h(z) = h_1(z) + \bar{h}(z) = \sum_{n=1}^\infty c_n z^n + \sum_{n=1}^\infty d_n \bar{z}^n \] (24)
are harmonic in $D$. As $L$ is linear and $L(|z|^2) = 0$, we may set
\[ H := L(F) = |z|^2 L(g) + L(h). \] (25)
Then we have
\[ H_z = 2|z|^2 g_z + |z|^2 z g_{zz} - z^2 g_z + h_z + z h_z, \]
\[ H_\bar{z} = -2|z|^2 g_\bar{z} - |z|^2 \bar{z} g_{\bar{z}z} + z^2 g_{\bar{z}} - h_\bar{z} - z h_\bar{z}. \] (26)
Note that $\lambda_F(0) = ||c_1| - |d_1|| = \lambda_h(0) = 1$; by Lemma 3, we have
\[ |c_n| + |d_n| \leq \frac{\Lambda^2 - 1}{n \Lambda}, \quad n = 2, 3, \ldots. \] (27)
Thus, for $z_1 \neq z_2$ in $D_r$ ($0 < r < r_0$), we have
\[ |H (z_1) - H (z_2)| \geq \int_{[z_1, z_2]} H_z (z) dz + H_\bar{z} (z) d\bar{z} \]
\[ \geq \int_{[z_1, z_2]} h_z (0) dz - h_\bar{z} (0) d\bar{z} - 2 \int_{[z_1, z_2]} |z|^2 (g_z dz - g_\bar{z} d\bar{z}) \]
\[ - \int_{[z_1, z_2]} |z|^2 (z g_{zz} dz - \bar{z} g_{\bar{z}\bar{z}} d\bar{z}) \]
\[ - \int_{[z_1, z_2]} z h_z dz - z h_\bar{z} d\bar{z} \]
\[ - \int_{[z_1, z_2]} z^2 g_z dz - \bar{z}^2 g_\bar{z} d\bar{z} \]
\[ - \int_{[z_1, z_2]} (h_z - h_\bar{z} (0)) dz \]
\[ - (h_z - h_\bar{z} (0)) d\bar{z}. \] (28)

Let $I_1 = \int_{[z_1, z_2]} h_z (0) dz - h_\bar{z} (0) d\bar{z}$,
\[ I_2 = \int_{[z_1, z_2]} |z|^2 (g_z dz - g_\bar{z} d\bar{z}) \]
\[ I_3 = \int_{[z_1, z_2]} |z|^2 (z g_{zz} dz - \bar{z} g_{\bar{z}\bar{z}} d\bar{z}) \]
\[ I_4 = \int_{[z_1, z_2]} z h_z dz - z h_\bar{z} d\bar{z} \]
\[ I_5 = \int_{[z_1, z_2]} z^2 g_z dz - \bar{z}^2 g_\bar{z} d\bar{z} \]
\[ I_6 = \int_{[z_1, z_2]} (h_z - h_\bar{z} (0)) dz - (h_z - h_\bar{z} (0)) d\bar{z}. \] (29)
By Lemmas 1, 2, and 3, elementary calculations yield that

\[
I_1 \geq \int_{|z_1,z_2|} \lambda_h(0) |dz| = \lambda_h(0) |z_1 - z_2| = |z_1 - z_2|,
\]

\[
I_2 \leq \int_{|z_1,z_2|} |z|^2 \left( |g_1| |dz| + |g_2| |d\bar{z}| \right)
\leq r^2 |z_1 - z_2| \Lambda g(z) \leq |z_1 - z_2| \frac{4Mr^2}{\pi (1 - r^2)},
\]

\[
I_3 \leq |z_1 - z_2| \sum_{n=2}^{\infty} n (n - 1) (|a_n| + |b_n|) r^{n+1}
\leq |z_1 - z_2| \frac{8Mr^3}{\pi (1 - r)^3},
\]

\[
I_4 \leq |z_1 - z_2| \sum_{n=2}^{\infty} n (n - 1) (|c_n| + |d_n|) r^{n-1}
\leq |z_1 - z_2| \frac{\Lambda^2 - 1}{\Lambda} \cdot \frac{r}{1 - r},
\]

\[
I_5 \leq \int_{|z_1,z_2|} \left( |z|^2 |g_1| |dz| + |\bar{z}|^2 |g_2| |d\bar{z}| \right)
\leq r^2 |z_1 - z_2| \Lambda g(z) \leq |z_1 - z_2| \frac{4Mr^2}{\pi (1 - r^2)},
\]

\[
I_6 \leq |z_1 - z_2| \sum_{n=2}^{\infty} n (|c_n| + |d_n|) r^{n-1}
\leq |z_1 - z_2| \frac{\Lambda^2 - 1}{\Lambda} \cdot \frac{r}{1 - r},
\]

Using these estimates and Lemma 6, we obtain

\[
|H(z_1) - H(z_2)|
\geq I_1 - 2I_2 - I_3 - I_4 - I_5 - I_6
\geq |z_1 - z_2| \left[ 1 - \frac{12Mr^2}{\pi (1 - r^2)} - \frac{8Mr^3}{\pi (1 - r)^3} - \frac{\Lambda^2 - 1}{\Lambda} \cdot \frac{2r - r^2}{(1 - r)^2} \right] > 0,
\]

which implies \(H(z_1) \neq H(z_2)\).

For any \(z\) such that \(z \in \partial \mathcal{D}_{r_3}\) by Lemmas 2, 4, and 5, we obtain

\[
|H(z)| = |z|^2 (zg_z - \bar{z}g_{\bar{z}}) + (zh_z - \bar{z}h_{\bar{z}})
\geq |zh_z(0) - \bar{z}h_{\bar{z}}(0)|
\geq r_0 \left[ 1 - \frac{\Lambda^2 - 1}{\Lambda} \cdot \frac{r_0}{1 - r_0} - \frac{4Mr_r^2}{\pi (1 - r_0^2)} \right] = \sigma_0.
\]

This completes the proof.

Next we improve Theorem A as follows.

**Theorem 11.** Let \(F(z) = |z|^2 g(z) + h(z)\) be a biharmonic mapping of the unit disk \(\mathcal{D}\), with \(F(0) = h(0) = F_{\cdot}(0) = 1, |g(z)| < M_1,\) and \(|h(z)| < M_2\) for \(z \in \mathcal{D}\), where \(M_1 > 0, M_2 > 1,\) then \(L(F)\) is univalent in the disk \(\mathcal{D}_{r_3}\), where \(r_3 = r_3\) is the unique root in \((0, 1)\) of the equation

\[
\lambda_0(M_2) - \frac{12M_1r^2}{\pi (1 - r^2)} - \frac{8Mr^3}{\pi (1 - r)^3} - \lambda_0(M_2) \sqrt{M_2^3 - 1}
\cdot \left[ \frac{2r\sqrt{4r^2 + r^4 + 1}}{(1 - r^2)^{5/2}} + \frac{r\sqrt{r^4 - 3r^2 + 4}}{(1 - r^2)^{5/2}} \right] = 0,
\]

and \(L(F)(\mathcal{D}_{r_3})\) contains a schlicht disk \(\mathcal{D}_{\sigma_3}\), where \(\lambda_0(M_2)\) is defined by (16) and

\[
\sigma_3 = r_3 \left[ \lambda_0(M_2) - \lambda_0(M_2) \sqrt{M_2^3 - 1} \right]
\cdot \frac{r_3\sqrt{r^4 - 3r^2 + 4}}{(1 - r_3^2)^{5/2}} - \frac{4Mr_3^2}{\pi (1 - r_3)^2}.
\]

**Proof.** Note that \(J_F(0) = |c_1|^2 + |d_1|^2 = J_h(0) = 1;\) by Lemma 4, we have

\[
\lambda_h(0) \geq \lambda_0(M_2).
\]
We adopt the same method in Theorem 10, for $z_1 \neq z_2$ in $D_\rho(0 < \rho < \rho_3)$; by Lemmas 1, 2, and 5, we get

\[ I_1 \geq \int_{[z_1, z_2]} \lambda_h(0) |dz| = \lambda_h(0) |z_1 - z_2|, \]

\[ I_2 \leq \int_{[z_1, z_2]} |z|^2 (|g_z| |dz| + |g_t| |d\bar{z}|) \]
\[ \leq r^2 |z_1 - z_2| \Lambda g(z) \leq |z_1 - z_2| \frac{4M_1 r^2}{\pi(1 - r^2)}, \]

\[ I_3 \leq |z_1 - z_2| \sum_{n=2}^{\infty} n(n-1) (|\alpha_n| + |b_n|) r^{n+1} \]
\[ \leq |z_1 - z_2| \frac{8M_1 r^3}{\pi(1 - r^2)}, \]

\[ I_4 \leq |z_1 - z_2| \sum_{n=2}^{\infty} n(n-1) (|\alpha_n| + |d_{sn}|) r^{n-1} \]
\[ \leq |z_1 - z_2| \left( \sum_{n=2}^{\infty} (|\alpha_n| + |d_{sn}|)^2 \right)^{1/2} \]
\[ \cdot \left( \sum_{n=2}^{\infty} n^2 r^{2(n-1)} \right)^{1/2}, \]

\[ I_5 \leq \int_{[z_1, z_2]} (|z|^2 |g_z| |dz| + |\bar{z}|^2 |g_t| |d\bar{z}|) \]
\[ \leq r^2 |z_1 - z_2| \Lambda g(z) \leq |z_1 - z_2| \frac{4M_1 r^2}{\pi(1 - r^2)}, \]

\[ I_6 \leq |z_1 - z_2| \sum_{n=2}^{\infty} n (|\alpha_n| + |d_{sn}|) r^{n-1} \]
\[ \leq |z_1 - z_2| \left( \sum_{n=2}^{\infty} (|\alpha_n| + |d_{sn}|)^2 \right)^{1/2} \]
\[ \cdot \left( \sum_{n=2}^{\infty} n^2 r^{2(n-1)} \right)^{1/2}, \]

\leq |z_1 - z_2| \lambda_h(0) \sqrt{M_2^2 - 1} \cdot \frac{r \sqrt{r^4 - 3r^2 + 4}}{(1 - r^2)^{3/2}}.

Using these estimates and Lemma 7, by (35), we obtain

\[ |H(z_1) - H(z_2)| \]
\[ \geq |z_1 - z_2| \left( \lambda_h(0) - \frac{12M_1 r^2}{\pi(1 - r^2)} \right) - \frac{8M_1 r^3}{\pi(1 - r^2)} - \lambda_h(0) \sqrt{M_2^2 - 1} \cdot \frac{2r \sqrt{4r^2 + r^4 + 1}}{(1 - r^2)^{3/2}} \]
\[ \geq |z_1 - z_2| \left( \lambda_0(M_2) - \frac{12M_1 r^2}{\pi(1 - r^2)} - \frac{8M_1 r^3}{\pi(1 - r^2)} - \lambda_0(M_2) \sqrt{M_2^2 - 1} \cdot \frac{2r \sqrt{4r^2 + r^4 + 1}}{(1 - r^2)^{3/2}} \right) > 0, \]

which implies $H(z_1) \neq H(z_2)$.

For any $z$ such that $z \in \partial D_{r_3}$, by (35) and Lemmas 2 and 5, we obtain

\[ |H(z)| \geq r_3 \left[ \lambda_h(0) - \sum_{n=2}^{\infty} (|\alpha_n| + |d_{sn}|) r_n^{n-1} \right. \]
\[ \left. - \sum_{n=1}^{\infty} (|\alpha_n| + |b_n|) r_3^{n+1} \right] \]
\[ \geq r_3 \left[ \lambda_h(0) - \lambda_h(0) \sqrt{M_2^2 - 1} \cdot \frac{r_3 \sqrt{r^2 - 3r^2 + 4}}{(1 - r_3)^{3/2}} \right. \]
\[ \left. - \frac{4M_1 r_3^2}{\pi(1 - r_3)^2} \right] \]
\[ \geq r_3 \left[ \lambda_0(M_2) - \lambda_0(M_2) \sqrt{M_2^2 - 1} \cdot \frac{r_3 \sqrt{r^2 - 3r^2 + 4}}{(1 - r_3)^{3/2}} \right. \]
\[ \left. - \frac{4M_1 r_3^2}{\pi(1 - r_3)^2} \right] = \sigma_3. \]

This completes the proof. \(\square\)

Setting $M_1 = M_2 = M$ in Theorem 11, we have the following corollary.

**Corollary 12.** Let $F(z) = |z|^2 g(z) + h(z)$ be a biharmonic mapping of the unit disk $D$, with $F(0) = h(0) = F'(0) - 1 = 0$, and both $g(z)$ and $h(z)$ are bounded by $M$. Then $L(F)$ is
univalent in the disk \( D_{r_1} \), where \( r_1 \) is the minimum root of the equation
\[
\lambda_0(M) - \frac{12Mr_2}{\pi(1-r_2^3)} - \frac{8Mr_3^3 - \lambda_0(M)\sqrt{M^4-1}}{\pi(1-r_1^3)} = 0, 
\]
and \( L(F)(D_{r_1}) \) contains a schlicht disk \( D_{s_1} \), where
\[
\sigma_1 = r_1 \left[ \lambda_0(M) - \lambda_0(M)\sqrt{M^4 - 1} \right] 
\]
\[
\cdot \frac{r_1\sqrt{r_1^4 - 3r_1^2 + 4} - 4Mr_1^2}{\pi(1-r_1^3)^{3/2}}. 
\]

In order to show Corollary 12 improves Theorem A, we use Mathematica to compute the approximate values for various choices of \( M \) as in Table 1.

**Remark 13.** From Table 1 we can see, for the same \( M \),
\[
r_2 > \rho_2, \quad \sigma_2 > R_2. 
\]
Finally we improve Theorems B as follows.

**Theorem 14.** Let \( F(z) = |z|^2g(z) \) be a biharmonic mapping in \( \mathbb{D} \) such that \( g(0) = 0 \), \( f(0) = 1 \) and \( |g(z)| < M \), where \( M \geq 1 \) and \( g(z) \) is harmonic in \( \mathbb{D} \). Then \( L(F) \) is univalent in the disk \( D_{r_2} \), where \( r_2 \) is the minimum positive root in \( (0,1) \) of the following equation:
\[
1 - \sqrt{M^4 - 1} \cdot \frac{3r\sqrt{r_1^4 - 3r_1^2 + 4} + 2\sqrt{r_1^4 + 4r_3^2 + 4}}{(1-r_2^3)^{3/2}} = 0, 
\]
and \( L(F)(D_{r_2}) \) contains a schlicht disk \( D_{s_2} \), with
\[
\sigma_2 = r_2^3\lambda_0(M) \left[ 1 - \sqrt{M^4 - 1} \cdot \frac{r_2\sqrt{r_2^4 - 3r_2^2 + 4}}{(1-r_2^3)^{3/2}} \right], 
\]
where \( \lambda_0(M) \) is defined by (16).

**Proof.** Let
\[
g(z) = g_1(z) + \overline{g_2}(z) = \sum_{n=1}^{\infty} a_nz^n + \sum_{n=1}^{\infty} b_n\overline{z}^n. 
\]

Let \( H(z) := L(F) = |z|^2L(g) \); then we have
\[
H_z = 2|z|^2g_z - \overline{z}^2g + |z|^2g_{zz}, 
\]
\[
H_{\overline{z}} = -2|z|^2g_{\overline{z}} + z^2g_{\overline{z}} - \overline{z}|z|^2g_{\overline{zz}}. 
\]
Table 1: The values of $r_1, \sigma_1$ are in Corollary 12. The values of $\rho_1, R_1$ are in Theorem A.

<table>
<thead>
<tr>
<th></th>
<th>$M = 1$</th>
<th>$M = 2$</th>
<th>$M = 3$</th>
<th>$M = 4$</th>
<th>$M = 5$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\rho_1$</td>
<td>0.0527621</td>
<td>0.0139445</td>
<td>0.00626165</td>
<td>0.00353488</td>
<td>0.0022661</td>
</tr>
<tr>
<td>$r_1$</td>
<td>0.357671</td>
<td>0.0593158</td>
<td>0.0269865</td>
<td>0.015355</td>
<td>0.00988556</td>
</tr>
<tr>
<td>$R_1$</td>
<td>0.013793</td>
<td>0.00164514</td>
<td>0.00048245</td>
<td>0.00020277</td>
<td>0.00010364</td>
</tr>
<tr>
<td>$\sigma_1$</td>
<td>0.216467</td>
<td>0.019479</td>
<td>0.00357231</td>
<td>0.00151701</td>
<td>0.00077955</td>
</tr>
</tbody>
</table>

Table 2: The values of $r_2, \sigma_2$ are in Theorem 14. The values of $\rho_2, R_2$ are in Theorem B.

<table>
<thead>
<tr>
<th></th>
<th>$M = 2$</th>
<th>$M = 3$</th>
<th>$M = 4$</th>
<th>$M = 5$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\rho_2$</td>
<td>0.00623234</td>
<td>0.00277176</td>
<td>0.00153948</td>
<td>0.00099817</td>
</tr>
<tr>
<td>$r_2$</td>
<td>0.032209</td>
<td>0.0139701</td>
<td>0.00782686</td>
<td>0.000500376</td>
</tr>
<tr>
<td>$R_2$</td>
<td>6.54254 × 10^{-8}</td>
<td>3.83564 × 10^{-9}</td>
<td>5.12297 × 10^{-10}</td>
<td>1.07466 × 10^{-10}</td>
</tr>
<tr>
<td>$\sigma_2$</td>
<td>9.84416 × 10^{-6}</td>
<td>5.35363 × 10^{-7}</td>
<td>7.06092 × 10^{-8}</td>
<td>1.47596 × 10^{-8}</td>
</tr>
</tbody>
</table>

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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