Research Article

Convergence Theorems for Accretive Operators with Nonlinear Mappings in Banach Spaces

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The purpose of this paper is to present two new forward-backward splitting schemes with relaxations and errors for finding a common element of the set of solutions to the variational inclusion problem with two accretive operators and the set of fixed points of strict pseudocontractions in infinite-dimensional Banach spaces. Under mild conditions, some weak and strong convergence theorems for approximating these common elements are proved. The methods in the paper are novel and different from those in the early and recent literature. Further, we consider the problem of finding a common element of the set of solutions of a mathematical model related to equilibrium problems and the set of fixed points of a strict pseudocontractions.

1. Introduction

The theory of nonexpansive mappings is very important because it is applied to convex optimization, the theory of nonlinear evolution equations, and others. Browder and Petryshyn [1] introduced a class of nonlinear mappings, called strict pseudocontractions, which includes the class of nonexpansive mappings. For strict pseudocontractions, we are interested in finding fixed points of the mappings. We also know the class of inverse-strongly accretive operators which is related to nonexpansive mappings. For inverse-strongly accretive operators, we are interested in finding zero points of the mappings.

Let C be a nonempty closed convex subset of a real Hilbert space H. Let $A : C \rightarrow H$ be a single-valued nonlinear mapping and let $B : H \rightarrow 2^H$ be a multivalued mapping. The so called quasi-variational inclusion problem is to find a $z \in H$ such that

$$0 \in (A + B) z. \qquad (1)$$

The set of solutions of (1) is denoted by $(A+B)^{-1}(0)$. A number of problems arising in structural analysis, mechanics, and economics can be studied in the framework of this kind of variational inclusions; see, for instance, [2–5]. The problem (1) includes many problems as special cases.

(1) If $B = \partial\phi : H \rightarrow 2^H$, where $\phi : H \rightarrow \mathbb{R} \cup +\infty$ is a proper convex lower semicontinuous function and $\partial\phi$ is the subdifferential and if only erential of $\partial\phi$, then the variational inclusion problem (1) is equivalent to finding $u \in H$ such that

$$\langle Au, y - u \rangle + \phi(y) - \phi(u) \geq 0, \quad \forall y \in H, \qquad (2)$$

which is called the mixed quasi-variational inequality (see, Noor [6]).

(2) If $B = \partial\delta_C$, where C is a nonempty closed convex subset of H and $\delta_C : H \rightarrow [0, \infty]$ is the indicator function of C, that is,

$$\delta_C = \begin{cases} 0, & x \in C, \\ +\infty, & x \notin C, \end{cases} \qquad (3)$$

then the variational inclusion problem (1) is equivalent to finding $u \in C$ such that

$$\langle Au, v - u \rangle \geq 0, \quad \forall v \in C. \tag{4}$$

This problem is called Hartman-Stampacchia variational inequality (see, e.g., [7]).

In [8], Zhang et al. investigated the problem of finding a common element of the set of solutions to the inclusion problem and the set of fixed points of nonexpansive mappings by considering the following iterative algorithm:

$$y_n = J_{M,\lambda} (x_n - \lambda A x_n),$$
$$x_{n+1} = \alpha_n x + (1 - \alpha_n) S y_n, \tag{5}$$

where $A : H \to H$ is an α-cocoercive mapping, $M : H \to 2^H$ is a maximal monotone mapping, $S : H \to H$ is a nonexpansive mapping, and $\{\alpha_n\}$ is a sequence in $[0, 1]$. Under mild conditions, they obtained a strong convergence theorem.

In [9], Manaka and Takahashi introduced the following iteration:

$$x_1 \in C, \qquad x_{n+1} = \alpha_n x_n + (1 - \alpha_n) S J_{\lambda_n} (I - \lambda_n A) x_n, \tag{6}$$
$$n \geq 1,$$

where $\{\alpha_n\}$ is a sequence in $(0, 1)$, $\{\lambda_n\}$ is a positive sequence, $S : C \to C$ is a nonexpansive mapping, $A : C \to H$ is an inverse-strongly monotone mapping, $B : D(B) \subset C \to 2^H$ is a maximal monotone operator, and $J_{\lambda_n} = (I + \lambda_n B)^{-1}$ is the resolvent of B. They showed that the sequence $\{x_n\}$ generated in (6) converges weakly to some $z \in (A + B)^{-1}(0) \cap F(S)$ provided that the control sequence satisfies some restrictions.

It is well known that the quasi-variational inclusion problem in the setting of Hilbert spaces has been extensively studied in the literature; see, for instance, [4–10]. However, there is little work in the existing literature on this problem in the setting of Banach spaces (though there was some work on finding a common zero of a finite family of accretive operators [11–13]). The main difficulties are due to the fact that the inner product structure of a Hilbert space fails to be true in a Banach space. To overcome these difficulties, López et al. [14] use the new technique to carry out certain initiative investigations on splitting methods for accretive operators in Banach spaces. They considered the following algorithms with errors in Banach spaces:

$$x_{n+1} = (1 - \alpha_n) x_n + \alpha_n \left(J_{r_n} (x_n - r_n (A x_n + a_n)) + b_n \right), \tag{7}$$

$$x_{n+1} = \alpha_n u + (1 - \alpha_n) \left(J_{r_n} (x_n - r_n (A x_n + a_n)) + b_n \right), \tag{8}$$

where $u \in E$, $\{a_n\}, \{b_n\} \subset E$, and $J_{r_n} = (I + r_n B)^{-1}$ is the resolvent of B. Then they studied the weak and strong convergence of algorithms (7) and (8), respectively.

Motivated and inspired by Zhang et al. [8], Manaka and Takahashi [9], Takahashi et al. [10], Chen and Fan [13], López et al. [14], and Cho et al. [15], the purpose of this paper is to introduce two iterative forward-backward splitting methods

for finding a common element of the set of solutions of the variational inclusion problem (1) with m-accretive operators and inverse-strongly accretive operators and the set of fixed points of strict pseudocontractions in the setting of Banach spaces. Under suitable conditions, some weak and strong convergence theorems for approximating to these common elements are proved. The results presented in the paper improve and extend the corresponding results in [8–10, 13–15].

2. Preliminaries

Throughout this paper, we denote by E and E^* a real Banach space and the dual space of E, respectively. Let C be a subset of E and let T be a mapping on C. We use $F(T)$ to denote the set of fixed points of T. The expressions $x_n \to x$ and $x_n \rightharpoonup x$ denote the strong and weak convergence of the sequence $\{x_n\}$, respectively, and $\omega_w(x_n)$ stands for the set of weak limit points of the sequence $\{x_n\}$. $\mathscr{B}_r$ will denote the closed ball with center zero and radius r.

Let $q > 1$ be a real number. The (generalized) duality mapping $J_q : E \to 2^{E^*}$ is defined by

$$J_q(x) = \left\{ x^* \in E^* : \langle x, x^* \rangle = \|x\|^q, \|x^*\| = \|x\|^{q-1} \right\} \tag{9}$$

for all $x \in E$, where $\langle \cdot, \cdot \rangle$ denotes the generalized duality pairing between E and E^*. In particular, $J = J_2$ is called the normalized duality mapping and $J_q(x) = \|x\|^{q-2} J_2(x)$ for $x \neq 0$. If E is a Hilbert space, then $J = I$ where I is the identity mapping. It is well known that if E is smooth, then J_q is single-valued, which is denoted by j_q.

A Banach space E is said to be uniformly convex if, for any $\varepsilon \in (0, 2]$, there exists $\delta > 0$ such that, for any $x, y \in S(E)$, $\|x - y\| \geq \varepsilon$ implies $\|(x + y)/2\| \leq 1 - \delta$. It is known that a uniformly convex Banach space is reflexive and strictly convex.

The norm of a Banach space E is said to be Gâteaux differentiable if the limit

$$\lim_{t \to 0} \frac{\|x + ty\| - \|x\|}{t} \tag{10}$$

exists for all x, y on the unit sphere $S(E) = \{x \in E : \|x\| = 1\}$. If, for each $y \in S(E)$, limit (10) is uniformly attained for $x \in S(E)$, then the norm of E is said to be uniformly Gâteaux differentiable. The norm of E is said to be Fréchet differentiable if, for each $x \in S(E)$, limit (10) is attained uniformly for $y \in S(E)$.

Let $\rho_E : [0, 1) \to [0, 1)$ be the modulus of smoothness of E defined by

$$\rho_E(t)$$
$$= \sup \left\{ \frac{1}{2} (\|x + y\| + \|x - y\|) - 1 : x \in S(E), \|y\| \leq t \right\}. \tag{11}$$

A Banach space E is said to be uniformly smooth if $\rho_E(t)/t \to 0$ as $t \to 0$. Let $q > 1$. A Banach space E is said to be q-uniformly smooth, if there exists a fixed constant $c > 0$

such that $\rho_E(t) \leq ct^q$. It is well known that E is uniformly smooth if and only if the norm of E is uniformly Fréchet differentiable. If E is q-uniformly smooth, then $q \leq 2$ and E is uniformly smooth, and hence the norm of E is uniformly Fréchet differentiable; in particular, the norm of E is Fréchet differentiable. Typical examples of both uniformly convex and uniformly smooth Banach spaces are L^p, where $p > 1$. More precisely, L^p is $\min\{p, 2\}$-uniformly smooth for every $p > 1$.

A Banach space E is said to satisfy Opial's condition if for any sequence $\{x_n\}$ in E the condition that $\{x_n\}$ converges weakly to $x \in E$ implies that the inequality

$$\liminf_{n \to \infty} \|x_n - x\| < \liminf_{n \to \infty} \|x_n - y\| \tag{12}$$

holds for every $y \in E$ with $y \neq x$.

A Banach space E is said to have the Kadec-Klee property if, for every sequence $\{x_n\}$ in E, $x_n \rightharpoonup x$ and $\|x_n\| \to \|x\|$ together imply $\|x_n - x\| \to 0$. As we know the duals of reflexive Banach spaces with Fréchet differentiable norms have the Kadec-Klee property.

Definition 1. A mapping $T : C \to E$ is said to be

  (1) nonexpansive if

$$\|Tx - Ty\| \leq \|x - y\| \quad \forall x, y \in C; \tag{13}$$

  (2) k-Lipschitz if there exists $k > 0$ such that

$$\|Tx - Ty\| \leq k\|x - y\| \quad \forall x, y \in C; \tag{14}$$

    in particular, if $0 < k < 1$, then T is called contractive and if $k = 1$, then T reduces to a nonexpansive mapping;

  (3) λ-strict pseudocontractive in the terminology of Browder and Petryshyn if for all $x, y \in C$, there exists $\lambda > 0$ and $j_q(x - y) \in J_q(x - y)$ such that

$$\begin{aligned} &\langle Tx - Ty, j_q(x - y) \rangle \\ &\leq \|x - y\|^q - \lambda\|(I - T)x - (I - T)y\|^q; \end{aligned} \tag{15}$$

  (4) accretive if for all $x, y \in C$, there exists $j_q(x - y) \in J_q(x - y)$ such that

$$\langle Tx - Ty, j_q(x - y) \rangle \geq 0; \tag{16}$$

  (5) η-strongly accretive if for all $x, y \in C$, there exists $\eta > 0$ and $j_q(x - y) \in J_q(x - y)$ such that

$$\langle Tx - Ty, j_q(x - y) \rangle \geq \eta\|x - y\|^q; \tag{17}$$

  (6) μ-inverse-strongly accretive if for all $x, y \in C$, there exists $\mu > 0$ and $j_q(x - y) \in J_q(x - y)$ such that

$$\langle Tx - Ty, j_q(x - y) \rangle \geq \mu\|Tx - Ty\|^q. \tag{18}$$

Remark 2. The conception of strict pseudocontractions was firstly introduced by Browder and Petryshyn [1] in a real Hilbert space. Let C be a nonempty subset of a real Hilbert space H, and let $T : C \to C$ be a mapping. In light of [1], T is said to be a K-strict pseudocontraction, if there exists a $K \in [0, 1)$ such that

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + K\|(I - T)x - (I - T)y\|^2 \tag{19}$$
$$\forall x, y \in C.$$

Remark 3. The class of strictly pseudocontractive mappings has been studied by several authors (see, e.g., [1, 16, 17]). However, their iterative methods are far less developed though Browder and Petryshyn [1] initiated their work in 1967. As a matter of fact, strictly pseudocontractive mappings have more powerful applications in solving inverse problems (see, e.g., [18]). Therefore it is interesting to develop the theory of iterative methods for strictly pseudocontractive mappings.

Remark 4. If $E := H$ is a real Hilbert space, then accretive and strongly accretive operators coincide with monotone and strongly monotone operators, respectively.

Definition 5. A set-valued mapping $T : D(T) \subseteq E \to 2^E$ is said to be

  (1) accretive if for any $x, y \in D(T)$, there exists $j(x - y) \in J(x - y)$, such that for all $u \in T(x)$ and $v \in T(y)$,

$$\langle u - v, j(x - y) \rangle \geq 0; \tag{20}$$

  (2) m-accretive if T is accretive and $(I + rT)(D(T)) = E$ for every (equivalently, for some) $r > 0$, where I is the identity mapping. In real Hilbert spaces, m-accretive operators coincide with maximal monotone operators.

Let $M : D(M) \to 2^E$ be m-accretive. The mapping $J_r^M : E \to D(M)$ defined by

$$J_r^M(u) = (I + rM)^{-1}(u), \quad \forall u \in E, \tag{21}$$

is called the resolvent operator associated with M, where r is any positive number and I is the identity mapping. It is well known that J_r^M is single-valued and nonexpansive.

In order to prove our main results, we need the following lemmas.

Lemma 6 (see [19]). *Let E be a Banach space and let J_q be a generalized duality mapping. Then for any given $x, y \in E$, the following inequality holds:*

$$\|x + y\|^q \leq \|x\|^q + q\langle y, j_q(x + y) \rangle,$$
$$j_q(x + y) \in J_q(x + y). \tag{22}$$

In particular, we have, for any given $x, y \in E$,

$$\|x + y\|^2 \leq \|x\|^2 + 2\langle y, j(x + y) \rangle,$$
$$j(x + y) \in J(x + y). \tag{23}$$

Lemma 7 (see [19]). *Let $1 < p < \infty$, $q \in (1, 2]$, $r > 0$ be given.*

    (i) *If E is uniformly convex, then there exists a continuous, strictly increasing, and convex function $\varphi : [0, \infty) \to [0, \infty)$ with $\varphi(0) = 0$ such that*

$$\|\lambda x + (1 - \lambda) y\|^p \leq \lambda \|x\|^p + \lambda \|y\|^p - W_p(\lambda) \varphi(\|x - y\|),$$
$$x, y \in \mathscr{B}_r, \ 0 \leq \lambda \leq 1, \tag{24}$$

    where $W_p(\lambda) = \lambda^p(1 - \lambda) + (1 - \lambda)^p \lambda$, $\mathscr{B}_r = \{z \in E : \|z\| \leq r\}$.

    (ii) *If E is a real q-uniformly smooth Banach space, then there exists a constant $C_q > 0$ such that*

$$\|x + y\|^q \leq \|x\|^q + q \langle y, J_q(x) \rangle + C_q \|y\|^q,$$
$$\forall x, y \in E. \tag{25}$$

Lemma 8 (see [20]). *Let $\{a_n\}$, $\{b_n\}$, and $\{\delta_n\}$ be sequences of nonnegative real numbers satisfying the inequality*

$$a_{n+1} \leq (1 + \delta_n) a_n + b_n, \quad \forall n = 1, 2, \ldots. \tag{26}$$

If $\sum_{n=0}^{\infty} \delta_n < \infty$ and $\sum_{n=0}^{\infty} b_n < \infty$, then $\lim_{n \to \infty} a_n$ exists. In particular, $\lim_{n \to \infty} a_n = 0$ whenever there exists a subsequence $\{a_{n_k}\}$ in $\{a_n\}$ which strongly converges to zero.

Lemma 9 (see [21]). *Let $\{\alpha_n\}$ be a sequence of nonnegative numbers satisfying the following property:*

$$\alpha_{n+1} \leq (1 - \gamma_n) \alpha_n + b_n + \gamma_n c_n, \quad n \in \mathbb{N}, \tag{27}$$

where $\{\gamma_n\}$, $\{b_n\}$, $\{c_n\}$ satisfy the restrictions

    (i) $\sum_{n=1}^{\infty} \gamma_n = \infty$,

    (ii) $b_n \geq 0$, $\sum_{n=1}^{\infty} b_n < \infty$,

    (iii) $\limsup_{n \to \infty} c_n \leq 0$.

Then, $\lim_{n \to \infty} \alpha_n = 0$.

Lemma 10 (see [16]). *Let C be a nonempty convex subset of a real q-uniformly smooth Banach space E and let $T : C \to C$ be a λ-strict pseudocontraction. For $\alpha \in (0, 1)$, we define $T_\alpha x = (1 - \alpha)x + \alpha Tx$. Then, as $\alpha \in (0, \rho]$, $\rho = \min\{1, \{q\lambda/C_q\}^{1/(q-1)}\}$, $T_\alpha : C \to C$ is nonexpansive such that $F(T_\alpha) = F(T)$.*

Lemma 11 (see [22]). *Let E be a uniformly convex Banach space, C a closed convex subset of E, and $T : C \to E$ a nonexpansive mapping with $F(T) \neq \emptyset$. Then, $I - T$ is demiclosed at zero.*

Lemma 12. *Let C be a nonempty closed convex subset of a real q-uniformly smooth Banach space E. Let the mapping $A : C \to E$ be an α-inverse-strongly accretive operator. Then the following inequality holds:*

$$\|(I - \lambda A) x - (I - \lambda A) y\|^q$$
$$\leq \|x - y\|^q - \lambda \left(q\alpha - C_q \lambda^{q-1} \right) \|Ax - Ay\|^q. \tag{28}$$

In particular, if $0 < \lambda \leq (q\alpha/C_q)^{1/(q-1)}$, then $\|I - \lambda A\|$ is nonexpansive.

Proof. Indeed, for all $x, y \in C$, it follows from Lemma 7 that

$$\|(I - \lambda A) x - (I - \lambda A) y\|^q$$
$$= \|(x - y) - \lambda (Ax - Ay)\|^q$$
$$\leq \|x - y\|^q - q\lambda \left\langle Ax - Ay, j_q(x - y) \right\rangle$$
$$+ C_q \lambda^q \|Ax - Ay\|^q \tag{29}$$
$$\leq \|x - y\|^q - q\alpha\lambda \|Ax - Ay\|^q$$
$$+ C_q \lambda^q \|Ax - Ay\|^q$$
$$\leq \|x - y\|^q - \lambda \left(q\alpha - C_q \lambda^{q-1} \right) \|Ax - Ay\|^q.$$

It is clear that if $0 < \lambda \leq (q\alpha/C_q)^{1/(q-1)}$, then $I - \lambda A$ is nonexpansive. This completes the proof. $\square$

Lemma 13 (see [23]). *If E is a uniformly convex Banach space and C is a closed convex bounded subset of E, there is a continuous strictly increasing function $g : [0, \infty) \to [0, \infty)$ with $g(0) = 0$ such that*

$$g \left(\|S(tx + (1 - t) y) - (\alpha Sx + (1 - \alpha) Sy)\| \right)$$
$$\leq \|x - y\| - \|Sx - Sy\| \tag{30}$$

for all $x, y \in C$, $t \in [0, 1]$ and nonexpansive mapping $S : C \to E$.

Lemma 14 (see [24]). *Let E be a real reflexive Banach space such that its dual E^* has the Kadec-Klee property. Let $\{x_n\}$ be a bounded sequence in E and $x^*, y^* \in \omega_w(x_n)$; here $\omega_w(x_n)$ denotes the weak w-limit set of $\{x_n\}$. Suppose $\lim_{n \to \infty} \|tx_n + (1 - t)x^* - y^*\|$ exists for all $t \in [0, 1]$. Then $x^* = y^*$.*

Lemma 15. *Assume that E is a real uniformly convex and q-uniformly smooth Banach space. Suppose that $A : E \to E$ is α-inverse-strongly accretive operator for some $\alpha > 0$ and $B : E \to 2^E$ is an m-accretive operator. Moreover, denote J_r by*

$$J_r \equiv J_r^B = (I + rB)^{-1} \tag{31}$$

and T_r by

$$T_r = J_r (I - rA) = (I + rB)^{-1} (I - rA). \tag{32}$$

Then, it holds for all $r > 0$ that $F(T_r) = (A + B)^{-1}(0)$.

Proof. From the definition of T_r, we have

$$x = T_r x \iff x = (I + rB)^{-1} (I - rA) x$$
$$\iff (I - rA) x \in (I + rB) x \tag{33}$$
$$\iff 0 \in (A + B) x.$$

This completes the proof. $\square$

Lemma 15 alludes to the fact that, in order to solve the inclusion problem (1), it suffices to find a fixed point of T_r. Since T_r is already split, an iterative algorithm for T_r corresponds to a splitting algorithm for (1). However, to guarantee convergence (weak or strong) of an iterative algorithm for T_r, we need good metric properties of T_r such as nonexpansivity. To this end, some related geometric conditions on the underlying space E are very necessary (see Lemmas 16 and 17 below).

Lemma 16 (see [14]). *Assume that E is a real uniformly convex and q-uniformly smooth Banach space. Suppose that $A : E \to E$ is α-inverse-strongly accretive operator for some $\alpha > 0$ and $B : E \to 2^E$ is an m-accretive operator. Then, the following relations hold.*

(i) *Given $0 < s \le r$ and $x \in E$,*

$$\left\| T_s x - T_r x \right\| \le \left| 1 - \frac{s}{r} \right| \left\| x - T_r x \right\|,$$
$$\left\| x - T_s x \right\| \le 2 \left\| x - T_r x \right\|. \tag{34}$$

(ii) *Given $s > 0$, there exists a continuous, strictly increasing, and convex function $\phi_q : [0, \infty) \to [0, \infty)$ with $\phi_q(0) = 0$ such that, for all $x, y \in \mathcal{B}_s$,*

$$\left\| T_r x - T_r y \right\|^q \le \left\| x - y \right\|^q$$
$$- r \left(\alpha q - r^{q-1} C_q \right) \left\| Ax - Ay \right\|^q \tag{35}$$
$$- \phi_q \left(\left\| (I - J_r)(I - rA)x - (I - J_r)(I - rA)y \right\| \right).$$

Lemma 17. *Let E be a real uniformly convex and q-uniformly smooth Banach space. Suppose that $S : E \to E$ is a nonexpansive mapping, $A : E \to E$ is an α-inverse-strongly accretive operator for some $\alpha > 0$, and $B : E \to 2^E$ is an m-accretive operator. Assume that $0 < r \le (q\alpha/C_q)^{1/(q-1)}$. Then $F(ST_r) = F(S) \cap F(T_r)$.*

Proof. Suppose that $x_1 \in F(ST_r)$; it is sufficient to show that $x_1 \in F(S) \cap F(T_r)$. Indeed, for $x_2 \in F(S) \cap F(T_r)$, we have by Lemma 16 that

$$\left\| x_1 - x_2 \right\|^q = \left\| ST_r x_1 - ST_r x_2 \right\|^q$$
$$\le \left\| T_r x_1 - T_r x_2 \right\|^q$$
$$\le \left\| x_1 - x_2 \right\|^q \tag{36}$$
$$- r \left(\alpha q - r^{q-1} C_q \right) \left\| Ax_1 - Ax_2 \right\|^q$$
$$- \phi_q \left\| (I - J_r)(I - rA)x_1 - (I - J_r)(I - rA)x_2 \right\|.$$

The property of ϕ and the condition $0 < r \le (q\alpha/C_q)^{1/(q-1)}$ together imply that

$$\left\| Ax_1 - Ax_2 \right\|$$
$$= \left\| (I - J_r)(I - rA)x_1 - (I - J_r)(I - rA)x_2 \right\| = 0. \tag{37}$$

It turns out that

$$\left\| x_1 - T_r x_1 - x_2 + T_r x_2 \right\| = 0, \tag{38}$$

which imply

$$T_r x_1 = x_1. \tag{39}$$

Noticing the assumption of $x_1 = ST_r x_1$, we can deduce $x_1 = Sx_1$. This means that $x_1 \in F(S) \cap F(T_r)$. $\qquad\square$

Lemma 18 (see [25]). *Let C be a nonempty, closed, and convex subset of a real q-uniformly smooth Banach space E. Let $V : C \to E$ be a k-Lipschitz and η-strongly accretive operator with constants $k, \eta > 0$. Let $0 < \mu < (q\eta/C_q k^q)^{1/(q-1)}$ and $\tau = \mu(\eta - (C_q \mu^{q-1} k^q / q))$. Then for $t \in (0, \min\{1, 1/\tau\})$, the mapping $S : C \to E$ defined by $S := (I - t\mu V)$ is a contraction with a constant $1 - t\tau$.*

Next we give a weak convergence theorem in a Banach space E.

3. Main Results

Theorem 19. *Let E be a uniformly convex and q-uniformly smooth Banach space. Let $A : E \to E$ be α-inverse-strongly accretive, $B : E \to 2^E$ m-accretive, and $S : E \to E$ λ-strict pseudocontractive. Assume that $F(S) \cap (A + B)^{-1}(0) \ne \emptyset$. Define a mapping $Tx := (1 - \sigma)x + \sigma Sx$ for all $x \in E$. For arbitrarily given $x_1 \in E$ and $\sigma \in (0, \rho]$, where $\rho = \min\{1, \{q\lambda/C_q\}^{1/(q-1)}\}$, let $\{x_n\}$ be the sequence generated iteratively by*

$$x_{n+1} = (1 - \alpha_n) x_n + \alpha_n T \left(J_{r_n} (x_n - r_n (Ax_n + a_n)) + b_n \right),$$
$$\forall n \ge 1, \tag{40}$$

where $J_{r_n} = (I + r_n B)^{-1}$, $\{a_n\}, \{b_n\} \subset E$, $\{\alpha_n\} \subset (0, 1]$, and $\{r_n\} \subset (0, +\infty)$. Assume that

(i) $\sum_{n=1}^{\infty} \|a_n\| < \infty$ and $\sum_{n=1}^{\infty} \|b_n\| < \infty$,

(ii) $0 < \liminf_{n \to \infty} \alpha_n \le \limsup_{n \to \infty} \alpha_n < 1$,

(iii) $0 < \liminf_{n \to \infty} r_n \le \limsup_{n \to \infty} r_n < (q\alpha/C_q)^{1/(q-1)}$.

Then $\{x_n\}$ converges weakly to some point $x \in F(S) \cap (A + B)^{-1}(0)$.

Proof. We divide the proof into several steps.

Step 1. We prove $\lim_{n \to \infty} \|x_n - z\|$ exists for any point $z \in F(S) \bigcap (A + B)^{-1}(0)$.

Putting $T_n = J_{r_n}(I - r_n A) = (I + r_n B)^{-1}(I - r_n A)$, one has

$$T \left(J_{r_n} (x_n - r_n (Ax_n + a_n)) + b_n \right) = TT_n x_n + g_n, \tag{41}$$

where

$$g_n = T \left(J_{r_n} (x_n - r_n (Ax_n + a_n)) + b_n \right) - TT_n x_n. \tag{42}$$

Then the iterative formula (40) turns into the form

$$x_{n+1} = (1 - \alpha_n) x_n + \alpha_n (TT_n x_n + g_n). \qquad (43)$$

Thus, by virtue of Lemmas 10 and 12 and nonexpansivity of J_{r_n}, we have

$$
\begin{aligned}
\|g_n\| &= \left\| T \left(J_{r_n} (x_n - r_n (Ax_n + a_n)) + b_n \right) - TT_n x_n \right\| \\
&\leq \left\| J_{r_n} (x_n - r_n (Ax_n + a_n)) - T_n x_n \right\| + \|b_n\| \qquad (44) \\
&\leq r_n \|a_n\| + \|b_n\|.
\end{aligned}
$$

By (44) and condition (i), we have that

$$\sum_{n=1}^{\infty} \|g_n\| < \infty. \qquad (45)$$

Since $z \in F(S) \cap (A+B)^{-1}(0)$, according to Lemmas 10 and 15, we can deduce $z \in F(T) \cap F(T_n)$. Lemma 16 and condition (iii) together imply T_n is nonexpansive. Therefore, we get from (43) that

$$
\begin{aligned}
&\|x_{n+1} - z\| \\
&\quad = (1 - \alpha_n) \|x_n - z\| + \alpha_n \|TT_n x_n + g_n - z\| \\
&\quad \leq (1 - \alpha_n) \|x_n - z\| + \alpha_n \|TT_n x_n - z\| + \alpha_n \|g_n\| \qquad (46) \\
&\quad \leq (1 - \alpha_n) \|x_n - z\| + \alpha_n \|x_n - z\| + \alpha_n \|g_n\| \\
&\quad \leq \|x_n - z\| + \alpha_n \|g_n\|.
\end{aligned}
$$

In view of (45), (46), and Lemma 8, we get that $\lim_{n \to \infty} \|x_n - z\|$ exists. Therefore $\{x_n\}$ is bounded.

Step 2. We show $\lim_{n \to \infty} \|T_n x_n - x_n\| = 0$.

Let $M_1 > 0$ be such that $\|x_n\| < M_1$, for all $n \in \mathbb{N}$ and let $s = q(M_1 + \|z\|)^{q-1}$. By (43), Lemmas 6, 10, and 16, we have

$$
\begin{aligned}
&\|x_{n+1} - z\|^q \\
&\quad = \left\| (1 - \alpha_n) (x_n - z) + \alpha_n (TT_n x_n + g_n - z) \right\|^q \\
&\quad \leq \left\| (1 - \alpha_n) (x_n - z) + \alpha_n (TT_n x_n - z) \right\|^q \\
&\qquad + \alpha_n q \left\langle g_n, j_q (x_{n+1} - z) \right\rangle \\
&\quad \leq (1 - \alpha_n) \|x_n - z\|^q \\
&\qquad + \alpha_n \|TT_n x_n - z\|^q + \alpha_n q \|g_n\| \|x_{n+1} - z\|^{q-1} \\
&\quad \leq (1 - \alpha_n) \|x_n - z\|^q + \alpha_n \|T_n x_n - z\|^q \\
&\qquad + \alpha_n q \|g_n\| \|x_{n+1} - z\|^{q-1} \\
&\quad \leq \|x_n - z\|^q - \alpha_n r_n \left(\alpha q - r_n^{q-1} C_q \right) \|Ax_n - Az\|^q \\
&\qquad - \alpha_n \phi_q \left(\left\| (I - J_{r_n}) (I - r_n A) x_n - (I - J_{r_n}) (I - r_n A) z \right\| \right) \\
&\qquad + \alpha_n s \|g_n\|
\end{aligned}
$$

$$
\begin{aligned}
&\leq \|x_n - z\|^q - \alpha_n r_n \left(\alpha q - r_n^{q-1} C_q \right) \|Ax_n - Az\|^q \\
&\quad - \alpha_n \phi_q \left(\|x_n - r_n Ax_n - T_n x_n + r_n Az\| \right) \\
&\quad + \alpha_n s \|g_n\|.
\end{aligned}
$$

$$\qquad (47)$$

Meanwhile, by the fact that $a^r - b^r \leq r a^{r-1} (a - b)$, $\forall r \geq 1$ and (47), we get that

$$
\begin{aligned}
&\alpha_n r_n \left(\alpha q - r_n^{q-1} C_q \right) \|Ax_n - Az\|^q \\
&\quad + \alpha_n \phi_q \left(\|x_n - r_n Ax_n - T_n x_n + r_n Az\| \right) \\
&\leq \|x_n - z\|^q - \|x_{n+1} - z\|^q + \alpha_n s \|g_n\| \\
&\leq q \|x_n - z\|^{q-1} \left(\|x_n - z\| - \|x_{n+1} - z\| \right) + \alpha_n s \|g_n\|.
\end{aligned}
$$

$$\qquad (48)$$

Thanks to (45), existence of $\lim_{n \to \infty} \|x_n - z\|$, (ii) and (iii), one has

$$\lim_{n \to \infty} \|Ax_n - Az\| = \lim_{n \to \infty} \|x_n - r_n Ax_n - T_n x_n + r_n Az\| = 0. \qquad (49)$$

It turns out that

$$\lim_{n \to \infty} \|T_n x_n - x_n\| = 0. \qquad (50)$$

Step 3. We prove $\lim_{n \to \infty} \|TT_n x_n - x_n\| = 0$.

Noticing (45) and Lemma 7, we have

$$
\begin{aligned}
&\|x_{n+1} - z\|^2 \\
&\quad = \left\| (1 - \alpha_n) (x_n - z) + \alpha_n (TT_n x_n + g_n - z) \right\|^2 \\
&\quad \leq (1 - \alpha_n) \|x_n - z\|^2 + \alpha_n \|TT_n x_n + g_n - z\|^2 \\
&\qquad - W_2 (\alpha_n) \varphi \left(\|TT_n x_n + g_n - x_n\| \right) \\
&\quad \leq (1 - \alpha_n) \|x_n - z\|^2 \\
&\qquad + \alpha_n \left(\|TT_n x_n - z\|^2 + 2 \|TT_n x_n - z\| \|g_n\| + \|g_n\|^2 \right) \\
&\qquad - W_2 (\alpha_n) \varphi \left(\|TT_n x_n + g_n - x_n\| \right) \\
&\quad \leq (1 - \alpha_n) \|x_n - z\|^2 \\
&\qquad + \alpha_n \left(\|x_n - z\|^2 + 2 \|x_n - z\| \|g_n\| + \|g_n\|^2 \right) \\
&\qquad - W_2 (\alpha_n) \varphi \left(\|TT_n x_n + g_n - x_n\| \right) \\
&\quad \leq \|x_n - z\|^2 + 2\alpha_n \|x_n - z\| \|g_n\| + \alpha_n \|g_n\|^2 \\
&\qquad - W_2 (\alpha_n) \varphi \left(\|TT_n x_n + g_n - x_n\| \right) \\
&\quad \leq \|x_n - z\|^2 + 2 \|x_n - z\| \|g_n\| + \|g_n\|^2 \\
&\qquad - W_2 (\alpha_n) \varphi \left(\|TT_n x_n + g_n - x_n\| \right),
\end{aligned}
$$

$$\qquad (51)$$

which implies

$$W_2(\alpha_n)\varphi(\|TT_nx_n + g_n - x_n\|)$$

$$\leq \|x_n - z\|^2 - \|x_{n+1} - z\|^2 \tag{52}$$

$$+ 2\|x_n - z\|\|g_n\| + \|g_n\|^2,$$

where $W_2(\alpha_n) = \alpha_n(1-\alpha_n)$. From (45), (52), (ii), and existence of $\lim_{n\to\infty}\|x_n - z\|$, it turns out that

$$\lim_{n\to\infty}\varphi(\|TT_nx_n + g_n - x_n\|) = 0. \tag{53}$$

It follows from the property of φ and (45) that

$$\lim_{n\to\infty}\|TT_nx_n - x_n\| = 0. \tag{54}$$

Step 4. We prove $\omega_w(x_n) \subset F(S) \cap (A+B)^{-1}(0)$.

Since $0 < \liminf_{n\to\infty}r_n \leq \limsup_{n\to\infty}r_n < 1$, there exists $\varepsilon > 0$ such that $r_n \geq \varepsilon$ for all $n \geq 1$. Then, by Lemma 16, we have

$$\lim_{n\to\infty}\|T_\varepsilon x_n - x_n\| \leq 2\lim_{n\to\infty}\|T_nx_n - x_n\| = 0. \tag{55}$$

It follows from (50), (54), and (55) that

$$\|TT_\varepsilon x_n - x_n\|$$

$$\leq \|TT_\varepsilon x_n - TT_nx_n\| + \|TT_nx_n - x_n\|$$

$$\leq \|T_\varepsilon x_n - T_nx_n\| + \|TT_nx_n - x_n\| \tag{56}$$

$$\leq \|T_\varepsilon x_n - x_n\| + \|x_n - T_nx_n\| + \|TT_nx_n - x_n\|$$

$$\longrightarrow 0.$$

By Lemmas 10, 11, and 17 and (56), we get

$$\omega_w(x_n) \subset F(TT_\varepsilon) = F(T) \cap F(T_\varepsilon)$$

$$= F(S) \cap (A+B)^{-1}(0). \tag{57}$$

Step 5. We show $\{x_n\}$ converges weakly to a fixed point of $x \in F(S) \cap (A+B)^{-1}(0)$.

Indeed, it suffices to show $\omega_w(x_n)$ consists of exactly only one point. To this end, we suppose that two different points x and y are in $\omega_w(x_n)$. Then there exist two different subsequences $\{n_i\}$ and $\{n_j\}$ such that $x_{n_i} \rightharpoonup x$ and $x_{n_j} \rightharpoonup y$ as $i \to \infty$ and $j \to \infty$. Define $S_{n,m} : E \to E$ by

$$S_{n,m} = V_{n+m-1}V_{n+m-2}\cdots V_n,$$

$$V_n = (1 - \alpha_n)I + \alpha_nTT_n. \tag{58}$$

Then x_n can be written

$$x_{n+m} = S_{n,m}x_n + c_{n,m}, \tag{59}$$

where

$$c_{n,m} = V_{n+m-1}$$

$$\times (V_{n+m-2}(\cdots V_{n+1}(V_nx_n + \alpha_ng_n) + \alpha_{n+1}g_{n+1}\cdots)$$

$$+ \alpha_{n+m-2}g_{n+m-2})$$

$$+ \alpha_{n+m-1}g_{n+m-1} - S_{n,m}x_n. \tag{60}$$

Thanks to the nonexpansivity of V_n, we have

$$\|c_{n,m}\| \leq \sum_{k=n}^{n+m-1}\|\alpha_kg_k\| \leq \sum_{k=n}^{n+m-1}\|g_k\|. \tag{61}$$

It follows from (45) that

$$\lim_{m,n\to\infty}\|c_{n,m}\| \longrightarrow 0. \tag{62}$$

Let

$$f_n(t) = \|tx_n + (1-t)x - y\|,$$

$$d_{n,m} = S_{n,m}(tx_n + (1-t)x) - (tS_{n,m}x_n + (1-t)x). \tag{63}$$

Applying Lemma 13 to the closed convex bounded subset $D := \overline{\mathrm{co}}(\{x_n\} \cup \{x, y\})$, we obtain

$$g(\|d_{n,m}\|) \leq \|x_n - x\| - \|S_{n,m}x_n - S_{n,m}x\|$$

$$\leq \|x_n - x\| - \|x_{n+m} - x - c_{n,m}\| \tag{64}$$

$$\leq \|x_n - x\| - \|x_{n+m} - x\| + \|c_{n,m}\|.$$

Since $\lim_{n\to\infty}\|x_n - x\|$ exists, (62), (64), and the property of g together imply that

$$\lim_{m,n\to\infty}\|d_{n,m}\| \longrightarrow 0. \tag{65}$$

Furthermore, we have

$$f_{n+m}(t) = \|tx_{n+m} + (1-t)x - y\|$$

$$\leq \|d_{n,m}\| + \|S_{n,m}(tx_n + (1-t)x) - y\| + t\|c_{m,n}\|$$

$$\leq \|d_{n,m}\| + \|tx_n + (1-t)x - y\| + t\|c_{m,n}\|$$

$$= \|d_{n,m}\| + f_n(t) + t\|c_{m,n}\|. \tag{66}$$

After taking first $\limsup_{m\to\infty}$ and then $\liminf_{n\to\infty}$ in (66) and using (62) and (65), we get

$$\limsup_{m\to\infty}f_m(t) \leq \liminf_{n\to\infty}f_n(t)$$

$$+ \lim_{m,n\to\infty}(\|d_{m,n}\| + \|c_{n,m}\|) = \liminf_{n\to\infty}f_n(t). \tag{67}$$

So $\lim_{n\to\infty}\|tx_n + (1-t)x - y\|$ exists for all $t \in [0, 1]$. It follows from Lemma 14 that $x = y$. This completes the proof. □

Remark 20. Compared with the known results in the literature, our results are very different from those in the following aspects.

(i) Theorem 19 improves and extends Theorem 3 of Kamimura and Takahashi [4] and Theorem 3.1 of Manaka and Takahashi [9] from Hilbert spaces to uniformly convex and q-uniformly smooth Banach spaces.

(ii) Theorem 19 also improves and extends Theorem 3.6 of López et al. [14] from the problem of finding an element of $(A + B)^{-1}(0)$ to the problem of finding an element of $(A + B)^{-1}(0) \cap F(S)$, where S is λ-strictly pseudocontractive on E.

In the following, we give a strong convergence theorem in a Banach space E.

Theorem 21. *Let E be a uniformly convex and q-uniformly smooth Banach space which admits a weakly sequentially continuous generalized duality mapping $j_q : E \to E^*$. Let $A : E \to E$ be α-inverse-strongly accretive, $B : E \to 2^E$ m-accretive, $G : E \to E$ k-Lipschitz and η-strongly accretive, $\psi : E \to E$ L-Lipschitz, and $S : E \to E$ λ-strictly pseudocontractive. Define a mapping $Tx := (1 - \sigma)x + \sigma Sx$ for all $x \in E$. For arbitrarily given $x_1 \in E$ and $\sigma \in (0, \rho]$, where $\rho = \min\{1, \{q\lambda/C_q\}^{1/(q-1)}\}$, let $\{x_n\}$ be the sequence generated iteratively by*

$$x_{n+1} = \alpha_n \gamma \psi (x_n) + (I - \alpha_n \mu G) T \times \left(J_{r_n} (x_n - r_n (Ax_n + a_n)) + b_n \right), \quad \forall n \geq 1. \tag{68}$$

Assume that $\{\alpha_n\} \subset [0, 1]$, $\{r_n\} \subset (0, +\infty)$ and $\{a_n\}, \{b_n\} \subset E$ satisfying the following conditions:

(i) $\sum_{n=1}^{\infty} \|a_n\| < \infty$ *and* $\sum_{n=1}^{\infty} \|b_n\| < \infty$,

(ii) $\sum_{n=1}^{\infty} \alpha_n = \infty$, $\lim_{n \to \infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$,

(iii) $0 < \liminf_{n \to \infty} r_n \leq \limsup_{n \to \infty} r_n < (q\alpha/C_q)^{1/(q-1)}$ *and* $\sum_{n=1}^{\infty} |r_{n+1} - r_n| < \infty$.

Suppose in addition that $(S) \cap (A + B)^{-1}(0) \neq \emptyset$, $0 < \mu < (q\eta/C_q k^q)^{1/(q-1)}$ and $0 \leq \gamma L < \tau$, where $\tau = \mu(\eta - (C_q \mu^{q-1} k^q / q))$. Then $\{x_n\}$ converges strongly to some point $z \in F(S) \cap (A + B)^{-1}(0)$ which solves the variational inequality: $\langle \gamma \psi(z) - \mu G(z), j_q(x - z) \rangle \leq 0$, for all $x \in F(S) \cap (A + B)^{-1}(0)$.

Proof. Let $\{y_n\}$ be a sequence generated by

$$y_{n+1} = \alpha_n \gamma \psi (y_n) + (I - \alpha_n \mu G) T T_n y_n, \tag{69}$$

where $T_n := J_{r_n}(I - r_n A)$. We show $\|y_n - x_n\| \to 0$.

It follows from Lemmas 10, 12, and 18 that

$$\|y_{n+1} - x_{n+1}\|$$
$$\leq \left\| \alpha_n \gamma \psi (y_n) + (I - \alpha_n \mu G) T J_{r_n} (y_n - r_n A y_n) \right.$$
$$- \alpha_n \gamma \psi (x_n) - (I - \alpha_n \mu G) T$$
$$\left. \times \left(J_{r_n} (x_n - r_n (Ax_n + a_n)) + b_n \right) \right\|$$

$$\leq (1 - \alpha_n \tau) \left\| T J_{r_n} (y_n - r_n A y_n) \right.$$
$$\left. - T \left(J_{r_n} (x_n - r_n (Ax_n + a_n)) + b_n \right) \right\|$$
$$+ \alpha_n \gamma L \|y_n - x_n\|$$

$$\leq (1 - \alpha_n \tau) \left\| J_{r_n} (y_n - r_n A y_n) \right.$$
$$\left. - J_{r_n} (x_n - r_n (Ax_n + a_n)) \right\|$$
$$+ \|b_n\| + \alpha_n \gamma L \|y_n - x_n\|$$

$$\leq [1 - \alpha_n (\tau - \gamma L)] \|y_n - x_n\|$$
$$+ r_n \|a_n\| + \|b_n\|. \tag{70}$$

By virtue of Lemma 8, (i), and (70), we have $\lim_{n \to \infty} \|y_n - x_n\| = 0$.

Hence, to show the desired result, it suffices to prove that $y_n \to z$.

Step 1. We prove that the sequence $\{y_n\}$ is bounded. Taking $x \in F(S) \cap (A + B)^{-1}(0)$, it follows from Lemmas 10, 12, 15, and 16 and condition (iii) that

$$\|y_{n+1} - x\| = \|\alpha_n \gamma \psi (y_n) + (I - \alpha_n \mu G) T T_n y_n - x\|$$
$$= \|\alpha_n \gamma (\psi (y_n) - \psi (x)) + \alpha_n (\gamma \psi (x) - \mu G (x))$$
$$+ (I - \alpha_n \mu G) T T_n y_n - (I - \alpha_n \mu G) x\|$$
$$\leq \alpha_n \gamma L \|y_n - x\| + \alpha_n \|\gamma \psi (x) - \mu G (x)\|$$
$$+ (1 - \alpha_n \tau) \|y_n - x\|$$
$$= [1 - \alpha_n (\tau - \gamma L)] \|y_n - x\|$$
$$+ \alpha_n \|\gamma \psi (x) - \mu G (x)\|$$
$$\leq \max \left\{ \frac{\|\gamma \psi (x) - \mu G (x)\|}{\tau - \gamma L}, \|y_n - x\| \right\}. \tag{71}$$

By induction, we have

$$\|y_n - x\| \leq \max \left\{ \frac{\|\gamma \psi (x) - \mu G (x)\|}{\tau - \gamma L}, \|y_1 - x\| \right\}, \quad \forall n \geq 1. \tag{72}$$

Hence, $\{y_n\}$ is bounded, and so are $\{\psi(y_n)\}$ and $\{T_n(y_n)\}$.

Step 2. We prove that

$$\lim_{n \to \infty} \|y_{n+1} - y_n\| \to 0. \tag{73}$$

Putting $z_n = T_n y_n = J_{r_n}(I - r_n A) y_n$, it follows from Lemma 16 that

$$\|z_{n+1} - z_n\|$$
$$= \|T_{n+1} y_{n+1} - T_n y_n\|$$
$$\leq \|T_{n+1} y_{n+1} - T_n y_{n+1}\| + \|T_n y_{n+1} - T_n y_n\|$$

$$\leq \left| 1 - \frac{r_{\alpha_n}}{r_{\beta_n}} \right| \left\| y_{n+1} - J_{r_{\beta_n}} \left(1 - r_{\beta_n} A \right) y_{n+1} \right\| + \left\| y_{n+1} - y_n \right\|$$

$$\leq \left| r_{\beta_n} - r_{\alpha_n} \right| \frac{\left\| y_{n+1} - J_{r_{\beta_n}} \left(1 - r_{\beta_n} A \right) y_{n+1} \right\|}{r_{\beta_n}} + \left\| y_{n+1} - y_n \right\|$$

$$\leq \left| r_{n+1} - r_n \right| M_2 + \left\| y_{n+1} - y_n \right\|, \tag{74}$$

where $M_2 > \sup_{n \geq 1} \{ \| y_{n+1} - J_{r_{\beta_n}} (1 - r_{\beta_n} A) y_{n+1} \| / r_{\beta_n} \}$, $r_{\alpha_n} = \min\{r_{n+1}, r_n\}$, and $r_{\beta_n} = \max\{r_{n+1}, r_n\}$. Hence from (69) and (74) we have

$$\left\| y_{n+1} - y_n \right\|$$

$$= \left\| \alpha_n \gamma \psi \left(y_n \right) + \left(I - \alpha_n \mu G \right) T z_n \right.$$

$$\left. - \alpha_{n-1} \gamma \psi \left(y_{n-1} \right) - \left(I - \alpha_{n-1} \mu G \right) T z_{n-1} \right\|$$

$$= \left\| \alpha_n \gamma \left(\psi \left(y_n \right) - \psi \left(y_{n-1} \right) \right) + \left(I - \alpha_n \mu G \right) T z_n \right.$$

$$\left. - \left(I - \alpha_n \mu G \right) T z_{n-1} + \left(\alpha_n - \alpha_{n-1} \right) \right.$$

$$\left. \times \left(\gamma \psi \left(y_{n-1} \right) - \mu G T z_{n-1} \right) \right\| \tag{75}$$

$$\leq \left(1 - \alpha_n \tau \right) \left\| T z_n - T z_{n-1} \right\|$$

$$+ \alpha_n \gamma L \left\| y_n - y_{n-1} \right\| + \left| \alpha_n - \alpha_{n-1} \right| M_3$$

$$\leq \left(1 - \alpha_n \tau \right) \left\| z_n - z_{n-1} \right\|$$

$$+ \alpha_n \gamma L \left\| y_n - y_{n-1} \right\| + \left| \alpha_n - \alpha_{n-1} \right| M_3$$

$$\leq \left[1 - \alpha_n \left(\tau - \gamma L \right) \right] \left\| y_n - y_{n-1} \right\|$$

$$+ \left| \alpha_n - \alpha_{n-1} \right| M_3 + \left| r_n - r_{n-1} \right| M_2,$$

where $M_3 > \sup_{n \geq 1} \{ \| \gamma \psi(y_n) - \mu G T z_n \| \}$. It follows from Lemma 9, (ii), and (iii) that $\lim_{n \to \infty} \| y_{n+1} - y_n \| = 0$.

Again from Lemmas 6 and 16, we obtain

$$\left\| y_{n+1} - x \right\|^q$$

$$= \left\| \alpha_n \gamma \psi \left(y_n \right) + \left(I - \alpha_n \mu G \right) T z_n - x \right\|^q$$

$$= \left\| \alpha_n \left(\gamma \psi \left(y_n \right) - \mu G \left(x \right) \right) + \left(I - \alpha_n \mu G \right) T z_n \right.$$

$$\left. - \left(I - \alpha_n \mu G \right) x \right\|^q$$

$$\leq \left(1 - \alpha_n \tau \right) \left\| T_n y_n - x \right\|^q \tag{76}$$

$$+ q \alpha_n \left\langle \gamma \psi \left(y_n \right) - \mu G \left(x \right), j_q \left(y_{n+1} - x \right) \right\rangle$$

$$\leq \left\| T_n y_n - x \right\|^q + q \alpha_n M_4$$

$$\leq \left\| y_n - x \right\|^q - r_n \left(\alpha q - r_n^{q-1} C_q \right) \left\| A y_n - A x \right\|^q$$

$$- \phi_q \left(\left\| y_n - r_n A y_n - T_n y_n + r_n A x \right\| \right) + q \alpha_n M_4,$$

where $M_4 > \sup_{n \geq 1} \{ \langle \gamma \psi(y_n) - \mu G(x), j_q(y_{n+1} - x) \rangle \}$. Meanwhile, by the fact that $a^r - b^r \leq r a^{r-1} (a - b)$ for all $r \geq 1$, we get that

$$r_n \left(\alpha q - r_n^{q-1} C_q \right) \left\| A y_n - A x \right\|^q$$

$$+ \phi_q \left(\left\| y_n - r_n A y_n - T_n y_n + r_n A x \right\| \right)$$

$$\leq \left\| y_n - x \right\|^q - \left\| y_{n+1} - x \right\|^q + q \alpha_n M_4 \tag{77}$$

$$\leq q \left\| y_n - x \right\|^{q-1} \left(\left\| y_n - x \right\| - \left\| y_{n+1} - x \right\| \right) + q \alpha_n M_4.$$

It follows immediately from (ii), (iii), (77), existence of $\lim_{n \to \infty} \| y_n - x \|$, and the property of ϕ_q that

$$\lim_{n \to \infty} \left\| A y_n - A x \right\| = \lim_{n \to \infty} \left\| y_n - r_n A y_n - T_n y_n + r_n A x \right\| = 0. \tag{78}$$

Hence we obtain that

$$\lim_{n \to \infty} \left\| T_n y_n - y_n \right\| = 0. \tag{79}$$

By condition (iii), there exists $\varepsilon > 0$ such that $r_n \geq \varepsilon$ for all $n \geq 1$. Then, by Lemma 16, we get

$$\lim_{n \to \infty} \left\| T_\varepsilon y_n - y_n \right\| \leq \lim_{n \to \infty} 2 \left\| T_n y_n - y_n \right\| = 0. \tag{80}$$

Step 3. We show $\lim_{n \to \infty} \| T T_\varepsilon y_n - y_n \| = 0$.

From (73), (79), (80), and (ii), we have

$$\left\| T T_\varepsilon y_n - y_n \right\|$$

$$\leq \left\| T T_\varepsilon y_n - T T_n y_n \right\| + \left\| T T_n y_n - y_n \right\|$$

$$\leq \left\| T_\varepsilon y_n - T_n y_n \right\| + \left\| T T_n y_n - y_n \right\|$$

$$\leq \left\| T_\varepsilon y_n - y_n \right\| + \left\| y_n - T_n y_n \right\| \tag{81}$$

$$+ \left\| T T_n y_n - y_{n+1} \right\| + \left\| y_{n+1} - y_n \right\|$$

$$\leq \left\| T_\varepsilon y_n - y_n \right\| + \left\| y_n - T_n y_n \right\|$$

$$+ \alpha_n \left\| \gamma \psi \left(y_n \right) - \mu G T T_n y_n \right\| + \left\| y_{n+1} - y_n \right\|$$

$$\longrightarrow 0.$$

Lemmas 10, 11, and 17 and (81) together imply that

$$\omega_w \left(y_n \right) \subset F \left(T T_\varepsilon \right) = F \left(T \right) \cap F \left(T_\varepsilon \right)$$

$$= F \left(S \right) \cap \left(A + B \right)^{-1} \left(0 \right). \tag{82}$$

By Song's Lemma 2.11 [25], we deduce directly that $\{z_t\}$ defined by $z_t = t \gamma \psi(z_t) - (I - t \mu G) T T_\varepsilon z_t$ converges strongly to some point $z \in F(T T_\varepsilon)$ which is the unique solution of the variational inequality:

$$\left\langle \gamma \psi \left(z \right) - \mu G \left(z \right), j_q \left(x - z \right) \right\rangle \leq 0, \quad \forall x \in F \left(T T_\varepsilon \right). \tag{83}$$

Step 4. We prove that

$$\limsup_{n \to \infty} \left\langle \gamma \psi \left(z \right) - \mu G \left(z \right), j_q \left(y_n - z \right) \right\rangle \leq 0. \tag{84}$$

We take a subsequence $\{y_{n_i}\}$ of $\{y_n\}$ such that

$$
\begin{aligned}
\limsup_{n \to \infty} & \left\langle \gamma\psi(z) - \mu G(z), j_q(y_n - z) \right\rangle \\
&= \lim_{i \to \infty} \left\langle \gamma\psi(z) - \mu G(z), j_q(y_{n_i} - z) \right\rangle.
\end{aligned}
\tag{85}
$$

Without loss of generality, we may further assume that $y_{n_i} \rightharpoonup \tilde{x}$ due to reflexivity of the Banach space E and boundedness of $\{y_n\}$. It follows from (82) that $\tilde{x} \in F(TT_\varepsilon)$. Since Banach space E has a weakly sequentially continuous generalized duality mapping $j_p : E \to E^*$, we obtain that

$$
\begin{aligned}
\limsup_{n \to \infty} & \left\langle \gamma\psi(z) - \mu G(z), j_q(y_n - z) \right\rangle \\
&= \lim_{i \to \infty} \left\langle \gamma\psi(z) - \mu G(z), j_q(y_{n_i} - z) \right\rangle \\
&= \left\langle \gamma\psi(z) - \mu G(z), j_q(\tilde{x} - z) \right\rangle \\
&\leq 0.
\end{aligned}
\tag{86}
$$

Step 5. We show $\|y_n - z\| \to 0$.

By Lemmas 9 and 16 and the fact that $ab \leq (1/q)a^q + ((q-1)/q)b^{q/(q-1)}$, we get

$$
\begin{aligned}
&\|y_{n+1} - z\|^q \\
&= \|\alpha_n \gamma\psi(y_n) + (I - \alpha_n \mu G) TT_n y_n - z\|^q \\
&= \left\langle \alpha_n \gamma\psi(y_n) + (I - \alpha_n \mu G) TT_n y_n - z, j_q(y_{n+1} - z) \right\rangle \\
&= \alpha_n \gamma \left\langle \psi(y_n) - \psi(z), j_q(y_{n+1} - z) \right\rangle \\
&\quad + \alpha_n \left\langle \gamma\psi(z) - \mu G(z), j_q(y_{n+1} - z) \right\rangle \\
&\quad + \left\langle (I - \alpha_n \mu G) TT_n y_n - (I - \alpha_n \mu G) z, j_q(y_{n+1} - z) \right\rangle \\
&\leq \alpha_n \gamma \|\psi(y_n) - \psi(z)\| \|y_{n+1} - z\|^{q-1} \\
&\quad + \alpha_n \left\langle \gamma\psi(z) - \mu G(z), j_q(y_{n+1} - z) \right\rangle \\
&\quad + (1 - \alpha_n \tau) \|y_n - z\| \|y_{n+1} - z\|^{q-1} \\
&\leq \alpha_n L \|y_n - z\| \|y_{n+1} - z\|^{q-1} \\
&\quad + \alpha_n \left\langle \gamma\psi(z) - \mu G(z), j_q(y_{n+1} - z) \right\rangle \\
&\quad + (1 - \alpha_n \tau) \|y_n - z\| \|y_{n+1} - z\|^{q-1} \\
&\leq [1 - \alpha_n(\tau - \gamma L)] \|y_n - z\| \|y_{n+1} - z\|^{q-1} \\
&\quad + \alpha_n \left\langle \gamma\psi(z) - \mu G(z), j_q(y_{n+1} - z) \right\rangle \\
&\leq [1 - \alpha_n(\tau - \gamma L)] \frac{1}{q}\|y_n - z\|^q + \frac{q-1}{q}\|y_{n+1} - z\|^q \\
&\quad + \alpha_n \left\langle \gamma\psi(z) - \mu G(z), j_q(y_{n+1} - z) \right\rangle,
\end{aligned}
\tag{87}
$$

which implies that

$$
\begin{aligned}
\|y_{n+1} - z\|^q &\leq [1 - \alpha_n(\tau - \gamma L)] \|y_n - z\|^q \\
&\quad + q\alpha_n \left\langle \gamma\psi(z) - \mu G(z), j_q(y_{n+1} - z) \right\rangle.
\end{aligned}
\tag{88}
$$

Apply Lemma 9 to (88) to conclude $y_n \to z$ as $n \to \infty$. This completes the proof. $\qquad \square$

Remark 22. Theorem 21 improves and extends Theorem 3.7 of López et al. [14] in the following aspects:

(i) from the problem of finding an element of $(A+B)^{-1}(0)$ to the problem of finding an element of $(A+B)^{-1}(0) \cap F(S)$, where S is λ-strictly pseudocontractive on E;

(ii) from a fixed element u to a Lipschitz mapping ψ.

Remark 23. Theorem 21 improves and extends Theorem 2.1 of Zhang et al. [8] in the following aspects:

(i) from Hilbert spaces to uniformly convex and q-uniformly smooth Banach spaces;

(ii) from finding a common element of the set of solutions to the variational inclusion problem and the set of fixed points of nonexpansive mappings to finding a common element of the set of solutions to the variational inclusion problem and the set of fixed points of λ-strict pseudocontractions;

(iii) from a fixed element u to a Lipschitz mapping ψ;

(iv) from a fixed positive number λ to a sequence positive number $\{r_n\}$.

As a direct consequence of Theorem 21, we obtain the following result.

Corollary 24. *Let H be a real Hilbert space. Let $A : H \to H$ be α-inverse-strongly monotone, $B : H \to 2^H$ maximal monotone, $G : H \to H$ k-Lipschitz and η-strongly monotone, $\psi : H \to H$ L-Lipschitz, and $S : H \to H$ K-strictly pseudocontractive. Define a mapping $Tx := (1 - \sigma)x + \sigma Sx$ for all $x \in H$. For arbitrarily given $x_1 \in H$ and $\sigma \in [K, 1)$, let $\{x_n\}$ be the sequence generated iteratively by*

$$
x_{n+1} = \alpha_n \gamma\psi(x_n) + (I - \alpha_n \mu G) TJ_{r_n}(x_n - r_n(Ax_n + a_n)),
$$

$$
\forall n \geq 1.
\tag{89}
$$

Assume that $\{\alpha_n\} \subset [0,1]$, $\{r_n\} \subset (0, +\infty)$, and $\{a_n\} \subset H$ satisfying the following conditions:

(i) $\sum_{n=1}^{\infty} \|a_n\| < \infty$,

(ii) $\sum_{n=1}^{\infty} \alpha_n = \infty$, $\lim_{n \to \infty} \alpha_n = 0$ and $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$,

(iii) $0 < \liminf_{n \to \infty} r_n \leq \limsup_{n \to \infty} r_n < 2\alpha$ and $\sum_{n=1}^{\infty} |r_{n+1} - r_n| < \infty$.

Suppose in addition that $F(S) \cap (A+B)^{-1}(0) \neq \emptyset$, $0 < \mu < 2\eta/k^2$ and $0 \leq \gamma L < \tau$, where $\tau = \mu(\eta - \mu k^2/2)$. Then $\{x_n\}$ converges strongly to some point $z \in F(S) \cap (A + B)^{-1}(0)$ which solves the variational inequality: $\langle \gamma\psi(z) - \mu G(z), x - z \rangle \leq 0$, for all $x \in F(S) \cap (A + B)^{-1}(0)$.

4. Applications

Using Corollary 24, we consider the problem for finding a common element of the set of solutions of a mathematical model related to equilibrium problems and the set of fixed points of a strict pseudocontraction in a Hilbert space. Let C be a nonempty, closed, and convex subset of a Hilbert space and let $f : C \times C \to \mathbb{R}$ be a bifunction satisfying the following conditions:

(A1) $f(x, x) = 0$ for all $x \in C$;

(A2) f is monotone, that is, $f(x, y) + f(y, x) \leq 0$ for all $x, y \in C$;

(A3) for all $x, y, z \in C$,

$$\limsup_{t \downarrow 0} f(tz + (1 - t)x, y) \leq f(x, y); \tag{90}$$

(A4) for all $x \in C$, $f(x, \cdot)$ is convex and lower semicontinuous.

Then, the mathematical model related to equilibrium problems (with respect to C) is to find $\widehat{x} \in C$ such that

$$f(\widehat{x}, y) = 0 \tag{91}$$

for all $y \in C$. The set of such solutions $\widehat{x}$ is denoted by $\text{EP}(f)$.

The following lemma appears implicitly in Blum and Oettli [26].

Lemma 25. *Let C be a nonempty, closed, and convex subset of H and let $f : C \times C \to \mathbb{R}$ be a bifunction satisfying (A1)–(A4). Let $r > 0$ and $x \in H$. Then, there exists $z \in C$ such that*

$$f(z, y) + \frac{1}{r} \langle y - z, z - x \rangle \geq 0, \quad \forall y \in C. \tag{92}$$

The following lemma was also given in Combettes and Hirstoaga [27].

Lemma 26. *Assume that $f : C \times C \to \mathbb{R}$ satisfies (A1)–(A4). For $r > 0$ and $x \in H$, define a mapping $S_r : H \to C$ as follows:*

$$S_r x = \left\{ z \in C : f(z, y) + \frac{1}{r} \langle y - z, z - x \rangle \geq 0, \forall y \in C \right\} \tag{93}$$

for all $x \in H$. Then, the following hold:

(1) *S_r is single-valued;*

(2) *S_r is a firmly nonexpansive mapping; that is, for all $x, y \in H$, $\|S_r x - S_r y\|^2 \leq \langle S_r x - S_r y, x - y \rangle$;*

(3) *$F(S_r) = \text{EP}(f)$;*

(4) *$\text{EP}(f)$ is closed and convex.*

We call such S_r the resolvent of f for $r > 0$. Using Lemmas 25 and 26, Takahashi et al. [10] proved the following theorem. See [10] for a more general result.

Theorem 27. *Let H be a Hilbert space and let C be a nonempty, closed, and convex subset of H. Let $f : C \times C \to \mathbb{R}$ satisfy (A1)–(A4). Let A_f be a multivalued mapping of H into itself defined by*

$$A_f x = \begin{cases} \{z \in H : f(x, y) \geq \langle y - x, z \rangle, \\ \qquad\qquad \forall y \in C\}, & x \in C, \\ \emptyset, & x \notin C. \end{cases} \tag{94}$$

Then, $\text{EP}(f) = A_f^{-1} 0$ and A_f is a maximal monotone operator with $\text{dom}(A_f) \subset C$. Further, for any $x \in H$ and $r > 0$, the resolvent S_r of f coincides with the resolvent of A_f; that is, $S_r x = (I + rA_f)^{-1} x$.

Theorem 28. *Let H be a real Hilbert space. Suppose $f : H \times H \to \mathbb{R}$ is a bifunction satisfying the following conditions:*

(B1) *$f(x, x) = 0$ for all $x \in H$;*

(B2) *f is monotone, that is, $f(x, y) + f(y, x) \leq 0$ for all $x, y \in H$;*

(B3) *for all $x, y, z \in H$,*

$$\limsup_{t \downarrow 0} f(tz + (1 - t)x, y) \leq f(x, y); \tag{95}$$

(B4) *for all $x \in H$, $f(x, \cdot)$ is convex and lower semicontinuous. Assume S_δ is the resolvent of f for $\delta > 0$, $G : H \to H$ is k-Lipschitz and η-strongly monotone, $\psi : H \to H$ is L-Lipschitz, and $S : H \to H$ is K-strictly pseudocontractive. Define a mapping $Tx := (1 - \sigma)x + \sigma Sx$ for all $x \in H$. For arbitrarily given $x_1 \in H$ and $\sigma \in [K, 1)$, let $\{x_n\}$ be the sequence generated iteratively by*

$$x_{n+1} = \alpha_n \gamma \psi(x_n) + (I - \alpha_n \mu G) TS_{r_n} x_n, \quad \forall n \geq 1. \tag{96}$$

Assume that $\{\alpha_n\} \subset [0, 1]$ and $\{r_n\} \subset (0, +\infty)$ satisfying the following conditions:

(i) *$\sum_{n=1}^{\infty} \alpha_n = \infty$, $\lim_{n \to \infty} \alpha_n = 0$ and $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$,*

(ii) *$\sum_{n=1}^{\infty} |r_{n+1} - r_n| < \infty$.*

Suppose in addition that $F(S) \cap \text{EP}(f) \neq \emptyset$, $0 < \mu < 2\eta/k^2$ and $0 \leq \gamma L < \tau$, where $\tau = \mu(\eta - \mu k^2/2)$. Then $\{x_n\}$ converges strongly to some point $z \in F(S) \cap \text{EP}(f)$ which solves the variational inequality: $\langle \gamma\psi(z) - \mu G(z), x - z \rangle \leq 0$, for all $x \in F(S) \cap \text{EP}(f)$.

Proof. Put $A = 0$ and $a_n = 0$ for all $n \in \mathbb{N}$ in Corollary 24. From Theorem 27, we also know that $J_{r_n}^{A_f} = S_{r_n}$ for all $n \in \mathbb{N}$. So, we obtain the desired result by Corollary 24. □

Conflict of Interests

The authors declare that they have no competing interests.

Acknowledgments

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