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Research Article

Nonfragile H_∞ Control for Stochastic Systems with Markovian Jumping Parameters and Random Packet Losses

Jing Wang[1] and Ke Zhang[2]

[1] *School of Electrical and Information Engineering, Anhui University of Technology, Ma'anshan 243002, China*
[2] *School of Information Science and Engineering, Northeastern University, Shenyang 110819, China*

Correspondence should be addressed to Jing Wang; jingwang08@126.com

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This paper is concerned with the nonfragile H_∞ control problem for stochastic systems with Markovian jumping parameters and random packet losses. The communication between the physical plant and controller is assumed to be imperfect, where random packet losses phenomenon occurs in a random way. Such a phenomenon is represented by a stochastic variable satisfying the Bernoulli distribution. The purpose is to design a nonfragile controller such that the resulting closed-loop system is stochastically mean square stable with a guaranteed H_∞ performance level γ. By using the Lyapunov function approach, some sufficient conditions for the solvability of the previous problem are proposed in terms of linear matrix inequalities (LMIs), and a corresponding explicit parametrization of the desired controller is given. Finally, an example illustrating the effectiveness of the proposed approach is presented.

1. Introduction

During the past several decades, stochastic systems have been the main focus of research receiving much attention since realistic models of most engineering systems involve random exogenous disturbances [1, 2]. As a simple yet significant mathematical model, stochastic systems have come to play a key role in many branches of science and engineering [3]. For this reason, many fundamental issues have been extensively addressed for stochastic systems, and consequently fruitful results have been presented in the literature; see, for example, [2, 4, 5] and the references therein.

In addition to stochastic systems, there have been great efforts in the research of the modeling of dynamic systems subject to random abrupt changes in their parameters [6–8]. Such random abrupt changes may be caused by various factors, including the switching between economic scenarios, abrupt changes in the operation point for nonlinear plant, and actuator/sensor failure or repairs, to name just a few. Fortunately, Markov jump systems provide a natural framework for modeling these practical systems subject to random abrupt changes. Since the pioneering work on Markov jump systems was introduced in [9], considerable research results related to Markov jump systems or system with Markovian jumping parameters have been presented in terms of a variety of methods. For more details, we refer to the literature [10–17]. When Markovian jumping parameters appear in stochastic systems, many control issues have been studied recently by researchers. For example, robust stability and stabilization problems were investigated in [18], passivity-based control problem was addressed in [19], optimal control problems were studied in [12, 20–22], and the sliding-mode control problem was solved in [23].

It is worth noting that the controller design methods proposed in the previous literature require two critical assumptions. One is that the controller can be implemented exactly, and the other is that the communication between the physical plant and controller is always perfect. Such two assumptions, however, may not be unreasonable in practice. Firstly, in the implementation of a design controller, uncertainties or inaccuracies do occur because of round-off errors in numerical computation. Some existing control

synthesis methods have proven to be sensitive, or fragile, with respect to small perturbations in controller parameters. Therefore, it is an important question to design a controller, which guarantees that the controller is insensitive to some amount of errors with its gain, that is, the nonfragile or resilient control problem [24]. Secondly, a modern control system can hardly work without the help of the networks and the computers and their intercommunication. They bring a lot of advantages, but the existence of network-induced phenomena is unavoidable [25]. For instance, packet losses may occur due to the unreliability of the network links. These may limit the scope of the applications of the existing results related to stochastic systems with Markovian jumping parameters. The main purpose of this paper, therefore, is to shorten such a gap.

In this paper, we make the first attempt to deal with the nonfragile H_∞ control for stochastic systems with Markovian jumping parameters and random packet losses. The packet losses phenomena are assumed to exist in communication links between the physical plant and controller. Attention is focused on the design of a nonfragile controller such that the resulting closed-loop system is stochastically mean square stable, and meanwhile a prescribed H_∞ performance is satisfied. Sufficient conditions for the existence of such a controller are given in terms of LMIs. By solving a convex optimization problem, a desired nonfragile controller can be constructed based on the use of standard numerical algorithms [26].

Notation. Throughout this paper, for symmetric matrices P, the notation $P \geq 0$ (resp., $P > 0$) means that the matrix P is positive semidefinite (resp., positive definite); I and 0 represent the identity matrix and zero matrix with appropriate dimension. The notation M^T represents the transpose of the matrix M; $\operatorname{diag}\{\cdots\}$ stands for a block-diagonal matrix. In symmetric block matrices or complex matrix expressions, we employ an asterisk $(*)$ to represent a term that is induced by symmetry; $\mathbb{E}\{\cdot\}$ denotes the expectation operator with respect to some probability measure $\mathscr{P}$; $(\Omega, \mathscr{F}, \mathscr{P})$ is a probability space; Ω is the sample space, $\mathscr{F}$ is the σ-algebra of subsets of the sample space, and $\mathscr{P}$ is the probability measure on $\mathscr{F}$; $l_2[0,\infty)$ is the space of square-summable infinite vector sequences over $[0,\infty)$; $|\cdot|$ refers to the Euclidean vector norm; $\|\cdot\|_2$ stands for the usual $l_2[0,\infty)$ norm. Matrices, if not explicitly stated, are assumed to have compatible dimensions. Z^+ represents $\{0,1,2,\ldots\}$.

2. Problem Formulations

Consider the following discrete-time stochastic Markov jump system over a probability space $(\Omega, \mathscr{F}, \mathscr{P})$:

$$\begin{aligned}(\Sigma): x(k+1) &= A(\delta_k)x(k) + B_1(\delta_k)u(k) + C(\delta_k)v(k) \\ &\quad + \left[E(\delta_k)x(k) + F(\delta_k)v(k)\right]\omega(k), \\ z(k) &= D(\delta_k)x(k) + B_2(\delta_k)u(k) + G(\delta_k)v(k),\end{aligned} \tag{1}$$

where $x(k) \in \mathbb{R}^n$ is the system state vector; $u(k) \in \mathbb{R}^m$ is the controlled input; $z(k) \in \mathbb{R}^p$ is the controlled output; $v(k) \in \mathbb{R}^q$ is the exogenous disturbance input that belongs to $l_2[0,\infty)$. For each δ_k, $A(\sigma_k)$, $B_1(\sigma_k)$, $C(\sigma_k)$, $E(\sigma_k)$, $F(\sigma_k)$, $D(\sigma_k)$, $B_2(\sigma_k)$, and $G(\sigma_k)$ are real constant matrices with appropriate dimensions. $\omega(k)$ is a one-dimensional zero mean Gaussian white noise sequence on a probability space $(\Omega, \mathscr{F}, \mathscr{P})$ with

$$\begin{aligned}&\mathbb{E}\{\omega(k)\} = 0; \qquad \mathbb{E}\left\{\omega^2(k)\right\} = 1; \\ &\mathbb{E}\{\omega(l)\omega(k)\} = 0, \quad l \neq k.\end{aligned} \tag{2}$$

In system (Σ), the system mode switching is governed by a discrete-time homogeneous Markov chain $\{\delta_k\}$ $(k \in Z^+)$, which takes values in a finite state space $\mathscr{S} = \{1,2,\ldots,\mathscr{N}\}$ with transition probability matrix $\Pi \triangleq \{\psi_{\alpha\beta}\}$ given by

$$\psi_{\alpha\beta} \triangleq \Pr\{\delta_{k+1} = \beta \mid \delta_k = \alpha\} \geq 0, \quad \forall \alpha, \beta \in \mathscr{S}, k \in Z^+, \tag{3}$$

with $0 \leq \psi_{\alpha\beta} \leq 1$, for any $\alpha, \beta \in \mathscr{S}$, and

$$\sum_{\beta=1}^{\mathscr{N}} \psi_{\alpha\beta} = 1, \quad \alpha \in \mathscr{S}. \tag{4}$$

In practice, it is usually of importance to require very accurate controllers to achieve given engineering specifications. However, the resulting closed-loop systems are sensitive to changes in controller gain. In this case, once there are some small perturbations in the controller parameters, the existence of these perturbations may cause a serious deterioration of system performance. Hence, it is imperative to consider the design of nonfragile controllers [24]. Consequently, in this paper, we are interested in designing the controller in the following form:

$$u_{oc} = \left(K_\alpha + \Delta K_\alpha(k)\right)x_{ic}(k), \quad \alpha \in \mathscr{S}, \tag{5}$$

where $x_{ic}(k)$ is the input of the controller; u_{oc} is the output of the controller; K_α are the gain matrices of the controller, which will be determined; $\Delta K_\alpha(k)$ are real-valued unknown matrices denoting the additive gain variations as follows:

$$\Delta K_\alpha(k) = M_\alpha H_\alpha(k) N_\alpha, \quad \alpha \in \mathscr{S}, \tag{6}$$

where M_α and N_α, $\alpha \in \mathscr{S}$, are the known real constant matrices of appropriate dimensions and $H_\alpha(k)$ are unknown time-varying matrix functions, which satisfy the following constraint:

$$H_\alpha^T(k) H_\alpha(k) \leq I, \quad \alpha \in \mathscr{S}. \tag{7}$$

Remark 1. Normally, under the implicit assumption that the communication between the plant and the controller is perfect, one can readily get that the controlled input $u(k)$ is equivalent to the the output of controller u_{oc} and the measurement state of the plant $x(k)$ is also equivalent to the input of the controller $x_{ic}(k)$. As noted in the previous section, such an assumption is sometimes unpractical especially under networked environments because of the existence of the packet losses.

Therefore, in this paper, the packet losses phenomena are considered in communication links. As a result, one has $u(k) \neq u_{oc}$ and $x(k) \neq x_{ic}(k)$, and the relations between them are modeled by using a stochastic method as follows:

$$x_{ic}(k) = \rho_k x(k), \qquad u(k) = \sigma_k u_{oc}. \tag{8}$$

Here, $\{\rho_k\}$ and $\{\sigma_k\}$ are two independent Bernoulli processes. As shown in (8), $\{\rho_k\}$ models the unreliable communication link from the sensor to the controller and $\{\sigma_k\}$ models the unreliable communication link from the controller to the actuator. Inspired by [27], a natural assumption on $\{\rho_k\}$ and $\{\sigma_k\}$ can be made as follows:

$$\begin{aligned} &\operatorname{Prob}\{\rho_k = 1\} = \mathbb{E}\{\rho_k\} = \overline{\rho}, \qquad \operatorname{Prob}\{\rho_k = 0\} = 1 - \overline{\rho}, \\ &\operatorname{Prob}\{\sigma_k = 1\} = \mathbb{E}\{\sigma_k\} = \overline{\sigma}, \qquad \operatorname{Prob}\{\sigma_k = 0\} = 1 - \overline{\sigma}, \end{aligned} \tag{9}$$

where either $\overline{\rho}$ or $\overline{\sigma}$ is a known constant satisfying $\overline{\rho} \in [0,1]$ and $\overline{\sigma} \in [0,1]$. Clearly, for $\{\rho_k\}$, when $\overline{\rho} = 0$ (resp., $\overline{\rho} = 1$), it means that the communication link from the sensor to the controller fails (resp., successful transmission), and $\{\sigma_k\}$ also has a similar inference. Throughout this paper, we also assume that the sequences $\omega(k)$, $\{\delta_k\}$, $\{\rho_k\}$, and $\{\sigma_k\}$ are mutually independent. Clearly, one can get

$$u(k) = \sigma_k \rho_k \left(K_\alpha + M_\alpha H_\alpha(k) N_\alpha\right) x(k). \tag{10}$$

In order to address the considered problem, before presenting further results, let us introduce a new Bernoulli process $\{\varrho_k\}$ satisfying $\varrho_k \equiv \sigma_k \rho_k$. Then, simple computation yields

$$\begin{aligned} &\operatorname{Prob}\{\varrho_k = 1\} = \mathbb{E}\{\varrho_k\} = \overline{\varrho} = \overline{\rho}\overline{\sigma}, \\ &\operatorname{Prob}\{\varrho_k = 0\} = 1 - \overline{\rho}\overline{\sigma}, \end{aligned} \tag{11}$$

which implies that

$$\begin{aligned} u(k) &= \sigma_k \rho_k \left(K_\alpha + M_\alpha H_\alpha(k) N_\alpha\right) x(k) \\ &= \varrho_k \left(K_\alpha + M_\alpha H_\alpha(k) N_\alpha\right) x(k) \\ &= \overline{\varrho} \left(K_\alpha + M_\alpha H_\alpha(k) N_\alpha\right) x(k) \\ &\quad + \left(\varrho_k - \overline{\varrho}\right) \left(K_\alpha + M_\alpha H_\alpha(k) N_\alpha\right) x(k). \end{aligned} \tag{12}$$

Under the control law (12), the resulting closed-loop system can be obtained as

$$\begin{aligned} \left(\overline{\Sigma}\right): x(k+1) &= \Omega_{1\alpha}(k) x(k) + \left(\varrho_k - \overline{\varrho}\right) \Omega_{2\alpha}(k) x(k) \\ &\quad + C_\alpha v(k) + \left[E_\alpha x(k) + F_\alpha v(k)\right] \omega(k), \\ z(k) &= \Omega_{3\alpha}(k) x(k) + \left(\varrho_k - \overline{\varrho}\right) \Omega_{4\alpha}(k) x(k) + G_\alpha v(k), \end{aligned} \tag{13}$$

where

$$\begin{aligned} \Omega_{1\alpha}(k) &= A_\alpha + \overline{\varrho} B_{1\alpha} \left(K_\alpha + M_\alpha H_\alpha(k) N_\alpha\right), \\ \Omega_{2\alpha}(k) &= B_{1\alpha} \left(K_\alpha + M_\alpha H_\alpha(k) N_\alpha\right), \\ \Omega_{3\alpha}(k) &= D_\alpha + \overline{\varrho} B_{2\alpha} \left(K_\alpha + M_\alpha H_\alpha(k) N_\alpha\right), \\ \Omega_{4\alpha}(k) &= B_{2\alpha} \left(K_\alpha + M_\alpha H_\alpha(k) N_\alpha\right). \end{aligned} \tag{14}$$

Before formulating the problem to be investigated, we first introduce the following definition for system $(\overline{\Sigma})$.

Definition 2. The closed-loop system in (13) with $v(k) \equiv 0$ is said to be stochastically mean square stable (SMSS) if there exists a $\kappa > 0$ such that

$$\mathscr{E}\{\|x(k)\|\} \longrightarrow 0 \quad \text{as } k \longrightarrow \infty, \tag{15}$$

for any initial condition $\|x(0)\| < \kappa$.

Definition 3. System $(\overline{\Sigma})$ is said to be SMSS with a guaranteed H_∞ performance level γ, if system $(\overline{\Sigma})$ is SMSS according to Definition 2, and the prescribed disturbance attenuation level γ is made small in the feasibility of

$$\|z(k)\|_E \leq \gamma \|v(k)\|_2, \tag{16}$$

for all nonzero $\omega(k) \in l_2[0,\infty)$ under zero initial conditions, where

$$\|z(k)\|_E \triangleq \mathscr{E}\left\{\sqrt{\sum_{k=0}^{\infty} z^T(k) z(k)}\right\}. \tag{17}$$

Now, let us state the problems concerned in this paper, which are listed as follows.

Problem I. Consider the stochastic system $(\overline{\Sigma})$, suppose that the controller gain matrices K_α and the additive gain variations $\Delta K_\alpha(k)$ are given, and determine under what condition the system $(\overline{\Sigma})$ is SMSS with a guaranteed H_∞ performance level γ.

Problem II. Consider the system $(\overline{\Sigma})$, and design a nonfragile controller in the form of (5) such that the resulting closed-loop system $(\overline{\Sigma})$ is SMSS with a guaranteed H_∞ performance level γ, in spite of the presence of packet losses phenomena.

3. Main Results

In this section, we will give an LMI approach to solving the nonfragile H_∞ control problem formulated in the previous section. Before proceeding further, we shall introduce the following lemmas, which will be used in the proof of the main results.

Lemma 4 (see [28]). *Given constant matrices $X = X^T, Y$ and $Z = Z^T > 0$ of appropriate dimensions, then*

$$X + Y^T Z Y < 0, \tag{18}$$

if and only if

$$\begin{bmatrix} X & Y^T \\ Y & -Z^{-1} \end{bmatrix} < 0, \tag{19}$$

or, equivalently,

$$\begin{bmatrix} -Z^{-1} & Y \\ Y^T & X \end{bmatrix} < 0. \tag{20}$$

Lemma 5 (see [29]). *Let A, L, E, H, and P be real matrices of appropriate dimensions with $H^T H \le I$. Then one has*

(1) *for any scalar $\epsilon > 0$ and vectors $x, y \in \mathbb{R}^n$,*

$$2x^T LHEy \le \epsilon^{-1} x^T LL^T x + \epsilon y^T E^T Ey, \tag{21}$$

(2) *for any scalar $\epsilon > 0$, such that $P - \epsilon LL^T > 0$,*

$$(A + LHE)^T P^{-1} (A + LHE) \le A^T \left(P - \epsilon LL^T\right)^{-1} A + \epsilon^{-1} E^T E. \tag{22}$$

Now, we first establish the following H_∞ performance analysis criterion, which will play a key role in derivation of the solution to the nonfragile H_∞ control problem.

Theorem 6. *Let the controller parameters in the filtering error system $(\overline{\Sigma})$, scalars $\gamma > 0$, $\overline{\varrho} > 0$, be given. Then, system $(\overline{\Sigma})$ is SMSS with a guaranteed H_∞ performance level γ, if there exist positive matrices $P_\alpha > 0$, such that, for each $\alpha \in \mathcal{S}$,*

$$\begin{bmatrix} -P_\alpha & 0 & \Omega_{1\alpha}^T(k) & \sqrt{\overline{\varrho}(1-\overline{\varrho})}\Omega_{2\alpha}^T(k) & E_\alpha^T & \Omega_{3\alpha}^T(k) & \sqrt{\overline{\varrho}(1-\overline{\varrho})}\Omega_{4\alpha}^T(k) \\ * & -\gamma^2 I & C_\alpha^T & 0 & F_\alpha^T & G_\alpha^T & 0 \\ * & * & -\mathcal{P}_\alpha^{-1} & 0 & 0 & 0 & 0 \\ * & * & * & -\mathcal{P}_\alpha^{-1} & 0 & 0 & 0 \\ * & * & * & * & -\mathcal{P}_\alpha^{-1} & 0 & 0 \\ * & * & * & * & * & -I & 0 \\ * & * & * & * & * & * & -I \end{bmatrix} < 0, \tag{23}$$

where $\mathcal{P}_\alpha \triangleq \sum_{\beta\in\mathcal{S}} \psi_{\alpha\beta} P_\beta$.

Proof. Firstly, we need to establish the stochastic mean square stability criterion of system $(\overline{\Sigma})$. For this purpose, we consider system $(\overline{\Sigma})$ with $v(k) \equiv 0$ and choose a stochastic Lyapunov function for system $(\overline{\Sigma})$ as follows:

$$V(x(k), k) = x^T(k) P_\alpha x(k), \tag{24}$$

where P_α are the positive matrices to be determined for each $\alpha \in \mathcal{S}$. Then, we have that, for each $\delta_k = \alpha \in \mathcal{S}$ and $\delta_{k+1} = \beta \in \mathcal{S}$,

$$\begin{aligned} &\mathcal{E}\{V(x(k+1), k+1) - V(x(k), k)\} \\ &\quad = \mathcal{E}\{[V(x(k+1), k+1) - V(x(k), k)] \mid \\ &\qquad (x(k), \delta_k = \alpha)\} \\ &\quad = \sum_{\beta\in\mathcal{S}} \Pr\{\delta_{k+1} = \beta \mid \delta_k = \alpha\} x^T(k+1) P_\beta x(k+1) \\ &\qquad - x^T(k) P_\alpha x(k) \\ &\quad = \mathcal{E}\left\{x^T(k+1) \mathcal{P}_\alpha x(k+1)\right\} - x^T(k) P_\alpha x(k) \\ &\quad = [\Omega_{1\alpha}(k) x(k)]^T \mathcal{P}_\alpha [\Omega_{1\alpha}(k) x(k)] \\ &\qquad + \overline{\varrho}(1-\overline{\varrho}) x^T(k) \Omega_{2\alpha}^T(k) \mathcal{P}_\alpha \Omega_{2\alpha}(k) x(k) \\ &\qquad + [E_\alpha x(k)]^T \mathcal{P}_\alpha [E_\alpha x(k)] - x^T(k) P_\alpha x(k). \end{aligned} \tag{25}$$

On the other hand, it can be deduced from (23) that

$$\begin{bmatrix} -P_\alpha & \Omega_{1\alpha}^T(k) & \sqrt{\overline{\varrho}(1-\overline{\varrho})}\Omega_{2\alpha}^T(k) & E_\alpha^T \\ * & -\mathcal{P}_\alpha^{-1} & 0 & 0 \\ * & * & -\mathcal{P}_\alpha^{-1} & 0 \\ * & * & * & -\mathcal{P}_\alpha^{-1} \end{bmatrix} < 0. \tag{26}$$

By applying the Schur complement formula (i.e., Lemma 4) to (26), for system $(\overline{\Sigma})$ with $v(k) \equiv 0$, one can readily obtain that

$$\mathcal{E}\{V(x(k+1), k+1) - V(x(k), k)\} < 0; \tag{27}$$

that is, system $(\overline{\Sigma})$ with $v(k) \equiv 0$ is SMSS according to [14, 16]. Next, we will show the H_∞ performance analysis of system $(\overline{\Sigma})$. To this end, we also obtain that each $\delta_k = \alpha \in \mathcal{S}$ and $\delta_{k+1} = \beta \in \mathcal{S}$,

$$\begin{aligned} &\mathcal{E}\{V(x(k+1), k+1) - V(x(k), k)\} \\ &\quad = [\Omega_{1\alpha}(k) x(k) + C_\alpha v(k)]^T \mathcal{P}_\alpha [\Omega_{1\alpha}(k) x(k) + C_\alpha v(k)] \\ &\qquad + \overline{\varrho}(1-\overline{\varrho}) x^T(k) \Omega_{2\alpha}^T(k) \mathcal{P}_\alpha \Omega_{2\alpha}(k) x(k) \\ &\qquad + [E_\alpha x(k) + F_\alpha v(k)]^T \mathcal{P}_\alpha [E_\alpha x(k) + F_\alpha v(k)] \\ &\qquad - x^T(k) P_\alpha x(k). \end{aligned} \tag{28}$$

Note that

$$\begin{aligned} &\mathcal{E}\left\{z^T(k) z(k) - \gamma^2 v^T(k) v(k)\right\} \\ &\quad = [\Omega_{3\alpha}(k) x(k) + G_\alpha v(k)]^T [\Omega_{3\alpha}(k) x(k) + G_\alpha v(k)] \\ &\qquad + \overline{\varrho}(1-\overline{\varrho}) x^T(k) \Omega_{4\alpha}^T(k) \Omega_{4\alpha}(k) x(k) - \gamma^2 v^T(k) v(k). \end{aligned} \tag{29}$$

It can be verified that

$$\begin{aligned}
&\mathscr{E}\left\{z^T(k)z(k)-\gamma^2 v^T(k)v(k)\right.\\
&\quad\left.+V(x(k+1),k+1)-V(x(k),k)\right\}\\
&=\left[\Omega_{1\alpha}(k)x(k)+C_\alpha v(k)\right]^T\mathscr{P}_\alpha\left[\Omega_{1\alpha}(k)x(k)+C_\alpha v(k)\right]\\
&\quad+\overline{\varrho}(1-\overline{\varrho})x^T(k)\Omega_{2\alpha}^T(k)\mathscr{P}_\alpha\Omega_{2\alpha}(k)x(k)\\
&\quad+\left[E_\alpha x(k)+F_\alpha v(k)\right]^T\mathscr{P}_\alpha\left[E_\alpha x(k)+F_\alpha v(k)\right]\\
&\quad-x^T(k)P_\alpha x(k)+\left[\Omega_{3\alpha}(k)x(k)+G_\alpha v(k)\right]^T\\
&\quad\times\left[\Omega_{3\alpha}(k)x(k)+G_\alpha v(k)\right]\\
&\quad+\overline{\varrho}(1-\overline{\varrho})x^T(k)\Omega_{4\alpha}^T(k)\Omega_{4\alpha}(k)x(k)-\gamma^2 v^T(k)v(k)\\
&=\begin{bmatrix}x(k)\\ v(k)\end{bmatrix}^T\begin{bmatrix}-P_\alpha & 0\\ 0 & -\gamma^2 I\end{bmatrix}\begin{bmatrix}x(k)\\ v(k)\end{bmatrix}\\
&\quad+\left[\Omega_{1\alpha}(k)x(k)+C_\alpha v(k)\right]^T\\
&\quad\times\mathscr{P}_\alpha\left[\Omega_{1\alpha}(k)x(k)+C_\alpha v(k)\right]\\
&\quad+\overline{\varrho}(1-\overline{\varrho})x^T(k)\Omega_{2\alpha}^T(k)\mathscr{P}_\alpha\Omega_{2\alpha}(k)x(k)\\
&\quad+\left[E_\alpha x(k)+F_\alpha v(k)\right]^T\mathscr{P}_\alpha\left[E_\alpha x(k)+F_\alpha v(k)\right]\\
&\quad+\left[\Omega_{3\alpha}(k)x(k)+G_\alpha v(k)\right]^T\left[\Omega_{3\alpha}(k)x(k)+G_\alpha v(k)\right]\\
&\quad+\overline{\varrho}(1-\overline{\varrho})x^T(k)\Omega_{4\alpha}^T(k)\Omega_{4\alpha}(k)x(k).
\end{aligned}\tag{30}$$

Similar to the derivation of (27), we apply the Schur complement to (23) and get

$$\begin{aligned}
&\mathscr{E}\left\{z^T(k)z(k)-\gamma^2 v^T(k)v(k)\right.\\
&\quad\left.+V(x(k+1),k+1)-V(x(k),k)\right\}<0.
\end{aligned}\tag{31}$$

For $k=0,1,2,\ldots$, summing up both sides of (31) under zero initial condition and noticing $V(x(\infty),\infty)\geq 0$, it can be verified that

$$\mathscr{E}\left\{\sum_{k=0}^{\infty}z^T(k)z(k)\right\}\leq\gamma^2\sum_{k=0}^{\infty}v^T(k)v(k),\tag{32}$$

or, equivalently, condition (16) is satisfied. This completes the proof. □

In the following, we will present a solution to the nonfragile H_∞ controller design problem for system (Σ) based on Theorem 6. The following theorem proposes a sufficient condition for the existence of such a controller for system (Σ).

Theorem 7. *Consider system (Σ), let scalars $\gamma>0$, $\overline{\varrho}>0$ be given, and let matrices $J_{1\alpha}$, $J_{2\alpha}$, and $J_{3\alpha}$ be fixed. Then, there exists an admissible controller in the form of (5) such that the resulting closed-loop system $(\overline{\Sigma})$ is SMSS with a guaranteed H_∞ performance level γ, in spite of the presence of packet losses phenomena if there exist matrices $Q_\alpha>0, X$ such that the following LMIs hold for each $\alpha\in\mathcal{S}$:*

$$\begin{bmatrix}
-Q_\alpha & 0 & \Gamma_{1\alpha}^T & \sqrt{\overline{\varrho}(1-\overline{\varrho})}Y_\alpha^T B_{1\alpha}^T & X^T E_\alpha^T & \Gamma_{6\alpha}^T & \sqrt{\overline{\varrho}(1-\overline{\varrho})}Y_\alpha^T B_{2\alpha}^T & X^T N_\alpha^T\\
* & -\gamma^2 I & C_\alpha^T & 0 & F_\alpha^T & G_\alpha^T & 0 & 0\\
* & * & \Gamma_{2\alpha} & \Gamma_{3\alpha} & 0 & \Gamma_{7\alpha} & \Gamma_{10\alpha} & 0\\
* & * & * & \Gamma_{4\alpha} & 0 & \Gamma_{8\alpha} & \Gamma_{11\alpha} & 0\\
* & * & * & * & \Gamma_{5\alpha} & 0 & 0 & 0\\
* & * & * & * & * & \Gamma_{9\alpha} & \Gamma_{12\alpha} & 0\\
* & * & * & * & * & * & \Gamma_{13\alpha} & 0\\
* & * & * & * & * & * & * & -\varepsilon_\alpha I
\end{bmatrix}<0,\tag{33}$$

where

$$\begin{aligned}
\Gamma_{1\alpha}&=A_\alpha X+\overline{\varrho}B_{1\alpha}Y_\alpha,\\
\Gamma_{2\alpha}&=\sum_{\beta\in\mathcal{S}}\psi_{\alpha\beta}J_{1\alpha}Q_\beta J_{1\alpha}^T-XJ_{1\alpha}^T-J_{1\alpha}X^T\\
&\quad+\varepsilon_\alpha\overline{\varrho}^2B_{1\alpha}M_\alpha M_\alpha^T B_{1\alpha}^T,\\
\Gamma_{3\alpha}&=\varepsilon_\alpha\overline{\varrho}\sqrt{\overline{\varrho}(1-\overline{\varrho})}B_{1\alpha}M_\alpha M_\alpha^T B_{1\alpha}^T,\\
\Gamma_{4\alpha}&=\sum_{\beta\in\mathcal{S}}\psi_{\alpha\beta}J_{2\alpha}Q_\beta J_{2\alpha}^T-XJ_{2\alpha}^T-J_{2\alpha}X^T\\
&\quad+\varepsilon_\alpha\overline{\varrho}(1-\overline{\varrho})B_{1\alpha}M_\alpha M_\alpha^T B_{1\alpha}^T,\\
\Gamma_{5\alpha}&=\sum_{\beta\in\mathcal{S}}\psi_{\alpha\beta}J_{3\alpha}Q_\beta J_{3\alpha}^T-XJ_{3\alpha}^T-J_{3\alpha}X^T,\\
\Gamma_{6\alpha}&=D_\alpha X+\overline{\varrho}B_{2\alpha}Y_\alpha,\\
\Gamma_{7\alpha}&=\varepsilon_\alpha\overline{\varrho}^2B_{1\alpha}M_\alpha M_\alpha^T B_{2\alpha}^T,\\
\Gamma_{8\alpha}&=\varepsilon_\alpha\overline{\varrho}\sqrt{\overline{\varrho}(1-\overline{\varrho})}B_{1\alpha}M_\alpha M_\alpha^T B_{2\alpha}^T,\\
\Gamma_{9\alpha}&=\varepsilon_\alpha\overline{\varrho}^2B_{2\alpha}M_\alpha M_\alpha^T B_{2\alpha}^T-I,
\end{aligned}$$

$$\begin{aligned}
\Gamma_{10\alpha} &= \varepsilon_\alpha \bar{\varrho}\sqrt{\bar{\varrho}(1-\bar{\varrho})}B_{1\alpha}M_\alpha M_\alpha^T B_{2\alpha}^T,\\
\Gamma_{11\alpha} &= \varepsilon_\alpha \bar{\varrho}(1-\bar{\varrho})B_{1\alpha}M_\alpha M_\alpha^T B_{2\alpha}^T,\\
\Gamma_{12\alpha} &= \varepsilon_\alpha \bar{\varrho}\sqrt{\bar{\varrho}(1-\bar{\varrho})}B_{2\alpha}M_\alpha M_\alpha^T B_{2\alpha}^T,\\
\Gamma_{13\alpha} &= \varepsilon_\alpha \bar{\varrho}(1-\bar{\varrho})B_{2\alpha}M_\alpha M_\alpha^T B_{2\alpha}^T - I.
\end{aligned} \tag{34}$$

In this case, a suitable nonfragile H_∞ controller in the form of (5) *is given by*

$$K_\alpha = Y_\alpha X^{-1}, \quad 1 \le \alpha \le \mathcal{N}. \tag{35}$$

Proof. Introduce the new variables $Q_\alpha = X^T P_\alpha X$; then one can find that

$$\mathcal{P}_\alpha \triangleq \sum_{\beta\in\mathcal{S}} \psi_{\alpha\beta}P_\beta = \sum_{\beta\in\mathcal{S}} \psi_{\alpha\beta}X^{-T}Q_\beta X^{-1}, \tag{36}$$

which implies that

$$-\mathcal{P}_\alpha^{-1} = -X\left(\sum_{\beta\in\mathcal{S}}\psi_{\alpha\beta}Q_\beta\right)^{-1}X^T. \tag{37}$$

Note that

$$\begin{aligned}
&\left(J_{l\alpha}\left(\sum_{\beta\in\mathcal{S}}\psi_{\alpha\beta}Q_\beta\right)-X\right)\left(\sum_{\beta\in\mathcal{S}}\psi_{\alpha\beta}Q_\beta\right)^{-1}\\
&\quad\times\left(J_{l\alpha}\left(\sum_{\beta\in\mathcal{S}}\psi_{\alpha\beta}Q_\beta\right)-X\right)\ge 0, \quad l=1,2,3.
\end{aligned} \tag{38}$$

It follows from (37) that

$$-\mathcal{P}_\alpha^{-1} \le \sum_{\beta\in\mathcal{S}}\psi_{\alpha\beta}J_{l\alpha}Q_\beta J_{l\alpha}^T - XJ_{l\alpha}^T - J_{l\alpha}X^T, \quad l=1,2,3. \tag{39}$$

In view of Lemma 5, one can get that

$$\Xi_{1\alpha}^T H_\alpha(k)\Xi_{2\alpha} + \Xi_{2\alpha}^T H_\alpha^T(k)\Xi_{1\alpha} \le \varepsilon_\alpha \Xi_{1\alpha}^T\Xi_{1\alpha} + \varepsilon_\alpha^{-1}\Xi_{2\alpha}^T\Xi_{2\alpha}, \tag{40}$$

where

$$\begin{aligned}
\Xi_{1\alpha} &= \left[0 \;\; 0 \;\; \bar{\varrho}M_\alpha^T B_{1\alpha}^T \;\; \sqrt{\bar{\varrho}(1-\bar{\varrho})}M_\alpha^T B_{1\alpha}^T \;\; 0 \;\; \bar{\varrho}M_\alpha^T B_{2\alpha}^T \;\; \sqrt{\bar{\varrho}(1-\bar{\varrho})}M_\alpha^T B_{2\alpha}^T\right],\\
\Xi_{2\alpha} &= \left[N_\alpha X \;\; 0 \;\; 0 \;\; 0 \;\; 0 \;\; 0 \;\; 0\right].
\end{aligned} \tag{41}$$

Using Lemma 4 and combining (33), (39), and (40) result in

$$\begin{bmatrix}
-Q_\alpha & 0 & \tilde{\Gamma}_{1\alpha}^T & \sqrt{\bar{\varrho}(1-\bar{\varrho})}\tilde{\Gamma}_{2\alpha}^T & X^T E_\alpha^T & \tilde{\Gamma}_{3\alpha}^T & \sqrt{\bar{\varrho}(1-\bar{\varrho})}\tilde{\Gamma}_{4\alpha}^T\\
* & -\gamma^2 I & C_\alpha^T & 0 & F_\alpha^T & G_\alpha^T & 0\\
* & * & -\mathcal{P}_\alpha^{-1} & 0 & 0 & 0 & 0\\
* & * & * & -\mathcal{P}_\alpha^{-1} & 0 & 0 & 0\\
* & * & * & * & -\mathcal{P}_\alpha^{-1} & 0 & 0\\
* & * & * & * & * & -I & 0\\
* & * & * & * & * & * & -I
\end{bmatrix} < 0, \tag{42}$$

where

$$\begin{aligned}
\tilde{\Gamma}_{1\alpha} &= A_\alpha X + \bar{\varrho}B_{1\alpha}\left(Y_\alpha + M_\alpha H_\alpha(k)N_\alpha X\right),\\
\tilde{\Gamma}_{2\alpha} &= B_{1\alpha}\left(Y_\alpha + M_\alpha H_\alpha(k)N_\alpha X\right),\\
\tilde{\Gamma}_{3\alpha} &= D_\alpha X + \bar{\varrho}B_{2\alpha}\left(Y_\alpha + M_\alpha H_\alpha(k)N_\alpha X\right),\\
\tilde{\Gamma}_{4\alpha} &= B_{2\alpha}\left(Y_\alpha + M_\alpha H_\alpha(k)N_\alpha X\right).
\end{aligned} \tag{43}$$

Then, by pre- and postmultiplying (42) by $\mathrm{diag}\{X^{-T}, I, I, I, I, I, I\}$ and its transpose, one has that inequality (23) holds. Therefore, in light of Theorem 6, we can conclude that the resulting closed-loop system is SMSS with a guaranteed H_∞ performance level γ. This completes the proof. □

4. An Illustrative Example

In this section, an example is used to illustrate the effectiveness of the presented nonfragile controller design method. Consider the discrete-time stochastic Markov jump system

(Σ) over a probability space $(\Omega, \mathscr{F}, \mathscr{P})$ with two modes ($\alpha = 1, 2$) and the following parameters:

$$A_1 = \begin{bmatrix} 1.1 & 0.31 & 0 \\ 0 & 0.33 & 0.21 \\ 0 & 0 & -0.52 \end{bmatrix},$$

$$B_{11} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 0 & 1 \end{bmatrix}, \qquad C_1 = \begin{bmatrix} 0.1 \\ 0 \\ 0 \end{bmatrix},$$

$$E_1 = \begin{bmatrix} 0.05 & 0 & 0 \\ 0 & 0.05 & 0 \\ 0 & 0 & 0.1 \end{bmatrix}, \qquad F_1 = \begin{bmatrix} 0.2 \\ 0 \\ 0 \end{bmatrix},$$

$$D_1 = \begin{bmatrix} 0.2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0.1 \end{bmatrix}, \qquad B_{21} = \begin{bmatrix} 0 & 1 \\ 0 & 0.1 \\ 0 & 0.1 \end{bmatrix},$$

$$G_1 = \begin{bmatrix} 0 \\ 0 \\ 0.1 \end{bmatrix},$$

$$M_1 = \begin{bmatrix} 0.1 \\ 0.2 \end{bmatrix}, \qquad N_1 = \begin{bmatrix} 0.1 & 0.2 & -0.1 \end{bmatrix}, \tag{44}$$

$$A_2 = \begin{bmatrix} 0.8 & -0.38 & 0 \\ -0.2 & 0 & 0.21 \\ 0.1 & 0 & -0.55 \end{bmatrix},$$

$$B_{12} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{bmatrix}, \qquad C_2 = \begin{bmatrix} 0 \\ 0.12 \\ 0 \end{bmatrix},$$

$$E_2 = \begin{bmatrix} 0.5 & 0 & 0 \\ 0 & 0.25 & 0 \\ 0 & 0 & -0.5 \end{bmatrix}, \qquad F_2 = \begin{bmatrix} 0.1 \\ 0 \\ 0 \end{bmatrix},$$

$$D_2 = \begin{bmatrix} -0.12 & 0 & 0.1 \\ 0 & 0 & 0 \\ 0 & 0 & 0.1 \end{bmatrix},$$

$$B_{22} = \begin{bmatrix} 0.1 & 0.1 \\ 0 & 0.2 \\ 0 & 0.2 \end{bmatrix}, \qquad G_2 = \begin{bmatrix} 0.2 \\ 0 \\ 0 \end{bmatrix},$$

$$M_2 = M_1, \qquad N_2 = N_1.$$

Here, our aim is to design a nonfragile controller in the form of (5) such that the resulting closed-loop system is SMSS with a guaranteed H_∞ performance level γ, in spite of the presence of packet losses phenomena. To this end, we suppose that $\overline{\varrho} = 0.7$, $\gamma = 1$, $J_{11} = J_{12} = \mathrm{diag}\{0.25, 0.25, 0.25\}$, $J_{21} = J_{22} = \mathrm{diag}\{0.5, 0.5, 0.5\}$, and $J_{31} = J_{32} = \mathrm{diag}\{0.05, 0.05, 0.05\}$, and choose the transition probability matrix Π as $\Pi = \left[\begin{smallmatrix} 0.9 & 0.1 \\ 0.3 & 0.7 \end{smallmatrix}\right]$. Then, by applying Theorem 7, one can get feasible solutions as follows:

$$Y_1 = \begin{bmatrix} -14.5256 & -10.2590 & -1.2361 \\ -1.9851 & -0.9283 & 0.6443 \end{bmatrix},$$

$$Y_2 = \begin{bmatrix} -6.8634 & 6.1819 & -0.9802 \\ 0.3373 & -0.0477 & 2.2476 \end{bmatrix}, \tag{45}$$

$$X = \begin{bmatrix} 12.7563 & 2.1242 & 0.7457 \\ 1.8940 & 26.0177 & -1.2706 \\ 0.6125 & 0.1118 & 13.4984 \end{bmatrix}.$$

Thus, the desired controller gains K_α ($\alpha = 1, 2,$) can be given by

$$K_1 = \begin{bmatrix} -1.0905 & -0.3050 & -0.0600 \\ -0.1548 & -0.0233 & 0.0541 \end{bmatrix},$$

$$K_2 = \begin{bmatrix} -0.5797 & 0.2850 & -0.0138 \\ 0.0191 & -0.0041 & 0.1651 \end{bmatrix}. \tag{46}$$

5. Conclusions

In this paper, we have studied the problem of nonfragile H_∞ control for stochastic systems with Markovian jumping parameters and random packet losses. An LMI approach has been developed to design a nonfragile controller which ensures both the stochastic mean square stability and a prescribed H_∞ performance level for the resulting closed-loop systems in the presence of random packet losses. The proposed approach has been illustrated to be effective by an example. It should be pointed out that the states of the system are assumed to be precisely known, but this is difficult to achieve in practice [30–32]. Therefore, one of our further research topics is to develop nonfragile output feedback controller design methods for stochastic Markov jump systems.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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