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Research Article

The Invertibility, Explicit Determinants, and Inverses of Circulant and Left Circulant and g-Circulant Matrices Involving Any Continuous Fibonacci and Lucas Numbers

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Circulant matrices play an important role in solving delay differential equations. In this paper, circulant type matrices including the circulant and left circulant and g-circulant matrices with any continuous Fibonacci and Lucas numbers are considered. Firstly, the invertibility of the circulant matrix is discussed and the explicit determinant and the inverse matrices by constructing the transformation matrices are presented. Furthermore, the invertibility of the left circulant and g-circulant matrices is also studied. We obtain the explicit determinants and the inverse matrices of the left circulant and g-circulant matrices by utilizing the relationship between left circulant, g-circulant matrices and circulant matrix, respectively.

1. Introduction

Circulant matrices have important applications in solving various differential equations [1–3]. The use of circulant preconditioners for solving structured linear systems has been studied extensively since 1986; see [4, 5]. Circulant matrices also play an important role in solving delay differential equations. In [6], Chan et al. proposed a preconditioner called the Strang-type block-circulant preconditioner for solving linear systems from IVPs. The Strang-type preconditioner was also used to solve linear systems from differential-algebraic equations and delay differential equations; see [7–14]. In [15], Jin et al. proposed the GMRES method with the Strang-type block-circulant preconditioner for solving singular perturbation delay differential equations.

The g-circulant matrices play an important role in various applications as well; please refer to [16, 17] for details. There are discussions about the convergence in probability and in distribution of the spectral norm of g-circulant matrices in [18, 19]. Ngondiep et al. showed the singular values of g-circulants in [20].

Recently, some scholars have given various algorithms for the determinants and inverses of nonsingular circulant matrices [21, 22]. Unfortunately, the computational complexity of these algorithms is increasing dramatically with the increasing order of matrices. However, some authors gave the explicit determinants and inverse of circulant involving Fibonacci and Lucas numbers. For example, Jaiswal evaluated some determinants of circulant whose elements are the generalized Fibonacci numbers [23]. Lind presented the determinants of circulant involving Fibonacci numbers [24]. Lin gave the determinant of the Fibonacci-Lucas quasicyclic matrices in [25]. Shen et al. considered circulant matrices with Fibonacci and Lucas numbers and presented their explicit determinants and inverses [26]. Bozkurt and Tam gave determinants and inverses of circulant matrices with Jacobsthal and Jacobsthal-Lucas numbers in [27].

The purpose of this paper is to obtain the explicit determinants, explicit inverses of circulant, left circulant, and g-circulant matrices involving any continuous Fibonacci numbers and Lucas numbers. And we generalize the result in [26].

In the following, let r be a nonnegative integer. We adopt the following two conventions $0^0 = 1$, and for any sequence $\{a_n\}$, $\sum_{k=i}^{n} a_k = 0$ in the case $i > n$.

The Fibonacci and Lucas sequences are defined by the following recurrence relations [23–26], respectively:

$$\begin{aligned} F_{n+1} &= F_n + F_{n-1} \text{ where } F_0 = 0, F_1 = 1, \\ L_{n+1} &= L_n + L_{n-1} \text{ where } L_0 = 2, L_1 = 1, \end{aligned} \tag{1}$$

for $n \geq 0$. The first few values of the sequences are given by the following table:

$$\begin{array}{c|cccccccccc} n & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & \cdots \\ \hline F_n & 0 & 1 & 1 & 2 & 3 & 5 & 8 & 13 & 21 & \cdots \\ L_n & 2 & 1 & 3 & 4 & 7 & 11 & 18 & 29 & 47 & \cdots \end{array}. \tag{2}$$

Let α and β be the roots of the characteristic equation $x^2 - x - 1 = 0$; then the Binet formulas of the sequences $\{F_{r+n}\}$ and $\{L_{r+n}\}$ have the form

$$F_{r+n} = \frac{\alpha^{r+n} - \beta^{r+n}}{\alpha - \beta}, \qquad L_{r+n} = \alpha^{r+n} + \beta^{r+n}. \tag{3}$$

Definition 1 (see [21, 22]). In a right circulant matrix (or simply, circulant matrix)

$$\operatorname{Circ}(a_1, a_2, \ldots, a_n) = \begin{bmatrix} a_1 & a_2 & \cdots & a_n \\ a_n & a_1 & \cdots & a_{n-1} \\ \vdots & \vdots & & \vdots \\ a_2 & a_3 & \cdots & a_1 \end{bmatrix}, \tag{4}$$

each row is a cyclic shift of the row above to the right. Right circulant matrix is a special case of a Toeplitz matrix. It is evidently determined by its first row (or column).

Definition 2 (see [22, 28]). In a left circulant matrix (or reverse circulant matrix)

$$\operatorname{LCirc}(a_1, a_2, \ldots, a_n) = \begin{bmatrix} a_1 & a_2 & \cdots & a_n \\ a_2 & a_3 & \cdots & a_1 \\ \vdots & \vdots & & \vdots \\ a_n & a_1 & \cdots & a_{n-1} \end{bmatrix}, \tag{5}$$

each row is a cyclic shift of the row above to the left. Left circulant matrix is a special Hankel matrix.

Definition 3 (see [19, 29]). A g-circulant matrix is an $n \times n$ complex matrix with the following form:

$$A_{g,n} = \begin{pmatrix} a_1 & a_2 & \cdots & a_n \\ a_{n-g+1} & a_{n-g+2} & \cdots & a_{n-g} \\ a_{n-2g+1} & a_{n-2g+2} & \cdots & a_{n-2g} \\ \vdots & \vdots & \ddots & \vdots \\ a_{g+1} & a_{g+2} & \cdots & a_g \end{pmatrix}, \tag{6}$$

where g is a nonnegative integer and each of the subscripts is understood to be reduced modulo n.

The first row of $A_{g,n}$ is $(a_1, a_2, \ldots, a_n)$; its $(j+1)$th row is obtained by giving its jth row a right circular shift by g positions (equivalently, g mod n positions). Note that $g = 1$ or $g = n+1$ yields the standard circulant matrix. If $g = n-1$, then we obtain the so called left circulant matrix.

Lemma 4 (see [26]). *Let $A = \operatorname{Circ}(a_1, a_2, \ldots, a_n)$ be circulant matrix; then one has*

(i) *A is invertible if and only if the eigenvalues of A*

$$\lambda_k = f\left(\omega^k\right) \neq 0, \quad (k = 0, 1, \ldots, n-1), \tag{7}$$

where $f(x) = \sum_{j=1}^{n} a_j x^{j-1}$ and $\omega = \exp(2\pi i/n)$;

(ii) *if A is invertible, then the inverse A^{-1} of A is a circulant matrix.*

Lemma 5. *Define*

$$\Delta := \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & 0 & \cdots & 1 & 0 \\ \vdots & \vdots & \ddots & & \vdots & \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \end{pmatrix}; \tag{8}$$

the matrix Δ is an orthogonal cyclic shift matrix (and a left circulant matrix). It holds that $\operatorname{LCirc}(a_1, a_2, \ldots, a_n) = \Delta \operatorname{Circ}(a_1, a_2, \ldots, a_n)$.

Lemma 6 (see [29]). *The $n \times n$ matrix $\mathbb{Q}_g$ is unitary if and only if $(n, g) = 1$, where $\mathbb{Q}_g$ is a g-circulant matrix with first row $e^* = [1, 0, \ldots, 0]$.*

Lemma 7 (see [29]). *$A_{g,n}$ is a g-circulant matrix with first row $[a_1, a_2, \ldots, a_n]$ if and only if $A_{g,n} = \mathbb{Q}_g C$, where $C = \operatorname{Circ}(a_1, a_2, \ldots, a_n)$.*

2. Determinant, Invertibility, and Inverse of Circulant Matrix with Any Continuous Fibonacci Numbers

In this section, let $A_{r,n} = \operatorname{Circ}(F_{r+1}, F_{r+2}, \ldots, F_{r+n})$ be a circulant matrix. Firstly, we give the determinant equation of the matrix $A_{r,n}$. Afterwards, we prove that $A_{r,n}$ is an invertible matrix for $n > 2$, and then we find the inverse of the matrix $A_{r,n}$. Obviously, when $n = 2$, $r \neq 0$, or $n = 1$, $A_{r,n}$ is also an invertible matrix.

Theorem 8. *Let $A_{r,n} = \operatorname{Circ}(F_{r+1}, F_{r+2}, \ldots, F_{r+n})$ be a circulant matrix. Then one has*

$$\begin{aligned} &\det A_{r,n} \\ &= F_{r+1} \cdot \left\{ \left(F_{r+1} - \frac{F_{r+2}}{F_{r+1}} F_{r+n} \right) \right. \\ &\quad + \sum_{k=1}^{n-2} \left[\left(F_{r+k+2} - \frac{F_{r+2}}{F_{r+1}} F_{r+k+1} \right) \right. \\ &\qquad \left. \left. \times \left(\frac{F_{r+n} - F_r}{F_{r+1} - F_{r+n+1}} \right)^{n-(k+1)} \right] \right\} \\ &\times \left(F_{r+1} - F_{r+n+1}\right)^{n-2}. \end{aligned} \tag{9}$$

where F_{r+n} is the $(r+n)$th Fibonacci number. Specially, when $r=0$, this result is the same as Theorem 2.1 in [26].

Proof. Obviously, $\det A_1 = (1-F_{n+1})^{n-1} + F_n^{n-2}\sum_{k=1}^{n-1} F_k((1-F_{n+1})/F_n)^{k-1}$ satisfies the formula. In the case $n>1$, let

$$\Gamma = \begin{pmatrix} 1 & & & & & & \\ -\frac{F_{r+2}}{F_{r+1}} & & & & & 1 & \\ -1 & & & & 1 & -1 & \\ 0 & & 0 & 1 & -1 & -1 & \\ \vdots & & & \iddots & \iddots & \iddots & \\ 0 & & 1 & \iddots & \iddots & & \\ 0 & 1 & -1 & \iddots & & 0 & \\ 0 & 1 & -1 & -1 & & & \end{pmatrix},$$

$$\Pi_1 = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & \left(\frac{F_{r+n}-F_r}{F_{r+1}-F_{r+n+1}}\right)^{n-2} & 0 & \cdots & 0 & 1 \\ 0 & \left(\frac{F_{r+n}-F_r}{F_{r+1}-F_{r+n+1}}\right)^{n-3} & 0 & \cdots & 1 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & \frac{F_{r+n}-F_r}{F_{r+1}-F_{r+n+1}} & 1 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \end{pmatrix} \tag{10}$$

be two $n\times n$ matrices; then we have

$$\Gamma A_{r,n}\Pi_1 = \begin{bmatrix} F_{r+1} & f'_{r,n} & F_{r+n-1} & \cdots & F_{r+2} \\ 0 & f_{r,n} & \tau_n & \cdots & \tau_3 \\ 0 & 0 & F_{r+1}-F_{r+n+1} & & 0 \\ 0 & 0 & F_r - F_{r+n} & & 0 \\ \vdots & \vdots & & \ddots & \\ 0 & 0 & 0 & & F_{r+1}-F_{r+n+1} \end{bmatrix}, \tag{11}$$

where

$$\begin{aligned} a_r &= \frac{F_{r+2}}{F_{r+1}}, f'_{r,n} \\ &= \sum_{k=1}^{n-1} F_{r+k+1}\left(\frac{F_{r+n}-F_r}{F_{r+1}-F_{r+n+1}}\right)^{n-(k+1)}, \\ \tau_k &= F_{r+k} - a_r F_{r+k-1}, \quad k=3,\dots,n, \\ f_{r,n} &= \left(F_{r+1} - \frac{F_{r+2}}{F_{r+1}}F_{r+n}\right) \\ &\quad + \sum_{k=1}^{n-2}\left[\left(F_{r+k+2} - \frac{F_{r+2}}{F_{r+1}}F_{r+k+1}\right)\right. \\ &\quad \left.\times\left(\frac{F_{r+n}-F_r}{F_{r+1}-F_{r+n+1}}\right)^{n-(k+1)}\right]. \end{aligned} \tag{12}$$

We obtain

$$\begin{aligned} &\det\Gamma \det A_{r,n} \det\Pi_1 \\ &= F_{r+1}\cdot\left\{\left(F_{r+1} - \frac{F_{r+2}}{F_{r+1}}F_{r+n}\right)\right. \\ &\quad + \sum_{k=1}^{n-2}\left[\left(F_{r+k+2} - \frac{F_{r+2}}{F_{r+1}}F_{r+k+1}\right)\right. \\ &\quad \left.\left.\times\left(\frac{F_{r+n}-F_r}{F_{r+1}-F_{r+n+1}}\right)^{n-(k+1)}\right]\right\} \\ &\quad \times (F_{r+1}-F_{r+n+1})^{n-2}; \end{aligned} \tag{13}$$

while

$$\det\Gamma = (-1)^{(n-1)(n-2)/2}, \det\Pi_1 = (-1)^{(n-1)(n-2)/2}, \tag{14}$$

we have

$$\begin{aligned} &\det A_{r,n} \\ &= F_{r+1}\cdot\left\{\left(F_{r+1} - \frac{F_{r+2}}{F_{r+1}}F_{r+n}\right)\right. \\ &\quad + \sum_{k=1}^{n-2}\left[\left(F_{r+k+2} - \frac{F_{r+2}}{F_{r+1}}F_{r+k+1}\right)\right. \\ &\quad \left.\left.\times\left(\frac{F_{r+n}-F_r}{F_{r+1}-F_{r+n+1}}\right)^{n-(k+1)}\right]\right\} \\ &\quad \times (F_{r+1}-F_{r+n+1})^{n-2}. \end{aligned} \tag{15}$$

□

Theorem 9. *Let $A_{r,n} = \mathrm{Circ}(F_{r+1}, F_{r+2}, \dots, F_{r+n})$ be a circulant matrix; if $n>2$, then $A_{r,n}$ is an invertible matrix. Specially, when $r=0$, one gets Theorem 2.2 in [26].*

Proof. When $n=3$ in Theorem 8, then we have $\det A_{r,n} = (F_{r+1}+F_{r+2}+F_{r+3})(F_{r+1}^2+F_rF_{r+2}) \neq 0$; hence $A_{r,n}$ is invertible.

In the case $n > 3$, since $F_{r+n} = (\alpha^{r+n} - \beta^{r+n})/(\alpha - \beta)$, where $\alpha + \beta = 1$, $\alpha\beta = -1$. We have

$$\begin{aligned} f\left(\omega^k\right) &= \sum_{j=1}^{n} F_{r+j}\left(\omega^k\right)^{j-1} \\ &= \frac{1}{\alpha-\beta}\sum_{j=1}^{n}\left(\alpha^{r+j}-\beta^{r+j}\right)\left(\omega^k\right)^{j-1} \\ &= \frac{1}{\alpha-\beta}\left[\frac{\alpha^{r+1}\left(1-\alpha^n\right)}{1-\alpha\omega^k} - \frac{\beta^{r+1}\left(1-\beta^n\right)}{1-\beta\omega^k}\right] \\ &= \frac{\left(\alpha^{r+1}-\beta^{r+1}\right)-\left(\alpha^{r+n+1}-\beta^{r+n+1}\right)}{(\alpha-\beta)\left(1-(\alpha+\beta)\omega^k+\alpha\beta\omega^{2k}\right)} \\ &\quad - \frac{\alpha\beta\left(\alpha^r-\beta^r-\alpha^{r+n}+\beta^{r+n}\right)\omega^k}{(\alpha-\beta)\left(1-(\alpha+\beta)\omega^k+\alpha\beta\omega^{2k}\right)} \\ &= \frac{F_{r+1}-F_{r+n+1}+\left(F_r-F_{r+n}\right)\omega^k}{1-\omega^k-\omega^{2k}} \\ &\qquad (k = 1, 2, \ldots, n-1). \end{aligned} \tag{16}$$

If there exists ω^l $(l = 1, 2, \ldots, n-1)$ such that $f(\omega^l) = 0$, we obtain $F_{r+1} - F_{r+n+1} + (F_r - F_{r+n})\omega^l = 0$ for $1 - \omega^l - \omega^{2l} \neq 0$; thus, $\omega^l = (F_{r+1} - F_{r+n+1})/(F_{r+n} - F_r)$ is a real number. While

$$\omega^l = \exp\left(\frac{2l\pi i}{n}\right) = \cos\left(\frac{2l\pi}{n}\right) + i\sin\left(\frac{2l\pi}{n}\right), \tag{17}$$

hence, $\sin(2l\pi/n) = 0$; so we have $\omega^l = -1$ for $0 < 2l\pi/n < 2\pi$. But $x = -1$ is not the root of the equation $F_{r+1} - F_{r+n+1} + (F_r - F_{r+n})x = 0$ $(n > 3)$. We obtain $f(\omega^k) \neq 0$ for any ω^k $(k = 1, 2, \ldots, n-1)$, while $f(1) = \sum_{j=1}^{n} F_{r+j} = -F_{r+1} + F_{r+n+1} - (F_r - F_{r+n}) = F_{r+n+2} - F_{r+2} \neq 0$. By Lemma 4, the proof is completed. □

Lemma 10. *Let the entries of the matrix $\mathcal{G} = [g_{i,j}]_{i,j=1}^{n-2}$ be of the form*

$$g_{i,j} = \begin{cases} F_{r+1} - F_{r+n+1}, & i = j, \\ F_r - F_{r+n}, & i = j+1, \\ 0, & \textit{otherwise}; \end{cases} \tag{18}$$

then the entries of the inverse $\mathcal{G}^{-1} = [g'_{i,j}]_{i,j=1}^{n-2}$ of the matrix $\mathcal{G}$ are equal to

$$g'_{i,j} = \begin{cases} \dfrac{\left(F_{r+n}-F_r\right)^{i-j}}{\left(F_{r+1}-F_{r+n+1}\right)^{i-j+1}}, & i \geq j, \\ 0, & i < j. \end{cases} \tag{19}$$

In particular, when $r = 0$, one gets Lemma 2.1 in [26].

Proof. Let $c_{i,j} = \sum_{k=1}^{n-2} g_{i,k}g'_{k,j}$. Obviously, $c_{i,j} = 0$ for $i < j$. In the case $i = j$, we obtain

$$c_{i,i} = g_{i,i}g'_{i,i} = \left(F_{r+1} - F_{r+n+1}\right)\cdot\frac{1}{F_{r+1}-F_{r+n+1}} = 1. \tag{20}$$

For $i \geq j+1$, we obtain

$$\begin{aligned} c_{i,j} &= \sum_{k=1}^{n-2} g_{i,k}g'_{k,j} = g_{i,i-1}g'_{i-1,j} + g_{i,i}g'_{i,j} \\ &= \left(F_r - F_{r+n}\right)\cdot\frac{\left(F_{r+n}-F_r\right)^{i-j-1}}{\left(F_{r+1}-F_{r+n+1}\right)^{i-j}} \\ &\quad + \left(F_{r+1}-F_{r+n+1}\right)\cdot\frac{\left(F_{r+n}-F_r\right)^{i-j}}{\left(F_{r+1}-F_{r+n+1}\right)^{i-j+1}} \\ &= 0. \end{aligned} \tag{21}$$

Hence, we verify $\mathcal{G}\mathcal{G}^{-1} = I_{n-2}$, where I_{n-2} is $(n-2)\times(n-2)$ identity matrix. Similarly, we can verify $\mathcal{G}^{-1}\mathcal{G} = I_{n-2}$. Thus, the proof is completed. □

Theorem 11. *Let $A_{r,n} = \mathrm{Circ}(F_{r+1}, F_{r+2}, \ldots, F_{r+n})$ $(n > 2)$ be a circulant matrix.*

Then one has

$$\begin{aligned} A_{r,n}^{-1} &= \frac{1}{f_{r,n}} \\ &\times \mathrm{Circ}\Bigg(1 + \sum_{i=1}^{n-2}\Bigg(\left(F_{r+n+2-i} - \frac{F_{r+2}}{F_{r+1}}F_{r+n+1-i}\right) \\ &\qquad \times \left(F_{r+n}-F_r\right)^{i-1}\Bigg) \\ &\qquad \times \left(\left(F_{r+1}-F_{r+n+1}\right)^i\right)^{-1}, -\frac{F_{r+2}}{F_{r+1}} \\ &\quad + \sum_{i=1}^{n-2}\Bigg(\left(F_{r+n+1-i} - \left(F_{r+2}/F_{r+1}\right)F_{r+n-i}\right) \\ &\qquad \times \left(F_{r+n}-F_r\right)^{i-1}\Bigg) \\ &\qquad \times \left(\left(F_{r+1}-F_{r+n+1}\right)^i\right), \\ &\quad - \frac{F_{r+3} - \left(F_{r+2}/F_{r+1}\right)F_{r+2}}{F_{r+1}-F_{r+n+1}}, \\ &\quad - \frac{\left(F_{r+3} - \left(F_{r+2}/F_{r+1}\right)F_{r+2}\right)\left(F_{r+n}-F_r\right)}{\left(F_{r+1}-F_{r+n+1}\right)^2}, \\ &\quad - \frac{\left(F_{r+3} - \left(F_{r+2}/F_{r+1}\right)F_{r+2}\right)\left(F_{r+n}-F_r\right)^2}{\left(F_{r+1}-F_{r+n+1}\right)^3}, \ldots, \\ &\quad - \frac{\left(F_{r+3} - \left(F_{r+2}/F_{r+1}\right)F_{r+2}\right)\left(F_{r+n}-F_r\right)^{n-3}}{\left(F_{r+1}-F_{r+n+1}\right)^{n-2}}\Bigg), \end{aligned} \tag{22}$$

where

$$f_{r,n} = \left(F_{r+1} - \frac{F_{r+2}}{F_{r+1}}F_{r+n}\right) + \sum_{k=1}^{n-2}\left(F_{r+k+2} - \frac{F_{r+2}}{F_{r+1}}F_{r+k+1}\right)\left(\frac{F_{r+n}-F_r}{F_{r+1}-F_{r+n+1}}\right)^{n-(k+1)}. \tag{23}$$

Specially, when $r = 0$, this result is the same as Theorem 2.3 in [26].

Proof. Let

$$\Pi_2 = \begin{pmatrix} 1 & -\frac{f'_{r,n}}{F_{r+1}} & x_3 & x_4 & \cdots & x_n \\ 0 & 1 & y_3 & y_4 & \cdots & y_n \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 \end{pmatrix}, \tag{24}$$

where $a_r = F_{r+2}/F_{r+1}$,

$$x_i = \frac{f'_{r,n}}{f_{r,n}}\frac{F_{r+n+3-i} - (F_{r+2}/F_{r+1})F_{r+n+2-i}}{F_{r+1}} - \frac{F_{r+n+2-i}}{F_{r+1}} \quad (i = 3, 4, \ldots, n),$$

$$y_i = -\frac{F_{r+n+3-i} - (F_{r+2}/F_{r+1})F_{r+n+2-i}}{f_{r,n}} \quad (i = 3, 4, \ldots, n),$$

$$f'_{r,n} = \sum_{k=1}^{n-1} F_{r+k+1}\left(\frac{F_{r+n}-F_r}{F_{r+1}-F_{r+n+1}}\right)^{n-(k+1)},$$

$$f_{r,n} = \left(F_{r+1} - \frac{F_{r+2}}{F_{r+1}}F_{r+n}\right) + \sum_{k=1}^{n-2}\left(F_{r+k+2} - \frac{F_{r+2}}{F_{r+1}}F_{r+k+1}\right)\left(\frac{F_{r+n}-F_r}{F_{r+1}-F_{r+n+1}}\right)^{n-(k+1)}. \tag{25}$$

We have

$$\Gamma A_{r,n}\Pi_1\Pi_2 = \mathscr{D}_1 \oplus \mathscr{G}, \tag{26}$$

where $\mathscr{D}_1 = \operatorname{diag}(F_{r+1}, f_{r,n})$ is a diagonal matrix and $\mathscr{D}_1 \oplus \mathscr{G}$ is the direct sum of $\mathscr{D}_1$ and $\mathscr{G}$. If we denote $\Pi = \Pi_1\Pi_2$, then we obtain

$$A_{r,n}^{-1} = \Pi\left(\mathscr{D}_1^{-1} \oplus \mathscr{G}^{-1}\right)\Gamma, \tag{27}$$

and the last row elements of the matrix Π are $0, 1, y_3, y_4, \ldots, y_{n-1}, y_n$. By Lemma 10, if let $A_{r,n}^{-1} = \operatorname{Circ}(u_1, u_2, \ldots, u_n)$, then its last row elements are given by the following equations:

$$\begin{aligned} u_2 &= -\frac{1}{f_{r,n}}\frac{F_{r+2}}{F_{r+1}} + \frac{1}{f_{r,n}}C_n^{(n-2)}, \\ u_3 &= -\frac{1}{f_{r,n}}C_n^{(1)}, \\ u_4 &= -\frac{1}{f_{r,n}}C_n^{(2)} + \frac{1}{f_{r,n}}C_n^{(1)}, \\ u_5 &= -\frac{1}{f_{r,n}}C_n^{(3)} + \frac{1}{f_{r,n}}C_n^{(2)} + \frac{1}{f_{r,n}}C_n^{(1)}, \\ &\vdots \\ u_n &= -\frac{1}{f_{r,n}}C_n^{(n-2)} + \frac{1}{f_{r,n}}C_n^{(n-3)} + \frac{1}{f_n}C_n^{(n-4)}, \\ u_1 &= \frac{1}{f_{r,n}} + \frac{1}{f_{r,n}}C_n^{(n-2)} + \frac{1}{f_{r,n}}C_n^{(n-3)}. \end{aligned} \tag{28}$$

Let

$$\begin{aligned} C_n^{(j)} &= \sum_{i=1}^{j}\frac{\left(F_{r+3+j-i} - (F_{r+2}/F_{r+1})F_{r+2+j-i}\right)(F_{r+n}-F_r)^{i-1}}{(F_{r+1}-F_{r+n+1})^i} \\ &= \sum_{i=1}^{j}\frac{a'_{j,r}}{(m_{1,r})^i}(m_{2,r})^{i-1} \quad (j = 1, 2, \ldots, n-2); \end{aligned} \tag{29}$$

we have

$$\begin{aligned} C_n^{(2)} - C_n^{(1)} &= \sum_{i=1}^{2}\frac{a'_{2,r}(m_{2,r})^{i-1}}{(m_{1,r})^i} - \frac{a'_{1,r}}{m_{1,r}} \\ &= \frac{a'_{1,r}}{(m_{1,r})^2}m_{2,r}, \\ C_n^{(n-2)} + C_n^{(n-3)} &= \sum_{i=1}^{n-2}\frac{a'_{n-2,r}(m_{2,r})^{i-1}}{(m_{1,r})^i} + \sum_{i=1}^{n-3}\frac{a'_{n-3,r}(m_{2,r})^{i-1}}{(m_{1,r})^i} \\ &= \sum_{i=1}^{n-3}\frac{(F_{r+n+2-i} - A_{r,n}F_{r+n+1-i})(m_{2,r})^{i-1}}{(m_{1,r})^i} \\ &\quad + \frac{a'_{1,r}}{(m_{1,r})^{n-2}}(m_{2,r})^{n-3} \\ &= \sum_{i=1}^{n-2}\frac{\left(F_{r+n+2-i} - (F_{r+2}/F_{r+1})F_{r+n+1-i}\right)(m_{2,r})^{i-1}}{(m_{1,r})^i}, \end{aligned}$$

$$\begin{aligned}
&C_n^{(j+2)} - C_n^{(j+1)} - C_n^{(j)} \\
&\quad = \sum_{i=1}^{j+2} \frac{a'_{j+2,r}(m_{2,r})^{i-1}}{(m_{1,r})^i} - \sum_{i=1}^{j+1} \frac{a'_{j+1,r}(m_{2,r})^{i-1}}{(m_{1,r})^i} \\
&\qquad - \sum_{i=1}^{j} \frac{a'_{j,r}(m_{2,r})^{i-1}}{(m_{1,r})^i} \\
&\quad = \frac{(F_{r+4} - A_{r,n}F_{r+3})(m_{2,r})^j}{(m_{1,r})^{j+1}} \\
&\qquad + \frac{(F_{r+3} - A_{r,n}F_{r+2})(m_{2,r})^{j+1}}{(m_{1,r})^{j+2}} \\
&\qquad - \frac{(F_{r+3} - A_{r,n}F_{r+2})(m_{2,r})^j}{(m_{1,r})^{j+1}} \\
&\quad = \frac{(F_{r+3} - A_{r,n}F_{r+2})(m_{2,r})^{j+1}}{(m_{1,r})^{j+2}} \\
&\qquad (j = 1, 2, \ldots, n-4).
\end{aligned} \tag{30}$$

We obtain

$$\begin{aligned}
&A_{r,n}^{-1} \\
&= \operatorname{Circ}\left(\frac{1 + C_n^{(n-2)} + C_n^{(n-3)}}{f_{r,n}}, \frac{C_n^{(n-2)} - F_{r+2}/F_{r+1}}{f_{r,n}},\right. \\
&\qquad -\frac{C_n^{(1)}}{f_{r,n}}, -\frac{C_n^{(2)} - C_n^{(1)}}{f_{r,n}}, \\
&\qquad -\frac{C_n^{(3)} - C_n^{(2)} - C_n^{(1)}}{f_{r,n}}, \ldots, \\
&\qquad \left. -\frac{C_n^{(n-2)} - C_n^{(n-3)} - C_n^{(n-4)}}{f_{r,n}}\right) \\
&= \frac{1}{f_{r,n}} \operatorname{Circ}\left(1 + \sum_{i=1}^{n-2}\left(\left(F_{r+n+2-i} - \frac{F_{r+2}}{F_{r+1}}F_{r+n+1-i}\right)\right.\right. \\
&\qquad \left.\times (m_{2,r})^{i-1}\right) \times \left((m_{1,r})^i\right)^{-1}, \\
&\qquad -\frac{F_{r+2}}{F_{r+1}} + \sum_{i=1}^{n-2} \frac{a'_{n-2,r}(m_{2,r})^{i-1}}{(m_{1,r})^i}, \\
&\qquad -\frac{a'_{1,r}}{m_{1,r}}, -\frac{a'_{1,r}}{(m_{1,r})^2} m_{2,r}, \\
&\qquad -\frac{a'_{1,r}}{(m_{1,r})^3}(m_{2,r})^2, \ldots, \\
&\qquad \left. -\frac{a'_{1,r}}{(m_{1,r})^{n-2}}(m_{2,r})^{n-3}\right) \\
&= \frac{1}{f_{r,n}} \operatorname{Circ}\left(1 + \sum_{i=1}^{n-2}\left(\left(F_{r+n+2-i} - \frac{F_{r+2}}{F_{r+1}}F_{r+n+1-i}\right)\right.\right. \\
&\qquad \left.\times (F_{r+n} - F_r)^{i-1}\right) \\
&\qquad \times \left((F_{r+1} - F_{r+n+1})^i\right)^{-1}, -\frac{F_{r+2}}{F_{r+1}} \\
&\qquad + \sum_{i=1}^{n-2}\left(\left(F_{r+n+1-i} - \frac{F_{r+2}}{F_{r+1}}F_{r+n-i}\right)\right. \\
&\qquad \left.\times (F_{r+n} - F_r)^{i-1}\right) \\
&\qquad \times \left((F_{r+1} - F_{r+n+1})^i\right)^{-1} \\
&\qquad - \frac{F_{r+3} - (F_{r+2}/F_{r+1})F_{r+2}}{F_{r+1} - F_{r+n+1}}, \\
&\qquad - \frac{(F_{r+3} - (F_{r+2}/F_{r+1})F_{r+2})(F_{r+n} - F_r)}{(F_{r+1} - F_{r+n+1})^2}, \\
&\qquad - \frac{(F_{r+3} - (F_{r+2}/F_{r+1})F_{r+2})(F_{r+n} - F_r)^2}{(F_{r+1} - F_{r+n+1})^3}, \ldots, \\
&\qquad \left. - \frac{(F_{r+3} - (F_{r+2}/F_{r+1})F_{r+2})(F_{r+n} - F_r)^{n-3}}{(F_{r+1} - F_{r+n+1})^{n-2}}\right),
\end{aligned} \tag{31}$$

where

$$\begin{aligned}
f_{r,n} &= \left(F_{r+1} - \frac{F_{r+2}}{F_{r+1}}F_{r+n}\right) \\
&\quad + \sum_{k=1}^{n-2}\left(F_{r+k+2} - \frac{F_{r+2}}{F_{r+1}}F_{r+k+1}\right)\left(\frac{F_{r+n} - F_r}{F_{r+1} - F_{r+n+1}}\right)^{n-(k+1)}.
\end{aligned} \tag{32}$$

□

3. Determinant, Invertibility, and Inverse of Circulant Matrix with Any Continuous Lucas Numbers

In this section, let $B_{r,n} = \operatorname{Circ}(L_{r+1}, L_{r+2}, \ldots, L_{r+n})$ be a circulant matrix. Firstly, we give a determinant formula for the matrix $B_{r,n}$. Afterwards, we prove that $B_{r,n}$ is an invertible matrix for any positive integer n, and then we find the inverse of the matrix $B_{r,n}$.

Theorem 12. *Let $B_{r,n} = \operatorname{Circ}(L_{r+1}, L_{r+2}, \ldots, L_{r+n})$ be a circulant matrix; then one has*

$$\begin{aligned}
&\det B_{r,n} \\
&= L_{r+1} \cdot \left\{ \left(L_{r+1} - \frac{L_{r+2}}{L_{r+1}} L_{r+n} \right) \right. \\
&\quad + \sum_{k=1}^{n-2} \left[\left(L_{r+k+2} - \frac{L_{r+2}}{L_{r+1}} L_{r+k+1} \right) \right. \\
&\quad \left. \left. \times \left(\frac{L_{r+n} - L_r}{L_{r+1} - L_{r+n+1}} \right)^{n-(k+1)} \right] \right\} \\
&\times (L_{r+1} - L_{r+n+1})^{n-2},
\end{aligned} \tag{33}$$

where L_{r+n} is the $(r+n)$th Lucas number. In particular, when $r = 0$, one gets Theorem 3.1 in [26].

Proof. Obviously, $\det B_1 = (1 - L_{n+1})^{n-1} + (L_n - 2)^{n-2} \sum_{k=1}^{n-1} (L_{k+2} - 3L_{k+1})((1 - L_{n+1})/(L_n - 2))^{k-1}$ satisfies the formula, when $n > 1$; let

$$\Sigma = \begin{pmatrix}
1 & & & & & & \\
-\frac{L_{r+2}}{L_{r+1}} & & & & & & 1 \\
-1 & & & & & 1 & -1 \\
0 & & 0 & & 1 & -1 & -1 \\
\vdots & & & \iddots & \iddots & \iddots & \\
0 & & 1 & \iddots & \iddots & & \\
0 & 1 & -1 & \iddots & & 0 & \\
0 & 1 & -1 & -1 & & & \\
\end{pmatrix},$$

$$\Omega_1 = \begin{pmatrix}
1 & 0 & 0 & \cdots & 0 & 0 \\
0 & \left(\frac{L_{r+n} - L_r}{L_{r+1} - L_{r+n+1}} \right)^{n-2} & 0 & \cdots & 0 & 1 \\
0 & \left(\frac{L_{r+n} - L_r}{L_{r+1} - L_{r+n+1}} \right)^{n-3} & 0 & \cdots & 1 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & \frac{L_{r+n} - L_r}{L_{r+1} - L_{r+n+1}} & 1 & \cdots & 0 & 0 \\
0 & 1 & 0 & \cdots & 0 & 0
\end{pmatrix}, \tag{34}$$

be two $n \times n$ matrices, we have

$$\begin{aligned}
&\Sigma B_{r,n} \Omega_1 \\
&= \begin{bmatrix}
L_{r+1} & l'_{r,n} & L_{r+n-1} & \cdots & L_{r+2} \\
0 & l_{r,n} & L_{r+n} - b_r L_{r+n-1} & \cdots & L_{r+3} - b_r L_{r+2} \\
0 & 0 & L_{r+1} - L_{r+n-1} & & 0 \\
0 & 0 & L_r - L_{r+n} & & 0 \\
\vdots & \vdots & & \ddots & \\
0 & 0 & 0 & & L_{r+1} - L_{r+n-1}
\end{bmatrix},
\end{aligned} \tag{35}$$

where

$$\begin{aligned}
b_r &= \frac{L_{r+2}}{L_{r+1}}, \\
l_{r,n} &= \left(L_{r+1} - \frac{L_{r+2}}{L_{r+1}} L_{r+n} \right) \\
&\quad + \sum_{k=1}^{n-2} \left(L_{r+k+2} - \frac{L_{r+2}}{L_{r+1}} L_{r+k+1} \right) \\
&\quad \times \left(\frac{L_{r+n} - L_r}{L_{r+1} - L_{r+n+1}} \right)^{n-(k+1)}, \\
l'_{r,n} &= \sum_{k=1}^{n-1} L_{r+k+1} \left(\frac{L_{r+n} - L_r}{L_{r+1} - L_{r+n+1}} \right)^{n-(k+1)}.
\end{aligned} \tag{36}$$

We obtain

$$\begin{aligned}
&\det \Sigma \det B_{r,n} \det \Omega_1 \\
&= L_{r+1} \cdot \left\{ \left(L_{r+1} - \frac{L_{r+2}}{L_{r+1}} L_{r+n} \right) \right. \\
&\quad + \sum_{k=1}^{n-2} \left[(L_{r+k+2} - b_r L_{r+k+1}) \right. \\
&\quad \left. \left. \times \left(\frac{L_{r+n} - L_r}{L_{r+1} - L_{r+n+1}} \right)^{n-(k+1)} \right] \right\} \\
&\times (L_{r+1} - L_{r+n+1})^{n-2},
\end{aligned} \tag{37}$$

while

$$\det \Sigma = \det \Omega_1 = (-1)^{(n-1)(n-2)/2}. \tag{38}$$

We have

$$\begin{aligned}
&\det B_{r,n} \\
&= L_{r+1} \cdot \left\{ \left(L_{r+1} - \frac{L_{r+2}}{L_{r+1}} L_{r+n} \right) \right. \\
&\quad + \sum_{k=1}^{n-2} \left[\left(L_{r+k+2} - \frac{L_{r+2}}{L_{r+1}} L_{r+k+1} \right) \right. \\
&\quad \left. \left. \times \left(\frac{L_{r+n} - L_r}{L_{r+1} - L_{r+n+1}} \right)^{n-(k+1)} \right] \right\} \\
&\times (L_{r+1} - L_{r+n+1})^{n-2}.
\end{aligned} \tag{39}$$

□

Theorem 13. *Let $B_{r,n} = \operatorname{Circ}(L_{r+1}, L_{r+2}, \ldots, L_{r+n})$ be a circulant matrix; then $B_{r,n}$ is invertible for any positive integer n. Specially, when $r = 0$, one gets Theorem 3.2 in [26].*

Proof. Since $L_{n+r} = \alpha^{n+r} + \beta^{n+r}$, where $\alpha + \beta = 1$, $\alpha \cdot \beta = -1$. Hence we have

$$\begin{aligned} f\left(\omega^k\right) &= \sum_{j=1}^{n} L_{n+r}\left(\omega^k\right)^{j-1} \\ &= \sum_{j=1}^{n}\left(\alpha^{r+j} + \beta^{r+j}\right)\left(\omega^k\right)^{j-1} \\ &= \frac{\alpha^{r+1}\left(1-\alpha^n\right)}{1-\alpha\omega^k} + \frac{\beta^{r+1}\left(1-\beta^n\right)}{1-\beta\omega^k} \\ &\quad \left(\text{because } 1-\alpha\omega^k \neq 0 \text{ and } 1-\beta\omega^k \neq 0\right) \\ &= \frac{\left(\alpha^{r+1}+\beta^{r+1}\right) - \left(\alpha^{r+n+1}+\beta^{r+n+1}\right)}{1-(\alpha+\beta)\omega^k + \alpha\beta\omega^{2k}} \\ &\quad - \frac{\alpha\beta\left(\alpha^r+\beta^r-\alpha^{r+n}-\beta^{r+n}\right)\omega^k}{1-(\alpha+\beta)\omega^k+\alpha\beta\omega^{2k}} \\ &= \frac{L_{r+1}-L_{r+n+1}+\left(L_r-L_{r+n}\right)\omega^k}{1-\omega^k-\omega^{2k}} \\ &\qquad (k = 1, 2, \ldots, n-1). \end{aligned} \tag{40}$$

If there exists ω^l $(l = 1, 2, \ldots, n-1)$ such that $f(\omega^l) = 0$, we obtain $L_{r+1} - L_{r+n+1} + (L_r - L_{r+n})\omega^l = 0$ for $1 - \omega^l - \omega^{2l} \neq 0$; thus, $\omega^l = (L_{r+1} - L_{r+n+1})/(L_{r+n} - L_r)$ is a real number, while

$$\omega^l = \exp\left(\frac{2l\pi i}{n}\right) = \cos\frac{2l\pi}{n} + i\sin\frac{2l\pi}{n}. \tag{41}$$

Hence, $\sin(2l\pi/n) = 0$; we have $\omega^l = -1$ for $0 < 2l\pi/n < 2\pi$. But $x = -1$ is not the root of the equation $L_{r+1} - L_{r+n+1} + (L_r - L_{r+n})x = 0$ for any positive integer n. We obtain $f(\omega^k) \neq 0$ for any ω^k $(k = 1, 2, \ldots, n-1)$, while $f(1) = \sum_{j=1}^{n} L_{r+j} = L_{r+n+2} - L_{r+2} \neq 0$. By Lemma 4, the proof is completed. □

Lemma 14. *Let the entries of the matrix $\mathscr{H} = [h_{i,j}]_{i,j=1}^{n-2}$ be of the form*

$$h_{i,j} = \begin{cases} L_{r+1} - L_{r+n+1}, & i = j, \\ L_r - L_{r+n}, & i = j+1, \\ 0, & \text{otherwise}; \end{cases} \tag{42}$$

then the entries of the inverse $\mathscr{H}^{-1} = [h'_{i,j}]_{i,j=1}^{n-2}$ of the matrix $\mathscr{H}$ are equal to

$$h'_{i,j} = \begin{cases} \dfrac{\left(L_{r+n}-L_r\right)^{i-j}}{\left(L_{r+1}-L_{r+n+1}\right)^{i-j+1}}, & i \geq j, \\ 0, & i < j. \end{cases} \tag{43}$$

Specially, when $r = 0$, one gets Lemma 3.1 in [26].

Proof. Let $r_{i,j} = \sum_{k=1}^{n-2} h_{i,k}h'_{k,j}$. Obviously, $r_{i,j} = 0$ for $i < j$. In the case $i = j$, we obtain

$$r_{i,i} = h_{i,i}h'_{i,i} = \left(L_{r+1} - L_{r+n+1}\right) \cdot \frac{1}{L_{r+1} - L_{r+n+1}} = 1. \tag{44}$$

For $i \geq j+1$, we obtain

$$\begin{aligned} r_{i,j} &= \sum_{k=1}^{n-2} h_{i,k}h'_{k,j} = h_{i,i-1}h'_{i-1,j} + h_{i,i}h'_{i,j} \\ &= \left(L_r - L_{r+n}\right) \cdot \frac{\left(L_{r+n}-L_r\right)^{i-j-1}}{\left(L_{r+1}-L_{r+n+1}\right)^{i-j}} \\ &\quad + \left(L_{r+1} - L_{r+n+1}\right) \cdot \frac{\left(L_{r+n}-L_r\right)^{i-j}}{\left(L_{r+1}-L_{r+n+1}\right)^{i-j+1}} \\ &= 0. \end{aligned} \tag{45}$$

Hence, we verify $\mathscr{H}\mathscr{H}^{-1} = I_{n-2}$, where I_{n-2} is $(n-2)\times(n-2)$ identity matrix. Similarly, we can verify $\mathscr{H}^{-1}\mathscr{H} = I_{n-2}$. Thus, the proof is completed. □

Theorem 15. *Let $B_{r,n} = \operatorname{Circ}(L_{r+1}, L_{r+2}, \ldots, L_{r+n})$ be a circulant matrix; then we have*

$$\begin{aligned} &B_{r,n}^{-1} \\ &= \frac{1}{l_{r,n}} \operatorname{Circ}\Bigg(1 \\ &\quad + \sum_{i=1}^{n-2}\left(\left(L_{r+n+2-i} - \frac{L_{r+2}}{L_{r+1}}L_{r+n+1-i}\right) \times \left(L_{r+n}-L_r\right)^{i-1}\right) \times \left(\left(L_{r+1}-L_{r+n+1}\right)^i\right)^{-1}, -\frac{L_{r+2}}{L_{r+1}} \\ &\quad + \sum_{i=1}^{n-2}\left(\left(L_{r+n+1-i} - \frac{L_{r+2}}{L_{r+1}}L_{r+n-i}\right) \times \left(L_{r+n}-L_r\right)^{i-1}\right) \times \left(\left(L_{r+1}-L_{r+n+1}\right)^i\right)^{-1}, \\ &\quad - \frac{L_{r+3} - \left(L_{r+2}/L_{r+1}\right)L_{r+2}}{L_{r+1}-L_{r+n+1}}, \\ &\quad - \frac{\left(L_{r+3} - \left(L_{r+2}/L_{r+1}\right)L_{r+2}\right)\left(L_{r+n}-L_r\right)}{\left(L_{r+1}-L_{r+n+1}\right)^2}, \\ &\quad - \frac{\left(L_{r+3} - \left(L_{r+2}/L_{r+1}\right)L_{r+2}\right)\left(L_{r+n}-L_r\right)^2}{\left(L_{r+1}-L_{r+n+1}\right)^3}, \ldots, \\ &\quad - \frac{\left(L_{r+3} - \left(L_{r+2}/L_{r+1}\right)L_{r+2}\right)\left(L_{r+n}-L_r\right)^{n-3}}{\left(L_{r+1}-L_{r+n+1}\right)^{n-2}}\Bigg), \end{aligned} \tag{46}$$

where

$$l_{r,n} = \left(L_{r+1} - \frac{L_{r+2}}{L_{r+1}} L_{r+n}\right) + \sum_{k=1}^{n-2} \left(L_{r+k+2} - \frac{L_{r+2}}{L_{r+1}} L_{r+k+1}\right) \left(\frac{L_{r+n} - L_r}{L_{r+1} - L_{r+n+1}}\right)^{n-(k+1)}. \tag{47}$$

In particular, when $r = 0$, the result is the same as Theorem 3.3 in [26].

Proof. Let Ω_2 be the form of

$$\begin{pmatrix} 1 & -\frac{l_{n\prime}}{L_{r+1}} & x_3' & x_4' & \cdots & x_n' \\ 0 & 1 & y_3' & y_4' & \cdots & y_n' \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 \end{pmatrix}, \tag{48}$$

where

$$b_r = \frac{L_{r+2}}{L_{r+1}},$$

$$x_i' = \frac{l_{r,n}'}{l_{r,n}} \frac{L_{r+n+3-i} - (L_{r+2}/L_{r+1}) L_{r+n+2-i}}{L_{r+1}} - \frac{L_{r+n+2-i}}{L_{r+1}} \quad (i = 3, 4, \ldots, n),$$

$$y_i' = -\frac{L_{r+n+3-i} - (L_{r+2}/L_{r+1}) L_{r+n+2-i}}{l_{r,n}} \quad (i = 3, 4, \ldots, n),$$

$$l_{r,n}' = \sum_{k=1}^{n-1} L_{r+k+1} \left(\frac{L_{r+n} - L_r}{L_{r+1} - L_{r+n+1}}\right)^{n-(k+1)},$$

$$l_{r,n} = \left(L_{r+1} - \frac{L_{r+2}}{L_{r+1}} L_{r+n}\right) + \sum_{k=1}^{n-2} \left(L_{r+k+2} - \frac{L_{r+2}}{L_{r+1}} L_{r+k+1}\right) \left(\frac{L_{r+n} - L_r}{L_{r+1} - L_{r+n+1}}\right)^{n-(k+1)}. \tag{49}$$

We have

$$\Sigma B_{r,n} \Omega_1 \Omega_2 = \mathscr{D}_2 \oplus \mathscr{H}, \tag{50}$$

where $\mathscr{D}_2 = \operatorname{diag}(L_{r+1}, l_{r,n})$ is a diagonal matrix and $\mathscr{D}_2 \oplus \mathscr{H}$ is the direct sum of $\mathscr{D}_2$ and $\mathscr{H}$. If we denote $\Omega = \Omega_1 \Omega_2$, we obtain

$$B_{r,n}^{-1} = \Omega \left(\mathscr{D}_2^{-1} \oplus \mathscr{H}^{-1}\right) \Sigma, \tag{51}$$

and the last row elements of the matrix Ω are $0, 1, y_3', y_4', \cdots, y_n'$. By Lemma 14, if let $B_{r,n}^{-1} = \operatorname{Circ}(v_1, v_2, \ldots, v_n)$, then its last row elements are given by the following equations:

$$\begin{aligned} v_2 &= -\frac{1}{l_{r,n}} \frac{L_{r+2}}{L_{r+1}} + \frac{1}{l_{r,n}} D_n^{(n-2)}, \\ v_3 &= -\frac{1}{l_{r,n}} D_n^{(1)}, \\ v_4 &= -\frac{1}{l_{r,n}} D_n^{(2)} + \frac{1}{l_{r,n}} C_n^{(1)}, \\ v_5 &= -\frac{1}{l_{r,n}} D_n^{(3)} + \frac{1}{l_{r,n}} D_n^{(2)} + \frac{1}{l_{r,n}} D_n^{(1)}, \\ &\vdots \\ v_n &= -\frac{1}{l_{r,n}} D_n^{(n-2)} + \frac{1}{l_{r,n}} D_n^{(n-3)} + \frac{1}{l_n} D_n^{(n-4)}, \\ v_1 &= \frac{1}{l_{r,n}} + \frac{1}{l_{r,n}} D_n^{(n-2)} + \frac{1}{l_{r,n}} D_n^{(n-3)}. \end{aligned} \tag{52}$$

Let

$$\begin{aligned} D_n^{(j)} &= \sum_{i=1}^{j} \left(\left(L_{r+3+j-i} - \frac{L_{r+2}}{L_{r+1}} L_{r+2+j-i}\right) \times \left(L_{r+n} - L_r\right)^{i-1}\right) \times \left(\left(L_{r+1} - L_{r+n+1}\right)^i\right)^{-1} \\ &= \sum_{i=1}^{j} \frac{b_{j,r}'}{(h_{1,r})^i} (h_{2,r})^{i-1} \quad (j = 1, 2, \ldots, n-2); \end{aligned} \tag{53}$$

we have

$$\begin{aligned} D_n^{(2)} - D_n^{(1)} &= \sum_{i=1}^{2} \frac{b_{2,r}' (h_{2,r})^{i-1}}{(h_{1,r})^i} - \frac{b_{1,r}'}{h_{1,r}} = \frac{b_{1,r}'}{(h_{1,r})^2} h_{2,r}, \\ D_n^{(n-2)} + D_n^{(n-3)} &= \sum_{i=1}^{n-2} \frac{b_{n-2,r}' (h_{2,r})^{i-1}}{(h_{1,r})^i} + \sum_{i=1}^{n-3} \frac{\left(b_{n-3,r}'\right) (h_{2,r})^{i-1}}{(h_{1,r})^i} \\ &= \sum_{i=1}^{n-3} \frac{\left(L_{r+n+2-i} - B_{r,n} L_{r+n+1-i}\right) (h_{2,r})^{i-1}}{(h_{1,r})^i} + \frac{b_{1,r}'}{(h_{1,r})^{n-2}} (h_{2,r})^{n-3} \\ &= \sum_{i=1}^{n-2} \frac{\left(L_{r+n+2-i} - B_{r,n} L_{r+n+1-i}\right) (h_{2,r})^{i-1}}{(h_{1,r})^i}, \end{aligned}$$

$$\begin{aligned}
&D_n^{(j+2)} - D_n^{(j+1)} - D_n^{(j)} \\
&\quad = \sum_{i=1}^{j+2} \frac{\left(b'_{j+2,r}\right)\left(h_{2,r}\right)^{i-1}}{\left(h_{1,r}\right)^{i}} - \sum_{i=1}^{j+1} \frac{\left(b'_{j+1,r}\right)\left(h_{2,r}\right)^{i-1}}{\left(h_{1,r}\right)^{i}} \\
&\qquad - \sum_{i=1}^{j} \frac{\left(b'_{j,r}\right)\left(h_{2,r}\right)^{i-1}}{\left(h_{1,r}\right)^{i}} \\
&\quad = \frac{\left(L_{r+4} - B_{r,n}L_{r+3}\right)\left(h_{2,r}\right)^{j}}{\left(h_{1,r}\right)^{j+1}} \\
&\qquad + \frac{\left(L_{r+3} - B_{r,n}L_{r+2}\right)\left(h_{2,r}\right)^{j+1}}{\left(h_{1,r}\right)^{j+2}} \\
&\qquad - \frac{\left(L_{r+3} - B_{r,n}L_{r+2}\right)\left(h_{2,r}\right)^{j}}{\left(h_{1,r}\right)^{j+1}} \\
&\quad = \frac{\left(L_{r+3} - B_{r,n}L_{r+2}\right)\left(h_{2,r}\right)^{j+1}}{\left(h_{1,r}\right)^{j+2}} \\
&\quad = \frac{b'_{1,r}\left(h_{2,r}\right)^{j+1}}{\left(h_{1,r}\right)^{j+2}} \qquad \left(j = 1, 2, \ldots, n-4\right).
\end{aligned} \tag{54}$$

We obtain

$$\begin{aligned}
&B_{r,n}^{-1} \\
&= \operatorname{Circ}\left(\frac{1 + D_n^{(n-3)} + D_n^{(n-2)}}{l_{r,n}}, \frac{D_n^{(n-2)} - L_{r+2}/L_{r+1}}{l_{r,n}},\right. \\
&\qquad -\frac{D_n^{(1)}}{l_{r,n}}, \frac{D_n^{(1)} - D_n^{(2)}}{l_{r,n}}, \frac{D_n^{(1)} + D_n^{(2)} - D_n^{(3)}}{l_{r,n}}, \ldots, \\
&\qquad \left.\frac{D_n^{(n-4)} + D_n^{(n-3)} - D_n^{(n-2)}}{l_{r,n}}\right) \\
&= \frac{1}{l_{r,n}} \operatorname{Circ}\left(1\right. \\
&\qquad + \sum_{i=1}^{n-2} \frac{\left(L_{r+n+2-i} - B_{r,n}L_{r+n+1-i}\right)\left(h_{2,r}\right)^{i-1}}{\left(h_{1,r}\right)^{i}}, \\
&\qquad - \frac{L_{r+2}}{L_{r+1}} + \sum_{i=1}^{n-2} \frac{b'_{n-2,r}\left(h_{2,r}\right)^{i-1}}{\left(h_{1,r}\right)^{i}}, \\
&\qquad - \frac{b'_{1,r}}{h_{1,r}}, -\frac{b'_{1,r}}{\left(h_{1,r}\right)^{2}} h_{2,r}, \\
&\qquad \left.- \frac{b'_{1,r}}{\left(h_{1,r}\right)^{3}}\left(h_{2,r}\right)^{2}, \ldots, -\frac{b'_{1,r}}{\left(h_{1,r}\right)^{n-2}}\left(h_{2,r}\right)^{n-3}\right) \\
&= \frac{1}{l_{r,n}} \operatorname{Circ}\left(1\right. \\
&\qquad + \sum_{i=1}^{n-2}\left(\left(L_{r+n+2-i} - \frac{L_{r+2}}{L_{r+1}} L_{r+n+1-i}\right)\right. \\
&\qquad\qquad \left.\times \left(L_{r+n} - L_r\right)^{i-1}\right) \\
&\qquad\qquad \times \left(\left(L_{r+1} - L_{r+n+1}\right)^{i}\right)^{-1}, \\
&\qquad - \frac{L_{r+2}}{L_{r+1}} + \sum_{i=1}^{n-2}\left(\left(L_{r+n+1-i} - \frac{L_{r+2}}{L_{r+1}} L_{r+n-i}\right)\right. \\
&\qquad\qquad \left.\times \left(L_{r+n} - L_r\right)^{i-1}\right) \\
&\qquad\qquad \times \left(\left(L_{r+1} - L_{r+n+1}\right)^{i}\right)^{-1}, \\
&\qquad - \frac{L_{r+3} - \left(L_{r+2}/L_{r+1}\right) L_{r+2}}{L_{r+1} - L_{r+n+1}}, \\
&\qquad - \frac{\left(L_{r+3} - \left(L_{r+2}/L_{r+1}\right) L_{r+2}\right)\left(L_{r+n} - L_r\right)}{\left(L_{r+1} - L_{r+n+1}\right)^{2}}, \\
&\qquad - \frac{\left(L_{r+3} - \left(L_{r+2}/L_{r+1}\right) L_{r+2}\right)\left(L_{r+n} - L_r\right)^{2}}{\left(L_{r+1} - L_{r+n+1}\right)^{3}}, \ldots, \\
&\qquad \left.- \frac{\left(L_{r+3} - \left(L_{r+2}/L_{r+1}\right) L_{r+2}\right)\left(L_{r+n} - L_r\right)^{n-3}}{\left(L_{r+1} - L_{r+n+1}\right)^{n-2}}\right).
\end{aligned} \tag{55}$$

□

4. Determinant, Invertibility, and Inverse of Left Circulant Matrix with Any Continuous Fibonacci and Lucas Numbers

In this section, let $A'_{r,n} = \operatorname{LCirc}\left(F_{r+1}, F_{r+2}, \ldots, F_{r+n}\right)$ and $B'_{r,n} = \operatorname{LCirc}\left(L_{r+1}, L_{r+2}, \ldots, L_{r+n}\right)$ be left circulant matrices. By using the obtained conclusions, we give a determinant formula for the matrices $A'_{r,n}$ and $B'_{r,n}$. Afterwards, we prove that $A'_{r,n}$ is an invertible matrix for $n > 2$ and $B'_{r,n}$ is an invertible matrix for any positive integer n. The inverse of the matrices $A'_{r,n}$ and $B'_{r,n}$ is also presented.

According to Lemma 5, Theorem 8, Theorem 9, and Theorem 11, we can obtain the following theorems.

Theorem 16. *Let $A'_{r,n} = \operatorname{LCirc}\left(F_{r+1}, F_{r+2}, \ldots, F_{r+n}\right)$ be a left circulant matrix; then one has*

$$\begin{aligned}
&\det A'_{r,n} \\
&\quad = (-1)^{(n-1)(n-2)/2} \cdot F_{r+1} \\
&\qquad \cdot \left[\left(F_{r+1} - \frac{F_{r+2}}{F_{r+1}} F_{r+n}\right)\right. \\
&\qquad + \sum_{k=1}^{n-2}\left(F_{r+k+2} - \frac{F_{r+2}}{F_{r+1}} F_{r+k+1}\right)
\end{aligned}$$

$$\times \left(\frac{F_{r+n}-F_r}{F_{r+1}-F_{r+n+1}}\right)^{n-(k+1)}\Bigg] \times \left(F_{r+1}-F_{r+n+1}\right)^{n-2}, \tag{56}$$

where F_{r+n} is the $(r+n)$th Fibonacci number.

Theorem 17. *Let $A'_{r,n} = \mathrm{LCirc}\,(F_{r+1}, F_{r+2}, \ldots, F_{r+n})$ be a left circulant matrix; if $n > 2$, then $\mathscr{A}'_{r,n}$ is an invertible matrix.*

Theorem 18. *Let $A'_{r,n} = \mathrm{LCirc}\,(F_{r+1}, F_{r+2}, \ldots, F_{r+n})$ $(n > 2)$ be a left circulant matrix; then one has*

$$\begin{aligned}
{A'_{r,n}}^{-1} &= \frac{1}{f_{r,n}} \\
&\times \mathrm{LCirc}\Bigg(1 + \sum_{i=1}^{n-2}\Bigg(\left(F_{r+n+2-i} - \frac{F_{r+2}}{F_{r+1}}F_{r+n+1-i}\right) \times \left(F_{r+n}-F_r\right)^{i-1}\Bigg) \times \left(\left(F_{r+1}-F_{r+n+1}\right)^{i}\right)^{-1}, \\
&\quad - \frac{\left(F_{r+3} - \left(F_{r+2}/F_{r+1}\right)F_{r+2}\right)\left(F_{r+n}-F_r\right)^{n-3}}{\left(F_{r+1}-F_{r+n+1}\right)^{n-2}}, \ldots, \\
&\quad - \frac{\left(F_{r+3} - \left(F_{r+2}/F_{r+1}\right)F_{r+2}\right)\left(F_{r+n}-F_r\right)^{2}}{\left(F_{r+1}-F_{r+n+1}\right)^{3}}, \\
&\quad - \frac{\left(F_{r+3} - \left(F_{r+2}/F_{r+1}\right)F_{r+2}\right)\left(F_{r+n}-F_r\right)}{\left(F_{r+1}-F_{r+n+1}\right)^{2}}, \\
&\quad - \frac{F_{r+3} - \left(F_{r+2}/F_{r+1}\right)F_{r+2}}{F_{r+1}-F_{r+n+1}}, \\
&\quad - \frac{F_{r+2}}{F_{r+1}} \\
&\quad + \sum_{i=1}^{n-2}\Bigg(\left(F_{r+n+1-i} - \frac{F_{r+2}}{F_{r+1}}F_{r+n-i}\right) \times \left(F_{r+n}-F_r\right)^{i-1}\Bigg) \times \left(\left(F_{r+1}-F_{r+n+1}\right)^{i}\right)^{-1}\Bigg),
\end{aligned} \tag{57}$$

where

$$f_{r,n} = \left(F_{r+1} - \frac{F_{r+2}}{F_{r+1}}F_{r+n}\right) + \sum_{k=1}^{n-2}\left(F_{r+k+2} - \frac{F_{r+2}}{F_{r+1}}F_{r+k+1}\right)\left(\frac{F_{r+n}-F_r}{F_{r+1}-F_{r+n+1}}\right)^{n-(k+1)}. \tag{58}$$

By Lemma 5, Theorem 12, Theorem 13, and Theorem 15, the following conclusions can be attained.

Theorem 19. *Let $B'_{r,n} = \mathrm{LCirc}\,(L_{r+1}, L_{r+2}, \ldots, L_{r+n})$ be a left circulant matrix; then one has*

$$\begin{aligned}
\det B'_{r,n} &= (-1)^{(n-1)(n-2)/2} \cdot L_{r+1} \\
&\quad \cdot \Bigg[\left(L_{r+1} - \frac{L_{r+2}}{L_{r+1}}L_{r+n}\right) + \sum_{k=1}^{n-2}\left(L_{r+k+2} - \frac{L_{r+2}}{L_{r+1}}L_{r+k+1}\right) \times \left(\frac{L_{r+n}-L_r}{L_{r+1}-L_{r+n+1}}\right)^{n-(k+1)}\Bigg] \\
&\quad \times \left(L_{r+1}-L_{r+n+1}\right)^{n-2},
\end{aligned} \tag{59}$$

where L_{r+n} is the $(r+n)$th Lucas number.

Theorem 20. *Let $B'_{r,n} = \mathrm{LCirc}\,(L_{r+1}, L_{r+2}, \ldots, L_{r+n})$ be a left circulant matrix; then $B'_{r,n}$ is invertible for any positive integer n.*

Theorem 21. *Let $B'_{r,n} = \mathrm{LCirc}\,(L_{r+1}, L_{r+2}, \ldots, L_{r+n})$ be a left circulant matrix; then one can obtain*

$$\begin{aligned}
{B'_{r,n}}^{-1} &= \frac{1}{l_{r,n}} \\
&\times \mathrm{LCirc}\Bigg(1 + \sum_{i=1}^{n-2}\Bigg(\left(L_{r+n+2-i} - \frac{L_{r+2}}{L_{r+1}}L_{r+n+1-i}\right) \times \left(L_{r+n}-L_r\right)^{i-1}\Bigg) \times \left(\left(L_{r+1}-L_{r+n+1}\right)^{i}\right)^{-1}, \\
&\quad - \frac{\left(L_{r+3} - \left(L_{r+2}/L_{r+1}\right)L_{r+2}\right)\left(L_{r+n}-L_r\right)^{n-3}}{\left(L_{r+1}-L_{r+n+1}\right)^{n-2}}, \ldots, \\
&\quad - \frac{\left(L_{r+3} - \left(L_{r+2}/L_{r+1}\right)L_{r+2}\right)\left(L_{r+n}-L_r\right)^{2}}{\left(L_{r+1}-L_{r+n+1}\right)^{3}}, \\
&\quad - \frac{\left(L_{r+3} - \left(L_{r+2}/L_{r+1}\right)L_{r+2}\right)\left(L_{r+n}-L_r\right)}{\left(L_{r+1}-L_{r+n+1}\right)^{2}}, \\
&\quad - \frac{L_{r+3} - \left(L_{r+2}/L_{r+1}\right)L_{r+2}}{L_{r+1}-L_{r+n+1}},
\end{aligned}$$

$$
\begin{aligned}
&- \frac{L_{r+2}}{L_{r+1}} \\
&+ \sum_{i=1}^{n-2} \left(\left(L_{r+n+1-i} - \frac{L_{r+2}}{L_{r+1}} L_{r+n-i} \right) \right. \\
&\qquad \left. \times \left(L_{r+n} - L_r \right)^{i-1} \right) \\
&\qquad \times \left(\left(L_{r+1} - L_{r+n+1} \right)^{i} \right)^{-1} \Bigg),
\end{aligned} \tag{60}
$$

where

$$
\begin{aligned}
l_{r,n} &= \left(L_{r+1} - \frac{L_{r+2}}{L_{r+1}} L_{r+n} \right) \\
&\quad + \sum_{k=1}^{n-2} \left(L_{r+k+2} - \frac{L_{r+2}}{L_{r+1}} L_{r+k+1} \right) \left(\frac{L_{r+n} - L_r}{L_{r+1} - L_{r+n+1}} \right)^{n-(k+1)}.
\end{aligned} \tag{61}
$$

5. Determinant, Invertibility, and Inverse of g-Circulant Matrix with Any Continuous Fibonacci and Lucas Numbers

In this section, let $A_{g,r,n} = g\text{-Circ}\,(F_{r+1}, F_{r+2}, \ldots, F_{r+n})$ and $B_{g,r,n} = g\text{-Circ}\,(L_{r+1}, L_{r+2}, \ldots, L_{r+n})$ be g-circulant matrices. By using the obtained conclusions, we give a determinant formula for the matrices $A_{g,r,n}$ and $B_{g,r,n}$. Afterwards, we prove that $A_{g,r,n}$ is an invertible matrix for $n > 2$ and $B_{g,r,n}$ is an invertible matrix if $(n, g) = 1$. The inverse of the matrices $A_{g,r,n}$ and $B_{g,r,n}$ is also presented.

From Lemma 6, Lemma 7, Theorem 8, Theorem 9, and Theorem 11, we deduce the following results.

Theorem 22. *Let $A_{g,r,n} = g\text{-Circ}\,(F_{r+1}, F_{r+2}, \ldots, F_{r+n})$ be a g-circulant matrix; then one has*

$$
\begin{aligned}
\det A_{g,r,n} &= \det \mathbb{Q}_g \cdot F_{r+1} \\
&\quad \cdot \left[\left(F_{r+1} - \frac{F_{r+2}}{F_{r+1}} F_{r+n} \right) \right. \\
&\qquad + \sum_{k=1}^{n-2} \left(F_{r+k+2} - \frac{F_{r+2}}{F_{r+1}} F_{r+k+1} \right) \\
&\qquad \left. \times \left(\frac{F_{r+n} - F_r}{F_{r+1} - F_{r+n+1}} \right)^{n-(k+1)} \right] \\
&\quad \times \left(F_{r+1} - F_{r+n+1} \right)^{n-2},
\end{aligned} \tag{62}
$$

where F_{r+n} is the $(r+n)$th Fibonacci number.

Theorem 23. *Let $A_{g,r,n} = g\text{-Circ}\,(F_{r+1}, F_{r+2}, \ldots, F_{r+n})$ be a g-circulant matrix and $(g, n) = 1$; if $n > 2$, then $A_{g,r,n}$ is an invertible matrix.*

Theorem 24. *Let $A_{g,r,n} = g\text{-Circ}\,(F_{r+1}, F_{r+2}, \ldots, F_{r+n})$ $(n > 2)$ be a g-circulant matrix and $(g, n) = 1$; then*

$$
\begin{aligned}
&A_{g,r,n}^{-1} \\
&= \Bigg[\frac{1}{f_{r,n}} \\
&\quad \times \operatorname{Circ} \Bigg(1 + \sum_{i=1}^{n-2} \left(\left(F_{r+n+2-i} - \frac{F_{r+2}}{F_{r+1}} F_{r+n+1-i} \right) \right. \\
&\qquad \left. \times \left(F_{r+n} - F_r \right)^{i-1} \right) \\
&\qquad \times \left(\left(F_{r+1} - F_{r+n+1} \right)^{i} \right)^{-1}, \\
&\qquad - \frac{F_{r+2}}{F_{r+1}} \\
&\qquad + \sum_{i=1}^{n-2} \left(\left(F_{r+n+1-i} - \frac{F_{r+2}}{F_{r+1}} F_{r+n-i} \right) \right. \\
&\qquad \left. \times \left(F_{r+n} - F_r \right)^{i-1} \right) \\
&\qquad \times \left(\left(F_{r+1} - F_{r+n+1} \right)^{i} \right)^{-1}, \\
&\qquad - \frac{F_{r+3} - \left(F_{r+2}/F_{r+1} \right) F_{r+2}}{F_{r+1} - F_{r+n+1}}, \\
&\qquad - \frac{\left(F_{r+3} - \left(F_{r+2}/F_{r+1} \right) F_{r+2} \right) \left(F_{r+n} - F_r \right)}{\left(F_{r+1} - F_{r+n+1} \right)^2}, \\
&\qquad - \frac{\left(F_{r+3} - \left(F_{r+2}/F_{r+1} \right) F_{r+2} \right) \left(F_{r+n} - F_r \right)^2}{\left(F_{r+1} - F_{r+n+1} \right)^3}, \ldots, \\
&\qquad - \frac{\left(F_{r+3} - \left(F_{r+2}/F_{r+1} \right) F_{r+2} \right) \left(F_{r+n} - F_r \right)^{n-3}}{\left(F_{r+1} - F_{r+n+1} \right)^{n-2}} \Bigg) \Bigg] \\
&\quad \times \mathbb{Q}_g^T,
\end{aligned} \tag{63}
$$

where

$$
\begin{aligned}
f_{r,n} &= \left(F_{r+1} - \frac{F_{r+2}}{F_{r+1}} F_{r+n} \right) \\
&\quad + \sum_{k=1}^{n-2} \left(F_{r+k+2} - \frac{F_{r+2}}{F_{r+1}} F_{r+k+1} \right) \left(\frac{F_{r+n} - F_r}{F_{r+1} - F_{r+n+1}} \right)^{n-(k+1)}.
\end{aligned} \tag{64}
$$

Taking Lemma 6, Lemma 7, Theorem 12, Theorem 13, and Theorem 15 into account, one has the following theorems.

Theorem 25. *Let $B_{g,r,n} = g\text{-Circ}(L_{r+1}, L_{r+2}, \ldots, L_{r+n})$ be a g-circulant matrix; then one has*

$$\begin{aligned}\det B_{g,r,n} = {} & \det \mathbb{Q}_g \cdot L_{r+1} \\ & \cdot \left[\left(L_{r+1} - \frac{L_{r+2}}{L_{r+1}} L_{r+n} \right) \right. \\ & + \sum_{k=1}^{n-2} \left(L_{r+k+2} - \frac{L_{r+2}}{L_{r+1}} L_{r+k+1} \right) \\ & \left. \times \left(\frac{L_{r+n} - L_r}{L_{r+1} - L_{r+n+1}} \right)^{n-(k+1)} \right] \\ & \times (L_{r+1} - L_{r+n+1})^{n-2},\end{aligned} \tag{65}$$

where L_{r+n} is the $(r+n)$th Lucas number.

Theorem 26. *Let $B_{g,r,n} = g\text{-Circ}(L_{r+1}, L_{r+2}, \ldots, L_{r+n})$ be a g-circulant matrix and $(g,n) = 1$; if $n > 2$, then $B_{g,r,n}$ is an invertible matrix.*

Theorem 27. *Let $B_{g,r,n} = g\text{-Circ}(L_{r+1}, L_{r+2}, \ldots, L_{r+n})$ $(n > 2)$ be a g-circulant matrix and $(g,n) = 1$; then*

$$\begin{aligned}& B_{g,r,n}^{-1} \\ & = \left[\frac{1}{l_{r,n}} \right. \\ & \times \operatorname{Circ} \left(1 + \sum_{i=1}^{n-2} \left(\left(L_{r+n+2-i} - \frac{L_{r+2}}{L_{r+1}} L_{r+n+1-i} \right) \right. \right. \\ & \qquad \left. \times (L_{r+n} - L_r)^{i-1} \right) \\ & \qquad \times \left((L_{r+1} - L_{r+n+1})^i \right)^{-1}, \\ & - \frac{L_{r+2}}{L_{r+1}} \\ & + \sum_{i=1}^{n-2} \left(\left(L_{r+n+1-i} - \frac{L_{r+2}}{L_{r+1}} L_{r+n-i} \right) \right. \\ & \qquad \left. \times (L_{r+n} - L_r)^{i-1} \right) \\ & \qquad \times \left((L_{r+1} - L_{r+n+1})^i \right)^{-1}, \\ & - \frac{L_{r+3} - (L_{r+2}/L_{r+1}) L_{r+2}}{L_{r+1} - L_{r+n+1}}, \\ & - \frac{(L_{r+3} - (L_{r+2}/L_{r+1}) L_{r+2})(L_{r+n} - L_r)}{(L_{r+1} - L_{r+n+1})^2}, \\ & - \frac{(L_{r+3} - (L_{r+2}/L_{r+1}) L_{r+2})(L_{r+n} - L_r)^2}{(L_{r+1} - L_{r+n+1})^3}, \ldots, \\ & - \left. \left. \frac{(L_{r+3} - (L_{r+2}/L_{r+1}) L_{r+2})(L_{r+n} - L_r)^{n-3}}{(L_{r+1} - L_{r+n+1})^{n-2}} \right) \right] \\ & \times \mathbb{Q}_g^T,\end{aligned} \tag{66}$$

where

$$\begin{aligned}l_{r,n} = {} & \left(L_{r+1} - \frac{L_{r+2}}{L_{r+1}} L_{r+n} \right) \\ & + \sum_{k=1}^{n-2} \left(L_{r+k+2} - \frac{L_{r+2}}{L_{r+1}} L_{r+k+1} \right) \left(\frac{L_{r+n} - L_r}{L_{r+1} - L_{r+n+1}} \right)^{n-(k+1)}.\end{aligned} \tag{67}$$

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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