Research Article

$\mathcal{Q}_K$ Spaces of Several Real Variables

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We introduce a new space, $\mathcal{Q}_K(\mathbb{R}^n)$ space, of several real variables with nondecreasing functions K . By giving basic properties of the weighted function K , by establishing a Stegenga-type estimate, and by introducing the K -Carleson measure on $\mathbb{R}_+^{n+1}$, we obtain various characterizations of $\mathcal{Q}_K(\mathbb{R}^n)$ space.

1. Introduction

Recall that a locally integrable function f belongs to $\text{BMO}(\mathbb{R}^n)$ if

$$\|f\|_{\text{BMO}(\mathbb{R}^n)} = \sup_{I \subset \mathbb{R}^n} |I|^{-1} \int_I |f(x) - f_I| dx < \infty, \quad (1)$$

where I denotes a cube in $\mathbb{R}^n$ with edges parallel to the coordinate axes and $|I|$ denotes the Lebesgue measure of I and

$$f_I = |I|^{-1} \int_I f(x) dx. \quad (2)$$

Via the John-Nirenberg inequality [1], one can show an equivalent condition of $\text{BMO}(\mathbb{R}^n)$ as follows:

$$\|f\|_{\text{BMO}(\mathbb{R}^n)}^2 \approx \sup_I |I|^{-2} \iint_I |f(x) - f(y)|^2 dx dy. \quad (3)$$

C. Fefferman's famous equation, $(H^1)^* = \text{BMO}$, describes a deep relation between BMO and the Hardy space (cf. [2, 3]). This leads quite naturally to increased study of these functions from the point of real variable theory and complex function theory views in the recent fifty years. See [2–9] for more results about $\text{BMO}(\mathbb{R}^n)$ space.

As a generalization of $\text{BMO}(\mathbb{R}^n)$, the space $\mathcal{Q}_\alpha(\mathbb{R}^n)$, $\alpha \in \mathbb{R}$, introduced by Essén et al. in [10], is defined to be the class of all locally integrable functions $f \in L_{\text{loc}}^2(\mathbb{R}^n)$ such that

$$\|f\|_{\mathcal{Q}_\alpha(\mathbb{R}^n)}^2 = \sup_I [\ell(I)]^{2\alpha-n} \iint_I \frac{|f(x) - f(y)|^2}{|x - y|^{n+2\alpha}} dx dy < \infty, \quad (4)$$

where $\ell(I) = |I|^{1/n}$ denotes the edge length of the cube I .

It is easy to see that $\mathcal{Q}_\alpha(\mathbb{R}^n)$ is always a subclass of $\text{BMO}(\mathbb{R}^n)$ and $\mathcal{Q}_\alpha(\mathbb{R}^n) = \text{BMO}(\mathbb{R}^n)$ by choosing $\alpha = -n/2$. Moreover, we know by [10] that $\mathcal{Q}_\alpha(\mathbb{R}^n) = \text{BMO}(\mathbb{R}^n)$ if and only if $\alpha < 0$. Also, we see that $\mathcal{Q}_\alpha(\mathbb{R})$ is trivial (containing a.e. constant functions only) if and only if $\alpha > 1/2$ and $\mathcal{Q}_\alpha(\mathbb{R}^n)$, $n \geq 2$, is trivial if and only if $\alpha \geq 1$.

In this paper, we introduce and develop a more general space $\mathcal{Q}_K(\mathbb{R}^n)$ of several real variables, which can be viewed as an extension and improvement of $\mathcal{Q}_\alpha(\mathbb{R}^n)$ spaces as well as $\text{BMO}(\mathbb{R}^n)$. A theory of $\mathcal{Q}_K(\mathbb{D})$ spaces on unit disc $\mathbb{D}$ has been extensively studied for recent years in the context of a wide class of function spaces; see, for example, [11–15]. Motivated by the theory of analytic $\mathcal{Q}_K(\mathbb{D})$ spaces, we define the following.

Definition 1. Let $K : [0, \infty) \rightarrow [0, \infty)$ be a nondecreasing function. A function $f \in L_{\text{loc}}^2(\mathbb{R}^n)$ is said to belong to the space $\mathcal{Q}_K(\mathbb{R}^n)$ if

$$\begin{aligned} \|f\|_{\mathcal{Q}_K(\mathbb{R}^n)}^2 &= \sup_{I \subset \mathbb{R}^n} \iint_I \frac{|f(x) - f(y)|^2}{|x - y|^{2n}} K\left(\frac{|x - y|}{\ell(I)}\right) dx dy < \infty. \end{aligned} \quad (5)$$

If we take $K(t) = t^{n-2\alpha}$, for $\alpha \in \mathbb{R}$, then $\mathcal{Q}_K(\mathbb{R}^n) = \mathcal{Q}_\alpha(\mathbb{R}^n)$. Modulo constants, $\mathcal{Q}_K(\mathbb{R}^n)$ is a Banach space under the norm defined in (5).

Our paper is organized as follows.

In Section 2, we investigate the relationship between $\mathcal{Q}_K(\mathbb{R}^n)$ and $\text{BMO}(\mathbb{R}^n)$ and give a sufficient and necessary condition for space $\mathcal{Q}_K(\mathbb{R}^n)$ which is nontrivial.

In Section 3, we give several results about the weight function K on which $\mathcal{Q}_K(\mathbb{R}^n)$ obviously depends. In the study of $\mathcal{Q}_K(\mathbb{R}^n)$, the auxiliary function φ_K defined by

$$\varphi_K(s) = \sup_{0 < t \leq 1} \frac{K(st)}{K(t)}, \quad 0 < s < \infty, \quad (6)$$

still works well as the analytic $\mathcal{Q}_K(\mathbb{D})$ spaces.

In Section 4, we define the K -Carleson measure on $\mathbb{R}_+^{n+1}$. By establishing a Stegenga-type estimate, we obtain a characterization of $\mathcal{Q}_K(\mathbb{R}^n)$ spaces in terms of the K -Carleson measure.

Throughout this note, $a \leq b$ means that there is a positive constant C such that $a \leq Cb$. Moreover, if both $a \leq b$ and $b \leq a$ hold, then one says that $a \approx b$. For the convenience of calculation, in this paper, we always assume that $K : [0, \infty) \rightarrow [0, \infty)$ is nondecreasing and $K(2t) \approx K(t)$.

2. Basic Properties of $\mathcal{Q}_K(\mathbb{R}^n)$

Our first observation is that $\mathcal{Q}_K(\mathbb{R}^n)$ is invariant under the conformal mappings and rotations; that is, for any conformal map $\phi(x) = \lambda x + x_0$, $\lambda \neq 0$ and $x_0 \in \mathbb{R}^n$, or any rotation $\psi(x) = xM$ for an orthogonal matrix M of order n ,

$$\begin{aligned} \|f \circ \phi\|_{\mathcal{Q}_K(\mathbb{R}^n)} &= \|f\|_{\mathcal{Q}_K(\mathbb{R}^n)}, \\ \|f \circ \psi\|_{\mathcal{Q}_K(\mathbb{R}^n)} &\approx \|f\|_{\mathcal{Q}_K(\mathbb{R}^n)} \end{aligned} \quad (7)$$

hold for any $f \in \mathcal{Q}_K(\mathbb{R}^n)$.

We say that the space $\mathcal{Q}_K(\mathbb{R}^n)$ is trivial if $\mathcal{Q}_K(\mathbb{R}^n)$ contains only a.e. constant functions. To discuss this problem, we recall the space $\text{CIS}(\mathbb{R}^n)$, the class of all functions $f \in C^1(\mathbb{R}^n)$, with

$$\|f\|_{\text{CIS}}^2 = \sup_I |I|^{((2-n)/n)} \int_I |\nabla f(x)|^2 dx < \infty, \quad (8)$$

where $\nabla f(x) = (\partial f(x)/\partial x_1, \dots, \partial f(x)/\partial x_n)$. By [16], we know that

$$(1 + |x|^2)^{-1} \in \text{CIS}(\mathbb{R}^2), \quad \ln(1 + |x|^2) \in \text{CIS}(\mathbb{R}^n), \quad (9)$$

$n > 2$.

Thus $\text{CIS}(\mathbb{R}^n)$ is not trivial for $n \geq 2$. However, $\text{CIS}(\mathbb{R})$ is trivial.

For any cube I , if $x, y \in I$, then $|x - y| \leq \sqrt{n}\ell(I)$. For $t > 0$, tI means that the cube has the same center as I and

the edge length $t\ell(I)$. If $x \in I$ and $|y| < \ell(I)$, then $x + y \in 3I$. By the change of variable, $f \in \mathcal{Q}_K(\mathbb{R}^n)$ if and only if

$$\begin{aligned} &\sup_I \int_{|y| < \ell(I)} \int_I |f(x + y) - f(x)|^2 \\ &\quad \times K\left(\frac{|y|}{\ell(I)}\right) |y|^{-2n} dx dy < \infty. \end{aligned} \quad (10)$$

Theorem 2. *The following statements are true.*

- (a) $\mathcal{Q}_{1/2}(\mathbb{R}) \subseteq \mathcal{Q}_K(\mathbb{R})$. Moreover, $\mathcal{Q}_K(\mathbb{R})$ is never trivial.
- (b) For $n \geq 2$, $\mathcal{Q}_K(\mathbb{R}^n)$ is not trivial if and only if

$$\int_0^{\sqrt{n}} \frac{K(t)}{t^{n-1}} dt < +\infty. \quad (11)$$

Moreover, if (11) holds, then

$$\text{CIS}(\mathbb{R}^n) \subset \mathcal{Q}_K(\mathbb{R}^n). \quad (12)$$

Proof. (a) For any cube I of $\mathbb{R}$ and $x, y \in I$, we have

$$|x - y| \leq \ell(I). \quad (13)$$

By assumption on K we have

$$\begin{aligned} &\iint_I \frac{|f(x) - f(y)|^2}{|x - y|^2} K\left(\frac{|x - y|}{\ell(I)}\right) dx dy \\ &\leq K(1) \iint_I \frac{|f(x) - f(y)|^2}{|x - y|^2} dx dy. \end{aligned} \quad (14)$$

Hence, $\mathcal{Q}_{1/2}(\mathbb{R}) \subseteq \mathcal{Q}_K(\mathbb{R})$. $\mathcal{Q}_{1/2}(\mathbb{R})$ is not trivial and so is $\mathcal{Q}_K(\mathbb{R})$.

(b) *Necessity.* It is enough to show that if

$$\int_0^{\sqrt{n}} \frac{K(t)}{t^{n-1}} dt = +\infty, \quad (15)$$

then $\mathcal{Q}_K(\mathbb{R}^n)$ is trivial. We will prove the necessity by two steps.

Step 1. If $f \in \mathcal{Q}_K(\mathbb{R}^n) \cap C^1(\mathbb{R}^n)$ and f is nonconstant, we may assume that f is real. Then there exists a point $x_0 = (x_1^0, x_2^0, \dots, x_n^0)$ such that $\nabla f(x_0) \neq 0$. By the Householder reflector [17, p. 71], there exists an orthogonal matrix $M = (a_{ij})$, $i, j = 1, 2, \dots, n$, such that

$$\nabla f(x_0) M = (|\nabla f(x_0)|, 0, \dots, 0), \quad (16)$$

which gives

$$\begin{aligned} \sum_{i=1}^n \frac{\partial f}{\partial x_i}(x_0) a_{i1} &= |\nabla f(x_0)|, \quad \sum_{i=1}^n \frac{\partial f}{\partial x_i}(x_0) a_{ij} = 0, \\ j &= 2, 3, \dots, n. \end{aligned} \quad (17)$$

Denote by M^T the transpose of the matrix M . Set $g(x) = f(xM^T)$. Since $\det(M^T) \neq 0$, there exists a point

$y_0 = (y_1^0, y_2^0, \dots, y_n^0)$ such that $y_0 M^T = x_0$. Write $y = x M^T$ for convenience as follows:

$$y_j = \sum_{i=1}^n x_i a_{ji}, \quad j = 1, 2, \dots, n. \quad (18)$$

Consequently,

$$\frac{\partial g}{\partial x_1}(y_0) = \sum_{j=1}^n \frac{\partial f}{\partial y_j}(y_0 M^T) a_{j1} = |\nabla f(x_0)|. \quad (19)$$

Similarly,

$$\frac{\partial g}{\partial x_i}(y_0) = 0, \quad i = 2, 3, \dots, n. \quad (20)$$

Thus

$$\nabla g(y_0) = (|\nabla f(x_0)|, 0, \dots, 0). \quad (21)$$

Note that $g \in C^1(\mathbb{R}^n)$. Then there exist a positive constant δ and a small cube I centered at y_0 on which $\partial g(x)/\partial x_1 > 2\delta$ and $\partial g(x)/\partial x_j < \delta$, $j \geq 2$. Define

$$D = \left\{ x = (x_1, \dots, x_n) \in \mathbb{R}^n : |x_2| + \dots + |x_n| < x_1 < \frac{\ell(I)}{4} \right\}. \quad (22)$$

If $x, y \in I$ and $x - y \in D$, using the mean value theorem, we get

$$g(x) - g(y) > \delta(x_1 - y_1). \quad (23)$$

Thus

$$\begin{aligned} & \iint_I \frac{|g(x) - g(y)|^2}{|x - y|^{2n}} K\left(\frac{|x - y|}{\ell(I)}\right) dx dy \\ & \geq \int_{I/2} dx \int_{I-x} \frac{|g(x+z) - g(x)|^2}{|z|^{2n}} K\left(\frac{|z|}{\ell(I)}\right) dz \\ & \geq \int_{I/2} dx \int_D \frac{\delta^2 |z_1|^2}{|z|^{2n}} K\left(\frac{|z|}{\ell(I)}\right) dz. \end{aligned} \quad (24)$$

If $z \in D$, then $|z| \approx z_1$. Hence

$$\begin{aligned} & \iint_I \frac{|g(x) - g(y)|^2}{|x - y|^{2n}} K\left(\frac{|x - y|}{\ell(I)}\right) dx dy \\ & \geq \int_{I/2} dx \int_D \frac{|z_1|^2}{|z_1|^{2n}} K\left(\frac{|z_1|}{\ell(I)}\right) dz \\ & \approx \int_0^{\ell(I)/4} \frac{1}{z_1^{2n-2}} K\left(\frac{z_1}{\ell(I)}\right) dz_1 \\ & \quad \times \int_{\{(z_2, \dots, z_n) : |z_2| + \dots + |z_n| < z_1\}} dz_2 \cdots dz_n \\ & \approx \int_0^{\sqrt{n}} \frac{K(t)}{t^{n-1}} dt = +\infty. \end{aligned} \quad (25)$$

It means that g is not an element of $\mathcal{Q}_K(\mathbb{R}^n)$. On the other hand, $g \in \mathcal{Q}_K(\mathbb{R}^n)$ since $\mathcal{Q}_K(\mathbb{R}^n)$ is invariant under rotations. This is a contradiction.

Step 2. Note that $\mathcal{Q}_K(\mathbb{R}^n)$ is conformal invariant. By Minkowski's inequality, if $f \in \mathcal{Q}_K(\mathbb{R}^n)$ and $g \in L^1(\mathbb{R}^n)$, then $f * g \in \mathcal{Q}_K(\mathbb{R}^n)$ and

$$\|f * g\|_{\mathcal{Q}_K(\mathbb{R}^n)} \leq \|f\|_{\mathcal{Q}_K(\mathbb{R}^n)} \int_{\mathbb{R}^n} |g(y)| dy, \quad (26)$$

where

$$(f * g)(x) = \int_{\mathbb{R}^n} f(x - y) g(y) dy. \quad (27)$$

In particular, if $g \in C_0^\infty$, the class of smooth functions with compact support, then $f * g \in \mathcal{Q}_K(\mathbb{R}^n) \cap C^1(\mathbb{R}^n)$. Thus $f * g$ is constant by Step 1. By [10], there exists a sequence $\{g_n\} \subset C_0^\infty$ with $g_n \geq 0$, $\int g_n = 1$, and $\text{supp } g_n$ shrinking to 0 such that $f * g_n \rightarrow f$ a.e. It follows that f is constant a.e. Thus, we complete the proof of necessity.

Sufficiency. Give a cube I and suppose $f \in \text{CIS}(\mathbb{R}^n)$. By

$$|f(x + y) - f(y)| \leq \int_0^1 |\nabla f(y + tx)| |x| dt, \quad (28)$$

we have

$$\begin{aligned} & \iint_I \frac{|f(x) - f(y)|^2}{|x - y|^{2n}} K\left(\frac{|x - y|}{\ell(I)}\right) dx dy \\ & \leq \int_I \int_{|z| < \sqrt{n}\ell(I)} \left(\int_0^1 |\nabla f(x + tz)| dt \right)^2 \\ & \quad \times \frac{1}{|z|^{2n-2}} K\left(\frac{|z|}{\ell(I)}\right) dz dx. \end{aligned} \quad (29)$$

It follows by Minkowski's inequality that

$$\begin{aligned} & \left(\iint_I \frac{|f(x) - f(y)|^2}{|x - y|^{2n}} K\left(\frac{|x - y|}{\ell(I)}\right) dx dy \right)^{1/2} \\ & \leq \int_0^1 \left(\int_I \int_{|z| < \sqrt{n}\ell(I)} |\nabla f(x + tz)|^2 \frac{1}{|z|^{2n-2}} \right. \\ & \quad \times K\left(\frac{|z|}{\ell(I)}\right) dz dx \Big)^{1/2} dt \\ & \leq \left(\int_{|z| < \sqrt{n}\ell(I)} \frac{1}{|z|^{2n-2}} K\left(\frac{|z|}{\ell(I)}\right) dz \right. \\ & \quad \times \int_{3\sqrt{n}I} |\nabla f(w)|^2 dw \Big)^{1/2} \\ & \leq \|f\|_{\text{CIS}} \\ & \quad \times \left(\ell(I)^{n-2} \int_0^{\sqrt{n}\ell(I)} \frac{1}{t^{n-1}} K\left(\frac{t}{\ell(I)}\right) dt \right)^{1/2} \\ & \approx \|f\|_{\text{CIS}} \left(\int_0^{\sqrt{n}} \frac{K(t)}{t^{n-1}} dt \right)^{1/2}. \end{aligned} \quad (30)$$

Thus, $\text{CIS}(\mathbb{R}^n) \subseteq \mathcal{Q}_K(\mathbb{R}^n)$ and $\mathcal{Q}_K(\mathbb{R}^n)$ is not trivial. $\square$

Theorem 3. The space $\mathcal{Q}_K(\mathbb{R}^n)$ is a subset of $BMO(\mathbb{R}^n)$. Furthermore,

- (a) if $\int_0^{\sqrt{n}} (K(t)/t^{n+1}) dt < \infty$, then $\mathcal{Q}_K(\mathbb{R}^n) = BMO(\mathbb{R}^n)$;
 (b) if $\mathcal{Q}_K(\mathbb{R}^n) = BMO(\mathbb{R}^n)$, then, for all $\beta < 1$, $\int_0^{\sqrt{n}} (K(t)/t^{n+\beta}) dt < \infty$.

Proof. Let $f \in \mathcal{Q}_K(\mathbb{R}^n)$. For any cube I and $x, y \in I$, if $r > 0$ is small enough, we know that the Lebesgue measure of the set

$$\{z \in I : \min(|x - z|, |y - z|) > r\ell(I)\} \quad (31)$$

is bigger than $|I|(1 - (4r^n \pi^{n/2}/n\Gamma(n/2)))$. Since K is nondecreasing,

$$\begin{aligned} & \int_I \min\left(K\left(\frac{|x - z|}{\ell(I)}\right), K\left(\frac{|y - z|}{\ell(I)}\right)\right) dz \\ & \geq \int_{\{z \in I : \min(|x - z|, |y - z|) > r\ell(I)\}} \min\left(K\left(\frac{|x - z|}{\ell(I)}\right), K\left(\frac{|y - z|}{\ell(I)}\right)\right) dz \\ & \geq K(r) |I| \left(1 - \frac{4r^n \pi^{n/2}}{n\Gamma(n/2)}\right). \end{aligned} \quad (32)$$

Consequently,

$$\begin{aligned} & K(r) \left(1 - \frac{4r^n \pi^{n/2}}{n\Gamma(n/2)}\right) |I|^{-2} \iint_I |f(x) - f(y)|^2 dx dy \\ & \leq |I|^{-3} \iiint_I |f(x) - f(y)|^2 \\ & \quad \times \min\left(K\left(\frac{|x - z|}{\ell(I)}\right), K\left(\frac{|y - z|}{\ell(I)}\right)\right) dx dy dz \\ & \leq 2|I|^{-3} \iiint_I |f(x) - f(z)|^2 K\left(\frac{|x - z|}{\ell(I)}\right) dx dy dz \\ & \quad + 2|I|^{-3} \iiint_I |f(y) - f(z)|^2 K\left(\frac{|y - z|}{\ell(I)}\right) dx dy dz \\ & \leq 4n^n \iint_I \frac{|f(y) - f(z)|^2}{|y - z|^{2n}} K\left(\frac{|y - z|}{\ell(I)}\right) dy dz. \end{aligned} \quad (33)$$

For a small enough $r > 0$, we obtain

$$\|f\|_{BMO(\mathbb{R}^n)}^2 \lesssim \sup_I \iint_I \frac{|f(y) - f(z)|^2}{|y - z|^{2n}} K\left(\frac{|y - z|}{\ell(I)}\right) dy dz. \quad (34)$$

Thus $\mathcal{Q}_K(\mathbb{R}^n) \subseteq BMO(\mathbb{R}^n)$.

(a) Note that

$$\begin{aligned} \|f\|_{BMO(\mathbb{R}^n)}^2 & \approx \sup_I |I|^{-2} \iint_I |f(x) - f(y)|^2 dx dy \\ & \approx \sup_I |I|^{-1} \int_I |f(x) - f_I|^2 dx. \end{aligned} \quad (35)$$

For a cube I and for every $y \in \mathbb{R}^n$ with $|y| < \sqrt{n}\ell(I)$,

$$\begin{aligned} & \int_I |f(x + y) - f(x)|^2 dx \\ & \leq \int_I |f(x + y) - f_{3\sqrt{n}I}|^2 + |f(x) - f_{3\sqrt{n}I}|^2 dx \\ & \leq |I| \|f\|_{BMO(\mathbb{R}^n)}^2. \end{aligned} \quad (36)$$

Therefore,

$$\begin{aligned} & \iint_I \frac{|f(x) - f(y)|^2}{|x - y|^{2n}} K\left(\frac{|x - y|}{\ell(I)}\right) dx dy \\ & \leq \int_{|y| < \sqrt{n}\ell(I)} \int_I |f(x + y) - f(x)|^2 \\ & \quad \times \frac{K(|y|/\ell(I))}{|y|^{2n}} dx dy \\ & \leq |I| \|f\|_{BMO(\mathbb{R}^n)}^2 \int_{|y| < \sqrt{n}\ell(I)} \frac{K(|y|/\ell(I))}{|y|^{2n}} dy \\ & \approx \|f\|_{BMO(\mathbb{R}^n)}^2 \int_0^{\sqrt{n}} \frac{K(t)}{t^{n+1}} dt. \end{aligned} \quad (37)$$

Thus $BMO(\mathbb{R}^n) \subseteq \mathcal{Q}_K(\mathbb{R}^n)$, and this deduces that $\mathcal{Q}_K(\mathbb{R}^n) = BMO(\mathbb{R}^n)$.

(b) Consider the Schwartz space $\mathcal{S}(\mathbb{R}^n)$ consisting of all those C^∞ functions φ on $\mathbb{R}^n$ such that

$$\sup_{x \in \mathbb{R}^n} |x^\alpha (D^\beta \varphi)(x)| < \infty \quad (38)$$

for all multi-indices $\alpha = (\alpha_1, \dots, \alpha_n)$ and $\beta = (\beta_1, \dots, \beta_n)$ of nonnegative integers, where $x^\alpha = x_1^{\alpha_1} \dots x_n^{\alpha_n}$ and $D^\beta = (\partial/\partial x_1)^{\beta_1} \dots (\partial/\partial x_n)^{\beta_n}$.

Let $\varphi \in \mathcal{S}(\mathbb{R}^n)$ be a fixed function such that the Fourier transform $\widehat{\varphi}$ of φ has support in the unit ball and $\varphi \neq 0$ on the cube $[-3\pi, 3\pi]^n$. Let $\{a_k\}$ be a sequence of real numbers and define

$$g(x) = \sum_{k=1}^{\infty} a_k \exp(2^k x_1 i), \quad (39)$$

where x_1 is the first coordinate of x . By [10], $f = \varphi g \in BMO(\mathbb{R}^n)$ if and only if $\sum_{k=1}^{\infty} a_k^2 < \infty$.

Suppose that $\mathcal{Q}_K(\mathbb{R}^n) = BMO(\mathbb{R}^n)$. Set $a_k^2 = (1/2^{k(1-\beta)})$ for $\beta < 1$. We know that $f \in \mathcal{Q}_K(\mathbb{R}^n)$. Choosing $I = [-\pi, \pi]^n$, we have

$$\int_{|y| < 2\pi} \int_I |f(x + y) - f(x)|^2 K(|y|) \frac{1}{|y|^{2n}} dx dy < \infty. \quad (40)$$

Since $|\varphi(x+y)| \geq c > 0$ for $x \in I$ and $|y| < 2\pi$,

$$\begin{aligned} & |g(x+y) - g(x)|^2 \\ & \leq |f(x+y) - f(x)|^2 \\ & \quad + |g(x)|^2 |\varphi(x+y) - \varphi(x)|^2. \end{aligned} \quad (41)$$

Since $\varphi \in \mathcal{S}(\mathbb{R}^n)$, we have $|\varphi(x+y) - \varphi(x)| \leq |y|$. Hence,

$$\begin{aligned} & \int_{|y| < 2\pi} \int_I |g(x)|^2 |\varphi(x+y) - \varphi(x)|^2 K(|y|) \frac{1}{|y|^{2n}} dx dy \\ & \leq \int_{|y| < 2\pi} K(|y|) \frac{1}{|y|^{2n-2}} dy \\ & \quad \times \int_{[-\pi, \pi]^n} \left| \sum_{k=1}^{\infty} a_k \exp(2^k x_1 i) \right|^2 dx \\ & \approx \sum_{k=1}^{\infty} a_k^2 \int_0^{\sqrt{n}} \frac{K(t)}{t^{n-1}} dt < \infty. \end{aligned} \quad (42)$$

Writing $y = (y_1, y')$, $y' \in \mathbb{R}^{n-1}$, we obtain

$$\begin{aligned} & \infty > \int_{|y| < 2\pi} \int_I |g(x+y) - g(x)|^2 K(|y|) \frac{1}{|y|^{2n}} dx dy \\ & \approx \int_{|y| < 2\pi} \sum_{k=1}^{\infty} |a_k (e^{2^k y_1 i} - 1)|^2 K(|y|) \frac{1}{|y|^{2n}} dy \\ & \geq \sum_{k=1}^{\infty} a_k^2 \int_{|y'| < y_1 < 2^{-k}} |2^k y_1|^2 K(|y|) \frac{1}{|y|^{2n}} dy \\ & \approx \sum_{k=1}^{\infty} 2^{2k} a_k^2 \int_0^{2^{-k}} \frac{K(t)}{t^{n-1}} dt \\ & \geq \int_0^{1/2} \frac{K(t)}{t^{n+\beta}} dt. \end{aligned} \quad (43)$$

□

For any cube I of $\mathbb{R}^n$, when $x, y \in I$, we have that $|x-y| \leq \sqrt{n}\ell(I)$. By the definition, the $\mathcal{Q}_K(\mathbb{R}^n)$ space depends on $K(t)$ when $0 \leq t \leq \sqrt{n}$. In fact, $\mathcal{Q}_K(\mathbb{R}^n)$ depends only on $K(t)$ when t is near origin, which can be found by the following theorem. Here the proof of the theorem is left to the reader.

Theorem 4. *The following statements are true.*

- (a) Suppose that $K(r) > 0$ for some $r > 0$. One defines $K_r(t) = \min(K(r), K(t))$. Then $\mathcal{Q}_{K_r}(\mathbb{R}^n) = \mathcal{Q}_K(\mathbb{R}^n)$.
- (b) Let

$$K_1(t) = \begin{cases} K(t), & 0 < t \leq 1, \\ K(1)t^{n-1}, & t \geq 1. \end{cases} \quad (44)$$

Then $\mathcal{Q}_{K_1}(\mathbb{R}^n) = \mathcal{Q}_K(\mathbb{R}^n)$.

Remark 5. If $K(0) > 0$, we have $\mathcal{Q}_K(\mathbb{R}^n) = \mathcal{Q}_{n/2}(\mathbb{R}^n)$. Since $\mathcal{Q}_{n/2}(\mathbb{R}^n)$ is trivial for $n \geq 2$, we only pay attention to the case $K(0) = 0$.

3. Weighted Functions

The characterization of $\mathcal{Q}_K(\mathbb{R}^n)$ depends on the properties of the weight function K obviously. In this section we give several results about the weight functions that are needed for the next section.

In the analytic $\mathcal{Q}_K(\mathbb{D})$ spaces, the auxiliary function

$$\varphi_K(s) = \sup_{0 < t \leq 1} \frac{K(st)}{K(t)}, \quad 0 < s < \infty, \quad (45)$$

plays a key role; see [12, 14, 15], for example.

To study $\mathcal{Q}_K(\mathbb{R}^n)$ spaces, we need some more constraints on K as follows:

$$\int_0^1 \varphi_K(s) \frac{ds}{\min(s, s^{n-1})} < \infty, \quad (46)$$

$$\int_1^\infty \varphi_K(s) \frac{ds}{s^{n+1}} < \infty. \quad (47)$$

Note that (46) implies the following two conditions:

$$\int_0^1 \varphi_K(s) \frac{ds}{s} < \infty, \quad (48)$$

$$\int_0^1 \varphi_K(s) \frac{ds}{s^{n-1}} < \infty, \quad n \geq 2. \quad (49)$$

In particular, if we choose $K(t) = t^{n-2\alpha}$, then condition (47) holds if and only if $\alpha > 0$; condition (48) holds if and only if $n > 2\alpha$; and condition (49) holds if and only if $\alpha < 1$.

Lemma 6. *Let K satisfy*

$$\sup_{0 < s < b} \int_s^b \frac{K(t)}{t^{n+1}} dt \int_0^s \frac{t^{n-1}}{K(t)} dt < \infty \quad (50)$$

for some $0 < b \leq \infty$. Then one can find another nonnegative weight function K^* with $K^*(0) = 0$ such that $\mathcal{Q}_{K^*}(\mathbb{R}^n) = \mathcal{Q}_K(\mathbb{R}^n)$, and the new weight function K^* has the following properties.

- (a) K^* is nondecreasing on $[0, \infty)$.
- (b) $K^* \approx K$ on $[0, \infty)$, and thus K^* satisfies condition (50).
- (c) $K^*(2t) \approx K^*(t)$ on $[0, \infty)$.
- (d) K^* is differentiable (up to any given order) on $(0, \infty)$.
- (e) For some small enough $c > 0$, $K^*(t)/t^{n-c}$ is nonincreasing on $(0, \infty)$. Consequently, $K^*(t)/t^n$ is also nonincreasing on $(0, \infty)$.

Proof. By Theorem 4, we may assume that $K(t) = K(1)t^{n-1}$ for $t \geq 1$. Since K satisfies condition (50), we claim that

$$\sup_{0 < s < \infty} \int_s^\infty \frac{K(t)}{t^{n+1}} dt \int_0^s \frac{t^{n-1}}{K(t)} dt < \infty. \quad (51)$$

If $b = \infty$, the claim is true. If $0 < b < \infty$, the claim will be confirmed by showing

$$\begin{aligned} \sup_{0 < s < b} \int_s^\infty \frac{K(t)}{t^{n+1}} dt \int_0^s \frac{t^{n-1}}{K(t)} dt &< \infty, \\ \sup_{b \leq s < \infty} \int_s^\infty \frac{K(t)}{t^{n+1}} dt \int_0^s \frac{t^{n-1}}{K(t)} dt &< \infty. \end{aligned} \quad (52)$$

For the case of $0 < s < b$, by (50),

$$\begin{aligned} \int_s^\infty \frac{K(t)}{t^{n+1}} dt \int_0^s \frac{t^{n-1}}{K(t)} dt \\ = \left(\int_s^b \frac{K(t)}{t^{n+1}} dt + \int_b^\infty \frac{K(t)}{t^{n+1}} dt \right) \int_0^s \frac{t^{n-1}}{K(t)} dt \\ \leq C + \int_b^\infty \frac{K(t)}{t^{n+1}} dt \int_0^b \frac{t^{n-1}}{K(t)} dt. \end{aligned} \quad (53)$$

Taking $s = (b/2)$ in (50), we have

$$\int_0^b \frac{t^{n-1}}{K(t)} dt < \infty. \quad (54)$$

Since $K(t) = K(1)t^{n-1}$, for $t \geq 1$, we have

$$\int_b^\infty \frac{K(t)}{t^{n+1}} dt \int_0^b \frac{t^{n-1}}{K(t)} dt < \infty. \quad (55)$$

Thus

$$\sup_{0 < s < b} \int_s^\infty \frac{K(t)}{t^{n+1}} dt \int_0^s \frac{t^{n-1}}{K(t)} dt < \infty. \quad (56)$$

Now, we prove

$$\sup_{b \leq s < \infty} \int_s^\infty \frac{K(t)}{t^{n+1}} dt \int_0^s \frac{t^{n-1}}{K(t)} dt < \infty. \quad (57)$$

In fact, if $b \leq s < \infty$, $b \geq 1$,

$$\begin{aligned} \int_s^\infty \frac{K(t)}{t^{n+1}} dt \int_0^s \frac{t^{n-1}}{K(t)} dt \\ \leq K(1) \int_0^b \frac{t^{n-1}}{K(t)} dt + K(1) s^{-1} \int_b^s \frac{1}{K(1)} dt \\ \leq K(1) \int_0^b \frac{t^{n-1}}{K(t)} dt + 1. \end{aligned} \quad (58)$$

If $b \leq s < \infty$, $0 < b < 1$ and $s < 1$,

$$\begin{aligned} \int_s^\infty \frac{K(t)}{t^{n+1}} dt \int_0^s \frac{t^{n-1}}{K(t)} dt \\ \leq \int_b^\infty \frac{K(t)}{t^{n+1}} dt \left(\int_0^b \frac{t^{n-1}}{K(t)} dt + \int_b^1 \frac{t^{n-1}}{K(t)} dt \right) \\ < \infty. \end{aligned} \quad (59)$$

If $b \leq s < \infty$, $0 < b < 1$ and $s \geq 1$,

$$\begin{aligned} \int_s^\infty \frac{K(t)}{t^{n+1}} dt \int_0^s \frac{t^{n-1}}{K(t)} dt \\ \leq \int_b^\infty \frac{K(t)}{t^{n+1}} dt \int_0^1 \frac{t^{n-1}}{K(t)} dt + \int_s^\infty \frac{K(t)}{t^{n+1}} dt \int_1^s \frac{t^{n-1}}{K(t)} dt \\ \leq \int_b^\infty \frac{K(t)}{t^{n+1}} dt \int_0^1 \frac{t^{n-1}}{K(t)} dt + 1. \end{aligned} \quad (60)$$

Hence,

$$\sup_{b \leq s < \infty} \int_s^\infty \frac{K(t)}{t^{n+1}} dt \int_0^s \frac{t^{n-1}}{K(t)} dt < \infty. \quad (61)$$

Therefore, our claim above has been confirmed. Let

$$K^*(t) = \begin{cases} t^n \int_t^\infty \frac{K(s)}{s^{n+1}} ds, & 0 < t < \infty, \\ 0, & t = 0. \end{cases} \quad (62)$$

Then $K^*(t) = K(1)t^{n-1}$ for $t \geq 1$.

(a) Fix $0 < t_1 < t_2 < \infty$ and consider the difference

$$\begin{aligned} K^*(t_2) - K^*(t_1) &= (t_2^n - t_1^n) \\ &\quad \times \int_{t_2}^\infty \frac{K(s)}{s^{n+1}} ds - t_1^n \int_{t_1}^{t_2} \frac{K(s)}{s^{n+1}} ds. \end{aligned} \quad (63)$$

Since K is nondecreasing and nonnegative, we have

$$\begin{aligned} K^*(t_2) - K^*(t_1) &\geq (t_2^n - t_1^n) K(t_2) \\ &\quad \times \int_{t_2}^\infty \frac{ds}{s^{n+1}} - t_1^n K(t_2) \int_{t_1}^{t_2} \frac{ds}{s^{n+1}} = 0. \end{aligned} \quad (64)$$

(b) Using the assumption that K is nondecreasing again, we obtain

$$K^*(t) \geq t^n K(t) \int_t^\infty \frac{ds}{s^{n+1}} = \frac{K(t)}{n} \quad (65)$$

for $0 < t < \infty$. On the other hand,

$$K^*(t) \leq t^n \left(\int_0^t \frac{s^{n-1}}{K(s)} ds \right)^{-1} \leq K(t), \quad 0 < t < \infty. \quad (66)$$

Thus, $K^* \approx K$ on $(0, \infty)$ and $\mathcal{Q}_{K^*}(\mathbb{R}^n) = \mathcal{Q}_K(\mathbb{R}^n)$.

(c) For any $t > 0$, we have

$$\frac{K^*(2t)}{K^*(t)} = 2^n \frac{\int_{2t}^\infty (K(s)/s^{n+1}) ds}{\int_t^\infty (K(s)/s^{n+1}) ds} \leq 2^n. \quad (67)$$

Since K^* is nondecreasing, $K^*(t) \leq K^*(2t)$.

(d) If we repeat the construction $K \mapsto K^*$, then we can make the new weight function differentiable up to any desired order.

(e) Note that if $c > 0$ is sufficiently small, then we have

$$(t^{c-n} K^*(t))' = t^{c-n-1} (cK^*(t) - K(t)) < 0, \quad 0 < t < \infty. \quad (68)$$

This means that $K^*(t)/t^{n-c}$ is nonincreasing. The proof is complete. $\square$

The following result shows that there is no essential difference between (47) and (50).

Lemma 7. *The following are equivalent.*

- (a) Equation (50) holds for K .
- (b) There exists a weight K_1 , comparable with K , such that, for some small enough $c > 0$, $K_1(t)/t^{n-c}$ is nonincreasing on $(0, \infty)$.
- (c) Equation (47) holds for K .

Proof. We assume that $K(t) = K(1)t^{n-1}$ for $t \geq 1$.

(a) $\Rightarrow$ (b) is obvious by Lemma 6.

Suppose that (b) holds. We have

$$\int_s^b \frac{K_1(t)}{t^{n+1}} dt \int_0^s \frac{t^{n-1}}{K_1(t)} dt \leq \int_s^\infty t^{-1-c} dt \int_0^s t^{c-1} dt = c^{-2}. \quad (69)$$

We obtain that (50) holds for K since K is comparable with K_1 . It means that (b) $\Rightarrow$ (a) holds.

For (c) $\Rightarrow$ (b), assume that (47) holds for K . We claim that

$$\liminf_{t \rightarrow 0} \frac{K(t)}{t^n} > 0. \quad (70)$$

If $s > 1$, it is clear that

$$\frac{K(1)}{K(1/s)} \leq \varphi_K(s) \quad (71)$$

and by (47)

$$\int_0^1 K(s)^{-1} \frac{ds}{s^{1-n}} = \int_1^\infty K\left(\frac{1}{s}\right)^{-1} \frac{ds}{s^{n+1}} < \infty. \quad (72)$$

Thus, we have

$$\frac{t^n}{K(t)} \leq K(t)^{-1} \int_0^t \frac{ds}{s^{1-n}} \leq \int_0^1 K(s)^{-1} \frac{ds}{s^{1-n}} < \infty, \quad (73)$$

which gives the claim. We define

$$K_1(t) = t^n \int_t^\infty \frac{K(s)}{s^{n+1}} ds, \quad 0 < t < \infty. \quad (74)$$

It is easy to check that $K_1(t)$ is nondecreasing. Since K is nondecreasing, it follows that $K(t) \leq K_1(t)$, $0 < t < \infty$. We note that, for $0 < t < 1$,

$$\begin{aligned} \int_t^1 \frac{K(s)}{s^{n+1}} ds &\leq K(t) \int_t^1 \frac{\varphi_K(s/t)}{s^{n+1}} ds \leq \frac{K(t)}{t^n} \int_1^\infty \frac{\varphi_K(r)}{r^{n+1}} dr, \\ \int_1^\infty \frac{K(s)}{s^{n+1}} ds &\approx K(1) \leq \frac{K(t)}{t^n}. \end{aligned} \quad (75)$$

Thus, we obtain that

$$K_1(t) \leq K(t) \left(\int_1^\infty \frac{\varphi_K(s)}{s^{n+1}} ds + 1 \right), \quad 0 < t < 1. \quad (76)$$

For $t \in [1, \infty)$, we have

$$K_1(t) = t^n \int_t^\infty \frac{K(s)}{s^{n+1}} ds = K(1) t^{n-1} = K(t). \quad (77)$$

Therefore, we get that $K_1 \approx K$ on $(0, \infty)$. Note that if c is sufficiently small, then we have

$$(t^{c-n} K_1(t))' = t^{c-1-n} (cK_1(t) - K(t)) < 0, \quad 0 < t < \infty. \quad (78)$$

This means that $K_1(t)/t^{n-c}$ is nonincreasing.

Suppose that $K_1(t)/t^{n-c}$ is nonincreasing on $(0, \infty)$. For $s \geq 1$,

$$\begin{aligned} \varphi_{K_1}(s) &= \sup_{0 < t \leq 1} \frac{(st)^{n-c} K_1(st) (st)^{c-n}}{K_1(t)} \\ &\leq \sup_{0 < t \leq 1} \frac{(st)^{n-c} K_1(t) t^{c-n}}{K_1(t)} = s^{n-c}, \end{aligned} \quad (79)$$

which gives

$$\int_1^\infty \varphi_{K_1}(s) \frac{ds}{s^{n+1}} < \infty. \quad (80)$$

Thus (b) $\Rightarrow$ (c) holds. $\square$

Lemma 8. *Let K satisfy (49). Then one can find another nonnegative weight function K_1 such that $\mathcal{Q}_{K_1}(\mathbb{R}^n) = \mathcal{Q}_K(\mathbb{R}^n)$, and the new weight function K_1 has the following properties.*

- (a) K_1 is nondecreasing on $(0, \infty)$.
- (b) $K_1 \approx K$ on $(0, \infty)$, and thus K_1 satisfies condition (49).
- (c) For some small enough $c > 0$, $K_1(t)/t^{n-2+c}$ is nondecreasing on $(0, \infty)$. Consequently, $K_1(t)/t^{n-2}$ is also nondecreasing on $(0, \infty)$. Conversely, if $K_1(t)/t^{n-2+c}$ is nondecreasing, for some $c > 0$, then (49) holds for K_1 .
- (d)

$$\sup_{0 < s < \infty} \int_0^s \frac{K_1(t)}{t^{n-1}} dt \int_s^\infty \frac{t^{n-3}}{K_1(t)} dt < \infty. \quad (81)$$

Proof. Assume that $K(t) = K(1)t^{n-1}$ for $t \geq 1$. Define

$$K_1(t) = \begin{cases} t^{n-2} \int_0^t K(s) \frac{ds}{s^{n-1}}, & 0 < t \leq 1, \\ t^{n-1} \int_0^1 K(s) \frac{ds}{s^{n-1}}, & t \geq 1. \end{cases} \quad (82)$$

Note that (49) is a condition for $n > 1$. Then, consider the following.

- (a) is obvious.
 (b) For $0 < t \leq 1$,

$$K_1(t) = \int_0^1 K(st) \frac{ds}{s^{n-1}} \leq K(t) \int_0^1 \varphi_K(s) \frac{ds}{s^{n-1}}. \quad (83)$$

On the other hand, since we always assume that $K(2t) \approx K(t)$, we obtain that

$$K_1(t) \geq t^{n-2} K\left(\frac{t}{2}\right) \int_{t/2}^t \frac{ds}{s^{n-1}} \geq K(t). \quad (84)$$

Thus, $K_1 \approx K$ on $(0, 1)$. For $t \geq 1$, clearly, $K_1(t) \approx K(t)$. Therefore, $K_1 \approx K$ on $(0, \infty)$ and we get that $\mathcal{Q}_{K_1}(\mathbb{R}^n) = \mathcal{Q}_K(\mathbb{R}^n)$.

- (c) If $0 < t \leq 1$, for some small enough $c > 0$,

$$\left(\frac{K_1(t)}{t^{n-2+c}}\right)' = t^{-c-n+1} (K(t) - cK_1(t)) > 0. \quad (85)$$

If $t \geq 1$, $K_1(t)/t^{n-2+c} = K_1(1)t^{1-c}$ is nondecreasing. Thus $K_1(t)/t^{n-2+c}$ is nondecreasing on $(0, \infty)$.

Conversely, if $K_1(t)/t^{n-2+c}$ is nondecreasing, for some $c > 0$, then, for $0 < s \leq 1$,

$$\begin{aligned} \varphi_{K_1}(s) &= \sup_{0 < t \leq 1} \frac{(st)^{n+c-2} K_1(st) (st)^{2-n-c}}{K_1(t)} \\ &\leq \sup_{0 < t \leq 1} \frac{(st)^{n+c-2} K_1(t) t^{2-n-c}}{K_1(t)} = s^{n+c-2}, \end{aligned} \quad (86)$$

which gives

$$\int_0^1 \varphi_{K_1}(s) \frac{ds}{s^{n-1}} < \infty. \quad (87)$$

(d) Note that $K_1(t)/t^{n-2+c}$ is nondecreasing. For $0 < s < \infty$, we have

$$\int_0^s \frac{K_1(t)}{t^{n-1}} dt \int_s^\infty \frac{t^{n-3}}{K_1(t)} dt \leq \int_0^s \frac{1}{t^{1-c}} dt \int_s^\infty \frac{1}{t^{1+c}} dt = c^{-2}. \quad (88)$$

The proof is complete. $\square$

We end this section by giving an example. Fix $0 < \beta < 1$, and set

$$K_\beta(t) = \begin{cases} t^{n+\beta-1}, & 0 < t \leq \frac{1}{e}, \\ \frac{1}{|\log t|}, & 0 < t \leq \frac{1}{e}, \\ e^{-\beta} t^{n-1}, & t > \frac{1}{e}. \end{cases} \quad (89)$$

Since

$$\int_0^{1/e} K_\beta(t) \frac{dt}{t^{n-1}} < \infty, \quad \int_0^{1/e} K_\beta(t) \frac{dt}{t^{n+\beta}} = \infty, \quad (90)$$

we obtain that $\mathcal{Q}_{K_\beta}(\mathbb{R}^n)$ is not trivial and

$$\mathcal{Q}_{K_\beta}(\mathbb{R}^n) \subsetneq \text{BMO}(\mathbb{R}^n). \quad (91)$$

Moreover, a direct calculation shows that (46) and (47) hold for K_β .

4. Carleson-Type Measures

Let I be a cube of $\mathbb{R}^n$ and let $\mathbb{R}_+^{n+1}$ denote the upper half space based on $\mathbb{R}^n$. Define the Carleson box as follows:

$$S(I) = \{(x, t) \in \mathbb{R}_+^{n+1} : x \in I, 0 < t < \ell(I)\}. \quad (92)$$

For $p > 0$ and a positive Borel measure μ on $\mathbb{R}_+^{n+1}$, μ is said to be a p -Carleson measure if

$$\mu(S(I)) \leq M \ell(I)^{pn} \quad (93)$$

for some $M < \infty$ and all cubes $I \subseteq \mathbb{R}^n$.

Denote by $\delta(x)$ the distance of the point $x \in \mathbb{R}_+^{n+1}$ to the boundary $\partial\mathbb{R}_+^{n+1}$. Also $\tilde{y}$ stands for the symmetric point of $y \in \mathbb{R}_+^{n+1}$ with respect to $\mathbb{R}^n$; that is, if $y = (y_1, \dots, y_n, y_{n+1})$, then $\tilde{y} = (y_1, \dots, y_n, -y_{n+1})$.

A positive Borel measure μ is said to be a K -Carleson measure on $\mathbb{R}_+^{n+1}$, as a modification of p -Carleson measure, provided

$$\sup_{I \subseteq \mathbb{R}^n} \int_{S(I)} K\left(\frac{\delta(x)}{\ell(I)}\right) (\delta(x))^{1-n} d\mu(x) < \infty. \quad (94)$$

Clearly, if $K(t) = t^{np}$, then μ is a K -Carleson measure on $\mathbb{R}_+^{n+1}$ if and only if $(\delta(x))^{np+1-n} d\mu(x)$ is a p -Carleson measure on $\mathbb{R}_+^{n+1}$. Now, we give a characterization of K -Carleson measure as follows.

Theorem 9. Let K satisfy (48). Let μ be a positive Borel measure on $\mathbb{R}_+^{n+1}$. Then μ is a K -Carleson measure if and only if

$$\sup_{y \in \mathbb{R}_+^{n+1}} \int_{\mathbb{R}_+^{n+1}} K\left(\frac{\delta(x) (\delta(y))^{1/n}}{|x - \tilde{y}|^{(n+1)/n}}\right) (\delta(x))^{1-n} d\mu(x) < \infty. \quad (95)$$

Proof (sufficiency). Let I be a cube and take y to be the center of the Carleson box $S(I)$. Then $\delta(y) = \ell(I)/2$. If $x \in S(I)$, then $|x - \tilde{y}| \leq \ell(I)$ and hence

$$\begin{aligned} &\int_{\mathbb{R}_+^{n+1}} K\left(\frac{\delta(x) (\delta(y))^{1/n}}{|x - \tilde{y}|^{(n+1)/n}}\right) (\delta(x))^{1-n} d\mu(x) \\ &\geq \int_{S(I)} K\left(\frac{\delta(x) (\delta(y))^{1/n}}{|x - \tilde{y}|^{(n+1)/n}}\right) (\delta(x))^{1-n} d\mu(x) \\ &\geq \int_{S(I)} K\left(\frac{\delta(x)}{\ell(I)}\right) (\delta(x))^{1-n} d\mu(x). \end{aligned} \quad (96)$$

Thus, if (95) holds, then μ is a K -Carleson measure.

Necessity. For $y = (y', y_{n+1}) \in \mathbb{R}_+^{n+1}$, let $I \subseteq \mathbb{R}^n$ be the cube with center y' and edge length $\delta(y)$. Set E_m to be the Carleson box $S(2^m I)$ for each positive integer m . It is clear that

$$\begin{aligned} |x - \tilde{y}| &\geq \delta(y), \quad x \in E_1, \\ |x - \tilde{y}| &\approx 2^m \delta(y), \quad x \in E_{m+1} \setminus E_m. \end{aligned} \quad (97)$$

Then

$$\begin{aligned} & \int_{\mathbb{R}_+^{n+1}} K \left(\frac{\delta(x) (\delta(y))^{1/n}}{|x - \tilde{y}|^{(n+1)/n}} \right) (\delta(x))^{1-n} d\mu(x) \\ &= \int_{E_1} K \left(\frac{\delta(x) (\delta(y))^{1/n}}{|x - \tilde{y}|^{(n+1)/n}} \right) (\delta(x))^{1-n} d\mu(x) \\ &+ \sum_{m=1}^{\infty} \int_{E_{m+1} \setminus E_m} K \left(\frac{\delta(x) (\delta(y))^{1/n}}{|x - \tilde{y}|^{(n+1)/n}} \right) (\delta(x))^{1-n} d\mu(x) \\ &\leq \int_{E_1} K \left(\frac{\delta(x)}{\ell(I)} \right) (\delta(x))^{1-n} d\mu(x) \\ &+ \sum_{m=1}^{\infty} \int_{E_{m+1}} K \left(\frac{\delta(x)}{2^{m/n} 2^{m+1} \ell(I)} \right) (\delta(x))^{1-n} d\mu(x). \end{aligned} \quad (98)$$

Since μ is a K -Carleson measure,

$$\int_{E_m} K \left(\frac{\delta(x)}{2^m \ell(I)} \right) (\delta(x))^{1-n} d\mu(x) \leq 1. \quad (99)$$

This together with (48) yields

$$\begin{aligned} & \int_{\mathbb{R}_+^{n+1}} K \left(\frac{\delta(x) (\delta(y))^{1/n}}{|x - \tilde{y}|^{(n+1)/n}} \right) (\delta(x))^{1-n} d\mu(x) \\ &\leq \sum_{m=1}^{\infty} \varphi_K \left(\frac{1}{2^{m/n}} \right) \approx \int_0^1 \varphi_K(s) \frac{ds}{s} < \infty. \end{aligned} \quad (100)$$

□

Let f be a measurable function on $\mathbb{R}^n$ satisfying

$$\int_{\mathbb{R}^n} \frac{|f(x)|}{1 + |x|^{n+1}} dx < \infty. \quad (101)$$

Its Poisson integral is defined by

$$f(x, t) = \int_{\mathbb{R}^n} P_t(x - y) f(y) dy, \quad (102)$$

where

$$P_t(x) = \frac{c_n t}{(t^2 + |x|^2)^{(n+1)/2}}, \quad c_n = \frac{\Gamma((n+1)/2)}{\pi^{(n+1)/2}}. \quad (103)$$

The gradient of $f(x, t)$ is

$$\nabla f(x, t) = \left(\frac{\partial f(x, t)}{\partial x_1}, \dots, \frac{\partial f(x, t)}{\partial x_n}, \frac{\partial f(x, t)}{\partial t} \right). \quad (104)$$

It is known that (101) holds for $f \in \text{BMO}(\mathbb{R}^n)$ (see [9]).

The following main theorem generalizes the result of $\mathcal{Q}_\alpha(\mathbb{R}^n)$ in [10].

Theorem 10. Suppose that (47) and (81) hold for K . Let $f \in L_{\text{loc}}^2(\mathbb{R}^n)$ with (101). Then $f \in \mathcal{Q}_K(\mathbb{R}^n)$ if and only if $|\nabla f(x, t)|^2 dx dt$ is a K -Carleson measure.

In order to prove Theorem 10, we need the following Hardy-type inequalities.

Lemma 11 (see [18]). Let $0 < b \leq \infty$, $1 < p < \infty$, and $p' = (p/(p-1))$. Assume that functions μ and ν are measurable and nonnegative in the interval $(0, b)$. Then

$$\int_0^b \left(\int_0^s f(t) dt \right)^p \mu(s) ds \leq C \int_0^b f^p(s) \nu(s) ds \quad (105)$$

holds for all measurable functions $f \geq 0$ if and only if

$$\begin{aligned} A &:= \sup_{0 < s < b} \left(\int_s^b \mu(t) dt \right)^{1/p} \left(\int_0^s (\nu(t))^{1-p'} dt \right)^{1/p'} < \infty, \\ \int_0^b \left(\int_s^b f(t) dt \right)^p \mu(s) ds &\leq C \int_0^b f^p(s) \nu(s) ds \end{aligned} \quad (106)$$

holds for all measurable functions $f \geq 0$ if and only if

$$B := \sup_{0 < s < b} \left(\int_0^s \mu(t) dt \right)^{1/p} \left(\int_s^b (\nu(t))^{1-p'} dt \right)^{1/p'} < \infty. \quad (107)$$

Here C depends only on p , A , or B .

The following Stegenga-type estimate will be used in the proof of Theorem 10.

Lemma 12. Suppose that (81) holds for K and

$$\sup_{0 < s < 1} \int_s^1 \frac{K(t)}{t^{3n+1}} dt \int_0^s \frac{t^{3n-1}}{K(t)} dt < \infty. \quad (108)$$

Let I and J be cubes in $\mathbb{R}^n$ centered at x_0 with $\ell(J) = 3\ell(I)$ and let $f \in L_{\text{loc}}^1(\mathbb{R}^n)$ satisfy (101). Then, there is a constant C independent of f , I , and J , such that

$$\begin{aligned} & \int_{S(I)} |\nabla f(x, t)|^2 K \left(\frac{t}{\ell(I)} \right) t^{1-n} dx dt \\ &\leq C \int_{|y| \leq \sqrt{n}\ell(J)} \int_J |f(x+y) - f(x)|^2 \\ &\quad \times K \left(\frac{|y|}{\ell(J)} \right) \frac{1}{|y|^{2n}} dx dy \\ &+ C \left(\int_0^{\sqrt{n}} \frac{K(s)}{s^{n-1}} ds + \int_{1/8}^\infty \frac{K(s)}{s^{n+1}} ds \right) \\ &\times |J|^{-1} \int_J |f(x) - f_J|^2 dx \\ &+ C \int_0^{\sqrt{n}} \frac{K(s)}{s^{n-1}} ds \left(\ell(J) \int_{\mathbb{R}^n \setminus (2/3)J} \frac{|f(x) - f_J|}{|x - x_0|^{n+1}} dx \right)^2. \end{aligned} \quad (109)$$

Proof. Without loss of generality, we may assume that $x_0 = 0$. Let φ be a function with $0 \leq \varphi \leq 1$ such that $\varphi = 1$ on $(2/3)J$, $\text{supp } \varphi \subseteq (3/4)J$, and

$$|\varphi(x) - \varphi(y)| \leq \ell(J)^{-1} |x - y|, \quad x, y \in \mathbb{R}^n. \quad (110)$$

Following Stegenga [19], we write

$$f = f_J + (f - f_J)\varphi + (f - f_J)(1 - \varphi) = f_1 + f_2 + f_3. \quad (111)$$

Then we have

$$f(x, t) = f_1(x, t) + f_2(x, t) + f_3(x, t) \quad (112)$$

for the corresponding Poisson integrals. Since f_1 is constant, it contributes nothing to the integral with the gradient square.

Note that

$$\begin{aligned} \frac{\partial P_s(y)}{\partial y_j} &= -(n+1) c_n s y_j (s^2 + |y|^2)^{-(n+3)/2}, \\ j &= 1, \dots, n. \end{aligned} \quad (113)$$

We obtain

$$\int_{\mathbb{R}^n} \frac{\partial P_s(y)}{\partial y_j} dy = 0, \quad j = 1, \dots, n. \quad (114)$$

Hence,

$$\begin{aligned} \frac{\partial f(x, s)}{\partial x_j} &= \int_{\mathbb{R}^n} \frac{\partial P_s(x - y)}{\partial x_j} f(y) dy \\ &= \int_{\mathbb{R}^n} (n+1) c_n s (y_j - x_j) \\ &\quad \times (s^2 + |y - x|^2)^{-(n+3)/2} f(y) dy \\ &= - \int_{\mathbb{R}^n} \frac{\partial P_s(y)}{\partial y_j} f(x + y) dy \\ &= \int_{\mathbb{R}^n} \frac{\partial P_s(y)}{\partial y_j} (f(x) - f(x + y)) dy. \end{aligned} \quad (115)$$

A direct calculation shows that

$$\begin{aligned} \left| \frac{\partial P_s(y)}{\partial y_j} \right| &\approx s |y_j| (s^2 + |y|^2)^{-(n+3)/2} \\ &\lesssim (s^2 + |y|^2)^{-(n+1)/2}, \end{aligned} \quad (116)$$

which gives

$$\left| \frac{\partial P_s(y)}{\partial y_j} \right| \lesssim s^{-n-1}, \quad \left| \frac{\partial P_s(y)}{\partial y_j} \right| \lesssim |y|^{-n-1}. \quad (117)$$

Therefore,

$$\begin{aligned} \left\| \frac{\partial f(x, s)}{\partial x_j} \right\|_{L^2(\mathbb{R}^n)} &\lesssim s^{-n-1} \int_{|y| \leq s} \|f(x + y) - f(x)\|_{L^2(\mathbb{R}^n)} dy \\ &\quad + \int_{|y| > s} \|f(x + y) - f(x)\|_{L^2(\mathbb{R}^n)} |y|^{-n-1} dy. \end{aligned} \quad (118)$$

We write $y = r\xi \in \mathbb{R}^n$ with $r = |y|$ and $|\xi| = 1$. Let

$$A(r) = \int_{|\xi|=1} \|f(x + r\xi) - f(x)\|_{L^2(\mathbb{R}^n)} d\xi. \quad (119)$$

Then

$$\left\| \frac{\partial f(x, s)}{\partial x_j} \right\|_{L^2(\mathbb{R}^n)} \lesssim s^{-n-1} \int_0^s A(r) r^{n-1} dr + \int_s^\infty A(r) r^{-2} dr. \quad (120)$$

Thus,

$$\begin{aligned} \int_{S(I)} \left| \frac{\partial f(x, s)}{\partial x_j} \right|^2 K\left(\frac{s}{\ell(I)}\right) s^{1-n} dx ds &\leq \int_0^{\ell(I)} \left\| \frac{\partial f(x, s)}{\partial x_j} \right\|_{L^2(\mathbb{R}^n)}^2 K\left(\frac{s}{\ell(I)}\right) s^{1-n} ds \\ &\leq \int_0^{\ell(I)} K\left(\frac{s}{\ell(I)}\right) s^{-3n-1} \left(\int_0^s A(r) r^{n-1} dr \right)^2 ds \\ &\quad + \int_0^{\ell(I)} K\left(\frac{s}{\ell(I)}\right) s^{1-n} \left(\int_s^\infty A(r) r^{-2} dr \right)^2 ds \\ &\approx \int_0^1 K(s) \frac{1}{s^{3n+1} \ell(I)^n} \left(\int_0^s A(\ell(I)r) r^{n-1} dr \right)^2 ds \\ &\quad + \int_0^1 K(s) \frac{1}{s^{n-1} \ell(I)^n} \left(\int_s^\infty A(\ell(I)r) r^{-2} dr \right)^2 ds. \end{aligned} \quad (121)$$

Note that

$$\sup_{0 < s < 1} \int_s^1 \frac{K(t)}{t^{3n+1}} dt \int_0^s \frac{t^{3n-1}}{K(t)} dt < \infty. \quad (122)$$

By Lemma 11,

$$\begin{aligned} \int_0^1 K(s) \frac{1}{s^{3n+1}} \left(\int_0^s A(\ell(I)r) r^{n-1} dr \right)^2 ds &\lesssim \int_0^1 K(s) \frac{1}{s^{n+1}} A^2(\ell(I)s) ds. \end{aligned} \quad (123)$$

Since (81) holds for K , by Lemma 11 again,

$$\begin{aligned} \int_0^1 K(s) \frac{1}{s^{n-1}} \left(\int_s^\infty A(\ell(I)r) r^{-2} dr \right)^2 ds &\lesssim \int_0^\infty K(s) \frac{1}{s^{n+1}} A^2(\ell(I)s) ds. \end{aligned} \quad (124)$$

Therefore,

$$\begin{aligned} & \int_{S(I)} \left| \frac{\partial f(x, s)}{\partial x_j} \right|^2 K\left(\frac{s}{\ell(I)}\right) s^{1-n} dx ds \\ & \leq \int_0^\infty K\left(\frac{s}{\ell(I)}\right) s^{-1-n} A^2(s) ds. \end{aligned} \quad (125)$$

By Hölder's inequality,

$$\begin{aligned} & \int_{S(I)} \left| \frac{\partial f(x, s)}{\partial x_j} \right|^2 K\left(\frac{s}{\ell(I)}\right) s^{1-n} dx ds \\ & \leq \int_0^\infty \int_{|\xi|=1} \|f(x + s\xi) - f(x)\|_{L^2(\mathbb{R}^n)}^2 \\ & \quad \times K\left(\frac{s}{\ell(I)}\right) s^{-1-n} d\xi ds \\ & \approx \int_{\mathbb{R}^n} \left[\int_{\mathbb{R}^n} |f(x + y) - f(x)|^2 dx \right] \\ & \quad \times K\left(\frac{|y|}{\ell(I)}\right) |y|^{-2n} dy. \end{aligned} \quad (126)$$

Note that

$$\frac{\partial P_s(y)}{\partial s} = c_n [s^2 + |y|^2 - s^2(n+1)] (s^2 + |y|^2)^{-(n+3)/2}. \quad (127)$$

Hence,

$$\left| \frac{\partial P_s(y)}{\partial s} \right| \leq (s^2 + |y|^2) (s^2 + |y|^2)^{-(n+3)/2}. \quad (128)$$

It follows that

$$\left| \frac{\partial P_s(y)}{\partial s} \right| \leq s^{-n-1}, \quad \left| \frac{\partial P_s(y)}{\partial s} \right| \leq |y|^{-n-1}. \quad (129)$$

Since

$$\int_{\mathbb{R}^n} P_s(y) dy = 1, \quad (130)$$

this deduces that

$$\int_{\mathbb{R}^n} \frac{\partial P_s(y)}{\partial s} dy = 0. \quad (131)$$

We have

$$\frac{\partial f(x, s)}{\partial s} = \int_{\mathbb{R}^n} \frac{\partial P_s(y)}{\partial s} (f(x + y) - f(x)) dy. \quad (132)$$

Repeating the procedure above, we also can obtain

$$\begin{aligned} & \int_{S(I)} \left| \frac{\partial f(x, s)}{\partial s} \right|^2 K\left(\frac{s}{\ell(I)}\right) s^{1-n} dx ds \\ & \leq \int_{\mathbb{R}^n} \left[\int_{\mathbb{R}^n} |f(x + y) - f(x)|^2 dx \right] K\left(\frac{|y|}{\ell(I)}\right) |y|^{-2n} dy. \end{aligned} \quad (133)$$

Therefore,

$$\begin{aligned} & \int_{S(I)} |\nabla f_2(x, s)|^2 K\left(\frac{s}{\ell(I)}\right) s^{1-n} dx ds \\ & \leq \iint_{\mathbb{R}^n} |f_2(x + y) - f_2(x)|^2 K\left(\frac{|y|}{\ell(I)}\right) |y|^{-2n} dx dy \\ & \approx \iint_{\mathbb{R}^n} \frac{|f_2(x) - f_2(y)|^2}{|x - y|^{2n}} K\left(\frac{|x - y|}{\ell(I)}\right) dx dy \\ & \approx \int \int_{x, y \in J} \cdots + \int \int_{x \notin J, y \in 3/4J} \cdots + \int \int_{y \notin J, x \in 3/4J} \\ & \approx B_1 + B_2 + B_3. \end{aligned} \quad (134)$$

To estimate B_1 , we note that

$$|\varphi(x) - \varphi(y)| \leq \ell(J)^{-1} |x - y|, \quad x, y \in \mathbb{R}^n. \quad (135)$$

Thus

$$\begin{aligned} |f_2(x) - f_2(y)| & \leq |f(x) - f(y)| \\ & + (\ell(J))^{-1} |x - y| |f(y) - f_J|. \end{aligned} \quad (136)$$

We have

$$\begin{aligned} & \iint_J \frac{|f(x) - f(y)|^2}{|x - y|^{2n}} K\left(\frac{|x - y|}{\ell(I)}\right) dx dy \\ & \leq \int_{|y| \leq \sqrt{n}\ell(J)} K\left(\frac{|y|}{\ell(J)}\right) |y|^{-2n} \\ & \quad \times \int_J |f(x + y) - f(x)|^2 dx dy, \\ & (\ell(J))^{-2} \iint_J \frac{|f(y) - f_J|^2}{|x - y|^{2n-2}} K\left(\frac{|x - y|}{\ell(I)}\right) dx dy \\ & \leq (\ell(J))^{-2} \int_{|y| \leq \sqrt{n}\ell(J)} K\left(\frac{|y|}{\ell(J)}\right) |y|^{2-2n} \\ & \quad \times \int_J |f(x + y) - f_J|^2 dx dy \\ & \leq \int_{|y| \leq \sqrt{n}\ell(J)} K\left(\frac{|y|}{\ell(J)}\right) |y|^{-2n} \\ & \quad \times \int_J |f(x + y) - f(x)|^2 dx dy \\ & + (\ell(J))^{-2} \int_{|y| \leq \sqrt{n}\ell(J)} K\left(\frac{|y|}{\ell(J)}\right) |y|^{2-2n} \\ & \quad \times \int_J |f(x) - f_J|^2 dx dy \end{aligned}$$

$$\begin{aligned}
&\leq \int_{|y| \leq \sqrt{n}\ell(I)} K\left(\frac{|y|}{\ell(I)}\right) |y|^{-2n} \\
&\quad \times \int_J |f(x+y) - f(x)|^2 dx dy \\
&\quad + \int_0^{\sqrt{n}} \frac{K(s)}{s^{n-1}} ds |J|^{-1} \int_J |f(x) - f_J|^2 dx.
\end{aligned} \tag{137}$$

Hence,

$$\begin{aligned}
B_1 &\leq \int_{|y| \leq \sqrt{n}\ell(I)} K\left(\frac{|y|}{\ell(I)}\right) |y|^{-2n} \int_J |f(x+y) - f(x)|^2 dx dy \\
&\quad + \int_0^{\sqrt{n}} \frac{K(s)}{s^{n-1}} ds |J|^{-1} \int_J |f(x) - f_J|^2 dx.
\end{aligned} \tag{138}$$

To handle B_2 , note that

$$\begin{aligned}
|x-y| &> \frac{1}{8}\ell(I), \quad |f_2(x) - f_2(y)| \leq |f(y) - f_J|, \\
&\quad x \notin J, \quad y \in 3/4J.
\end{aligned} \tag{139}$$

We obtain

$$\begin{aligned}
B_2 &= \int_{x \notin J} \int_{y \in 3/4J} \frac{|f_2(x) - f_2(y)|^2}{|x-y|^{2n}} K\left(\frac{|x-y|}{\ell(I)}\right) dx dy \\
&\leq \int_{y \in 3/4J} |f(y) - f_J|^2 dy \int_{|z| > (1/8)\ell(I)} K\left(\frac{|z|}{\ell(I)}\right) |z|^{-2n} dz \\
&\leq \int_{1/8}^{\infty} \frac{K(s)}{s^{n+1}} ds |J|^{-1} \int_J |f(y) - f_J|^2 dy.
\end{aligned} \tag{140}$$

Similarly,

$$B_3 \leq \int_{1/8}^{\infty} \frac{K(s)}{s^{n+1}} ds |J|^{-1} \int_J |f(y) - f_J|^2 dy. \tag{141}$$

Moreover, if $(x, s) \in S(I)$ and $y \in \mathbb{R}^n \setminus (2/3)J$, then

$$\frac{1}{(s+|x-y|)^{n+1}} \leq |y|^{-n-1}, \tag{142}$$

and, by $|\nabla P_s(y)| \leq (s+|y|)^{-n-1}$, we have

$$\begin{aligned}
&\int_{S(I)} |\nabla f_3(x, s)|^2 K\left(\frac{s}{\ell(I)}\right) s^{1-n} dx ds \\
&\leq \int_{S(I)} \left(\int_{\mathbb{R}^n} |\nabla P_s(x-y)| |f_3(y)| dy \right)^2 \\
&\quad \times K\left(\frac{s}{\ell(I)}\right) s^{1-n} dx ds
\end{aligned}$$

$$\begin{aligned}
&\leq \int_{S(I)} \left(\int_{\mathbb{R}^n \setminus (2/3)J} \frac{|f(y) - f_J|}{(s+|x-y|)^{n+1}} dy \right)^2 \\
&\quad \times K\left(\frac{s}{\ell(I)}\right) s^{1-n} dx ds \\
&\leq \int_0^1 \frac{K(s)}{s^{n-1}} ds \left(\ell(I) \int_{\mathbb{R}^n \setminus (2/3)J} \frac{|f(y) - f_J|}{|y|^{n+1}} dy \right)^2.
\end{aligned} \tag{143}$$

Combining the inequalities above, Lemma 12 is proved. $\square$

Proof of Theorem 10. We assume that $K(t) = K(1)t^{n-1}$ for $t \geq 1$. By Lemma 7, K satisfies (47) if and only if

$$\sup_{0 < s < 1} \int_s^1 \frac{K(t)}{t^{n+1}} dt \int_0^s \frac{t^{n-1}}{K(t)} dt < \infty. \tag{144}$$

Sufficiency. Let I be a cube and $f(x, t)$ is the Poisson integral of f . Note that

$$f(x, |y|) - f(x) = \int_0^{|y|} \frac{\partial f(x, t)}{\partial t} dt. \tag{145}$$

By Minkowski's inequality, we have

$$\begin{aligned}
&\left(\int_I |f(x, |y|) - f(x)|^2 dx \right)^{1/2} \\
&\leq \int_0^{|y|} \left(\int_I \left| \frac{\partial f(x, t)}{\partial t} \right|^2 dx \right)^{1/2} dt \\
&\leq \int_0^{|y|} \left(\int_I |\nabla f(x, t)|^2 dx \right)^{1/2} dt.
\end{aligned} \tag{146}$$

Hence,

$$\begin{aligned}
&\int_{|y| < \ell(I)} K\left(\frac{|y|}{\ell(I)}\right) \frac{1}{|y|^{2n}} \left(\int_I |f(x, |y|) - f(x)|^2 dx \right) dy \\
&\leq (\ell(I))^{2-n} \int_0^1 \frac{K(r)}{r^{n+1}} \left(\int_0^r \left(\int_I |\nabla f(x, \ell(I)s)|^2 dx \right)^{1/2} ds \right)^2 dr \\
&\leq \int_{S(I)} |\nabla f(x, t)|^2 K\left(\frac{t}{\ell(I)}\right) t^{1-n} dx dt.
\end{aligned} \tag{147}$$

The last inequality above holds by Lemma 11 since K satisfies

$$\sup_{0 < s < 1} \int_s^1 \frac{K(t)}{t^{n+1}} dt \int_0^s \frac{t^{n-1}}{K(t)} dt < \infty. \tag{148}$$

Thus,

$$\begin{aligned}
&\sup_I \int_{|y| < \ell(I)} K\left(\frac{|y|}{\ell(I)}\right) \frac{1}{|y|^{2n}} \\
&\quad \times \left(\int_I |f(x, |y|) - f(x)|^2 dx \right) dy < \infty.
\end{aligned} \tag{149}$$

For $|y| < \ell(I)$,

$$\begin{aligned} & \int_I |f(x+y, |y|) - f(x+y)|^2 dx \\ & \leq \int_{3I} |f(x, |y|) - f(x)|^2 dx. \end{aligned} \quad (150)$$

Similarly, we get

$$\begin{aligned} & \sup_I \int_{|y| < \ell(I)} K\left(\frac{|y|}{\ell(I)}\right) \frac{1}{|y|^{2n}} \\ & \times \left(\int_I |f(x+y, |y|) - f(x+y)|^2 dx \right) dy < \infty. \end{aligned} \quad (151)$$

Note that

$$\begin{aligned} & |f(x+y, |y|) - f(x, |y|)| \\ & \leq \int_0^{|y|} |\nabla f(x+te_y, |y|)| dt, \\ & e_y = \frac{y}{|y|}. \end{aligned} \quad (152)$$

When $|y| < \ell(I)$, we employ Minkowski's inequality to get

$$\begin{aligned} & \left(\int_I |f(x+y, |y|) - f(x, |y|)|^2 dx \right)^{1/2} \\ & \leq \int_0^{|y|} \left(\int_I |\nabla f(x+te_y, |y|)|^2 dx \right)^{1/2} dt \\ & \leq |y| \left(\int_{3I} |\nabla f(x, |y|)|^2 dx \right)^{1/2}. \end{aligned} \quad (153)$$

Hence,

$$\begin{aligned} & \sup_I \int_{|y| < \ell(I)} K\left(\frac{|y|}{\ell(I)}\right) \frac{1}{|y|^{2n}} \\ & \times \left(\int_I |f(x+y, |y|) - f(x, |y|)|^2 dx \right) dy \\ & \leq \sup_I \int_{S(I)} |\nabla f(x, t)|^2 K\left(\frac{t}{\ell(I)}\right) t^{1-n} dx dt < \infty. \end{aligned} \quad (154)$$

By the triangle inequality, we get

$$\begin{aligned} & \sup_I \int_{|y| < \ell(I)} K\left(\frac{|y|}{\ell(I)}\right) \frac{1}{|y|^{2n}} \\ & \times \left(\int_I |f(x+y) - f(x)|^2 dx \right) dy < \infty. \end{aligned} \quad (155)$$

Thus $f \in \mathcal{Q}_K(\mathbb{R}^n)$.

Necessity. Note that

$$\begin{aligned} & \sup_{0 < s < 1} \int_s^1 \frac{K(t)}{t^{3n+1}} dt \int_0^s \frac{t^{3n-1}}{K(t)} dt \\ & \leq \sup_{0 < s < 1} \int_s^1 \frac{K(t)}{t^{n+1}} dt \int_0^s \frac{t^{n-1}}{K(t)} dt. \end{aligned} \quad (156)$$

Thus K satisfies the conditions of Lemma 12. Let I and J be cubes in $\mathbb{R}^n$ centered at x_0 with $\ell(J) = 3\ell(I)$. Since $f \in \mathcal{Q}_K(\mathbb{R}^n) \subseteq \text{BMO}(\mathbb{R}^n)$, by [10, p. 590], we have

$$\ell(J) \int_{\mathbb{R}^n \setminus (2/3)J} \frac{|f(x) - f_J|}{|x - x_0|^{n+1}} dx \leq \|f\|_{\text{BMO}(\mathbb{R}^n)}. \quad (157)$$

Note that

$$\begin{aligned} & |J|^{-1} \int_J |f(x) - f_J|^2 dx \\ & \leq \int_{|y| \leq \sqrt{n}\ell(J)} \int_J |f(x+y) - f(x)|^2 K\left(\frac{|y|}{\ell(J)}\right) \frac{1}{|y|^{2n}} dx dy. \end{aligned} \quad (158)$$

Lemma 12 gives

$$\begin{aligned} & \int_{S(I)} |\nabla f(x, t)|^2 K\left(\frac{t}{\ell(I)}\right) t^{1-n} dx dt \\ & \leq \|f\|_{\mathcal{Q}_K(\mathbb{R}^n)}^2 + \|f\|_{\text{BMO}(\mathbb{R}^n)}^2. \end{aligned} \quad (159)$$

The proof of Theorem 10 is complete. $\square$

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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