We study the asymptotic properties of minimum distance estimator of drift parameter for a class of nonlinear scalar stochastic differential equations driven by mixed fractional Brownian motion. The consistency and limit distribution of this estimator are established as the diffusion coefficient tends to zero under some regularity conditions.

1. Introduction

Stochastic differential equations (SDEs) are a natural choice to model the time evolution of dynamic systems which are subject to random influences. For existence and uniqueness of solutions of finite dimensional stochastic differential equations and properties of stochastic integrals, we refer to [1–3] and the references therein.

It is natural that a model contains unknown parameters. The parametric estimation problems for diffusion processes satisfying SDEs driven by Brownian motion (hereafter Bm) have been studied earlier. For a more recent comprehensive discussion, we refer to [4, 5] and the references therein. In case of statistical inference for diffusion processes satisfying SDEs driven by a fractional Brownian motion (hereafter fBm), substantial progress has been made in this direction; we refer to [6–9] for more details.

The mixed fractional Brownian motion (hereafter mfBm) was introduced by Patrick [10] to present a stochastic model of the discounted stock price in some arbitrage-free and complete financial markets. As a result, in order to take into account long memory and exclude arbitrage, it is natural to use mfBm to replace the standard Brownian motion. Consequently, there has been a growing interest in parameter estimation for stochastic processes driven by mfBm.

There are several heuristic methods available for use in case of SDEs driven by mfBm, such as MLE, LSE, and sequential estimation. In the continuous case, since the MLE has desirable asymptotic properties of consistency, normality, and efficiency under broad conditions, perhaps the most direct method is the MLE. However, MLE has some shortcomings; MLE’s calculation is often cumbersome as the expressions for MLE involve stochastic integrals which need good approximations for computational purposes. Moreover, the generally good asymptotic properties are not always satisfied in the discrete case. Paper [11] showed that the approach of estimating parameters of an Itô process by applying MLE to a discretization of the SDE does not yield consistent estimators. LSE is asymptotically equivalent to the MLE. It is well known that the sequential estimation methods might lead to equally efficient estimators from the process observed possibly over a shorter expected period of observation time. Although there exists a wide range of estimation techniques developed for the problem of parameters estimation for SDEs driven by mfBm, we should choose a suitable estimation method.

Though mfBm has stationary self-similar increments, it does not have stationary increments and is not a Markov process. So, state-space models and Kalman filter estimators cannot be applied to the parameters of this process. Under the circumstances, in order to overcome those difficulties, the minimum distance approach is proposed.

The interest for this method of parametric estimation is that the minimum distance estimation (hereafter MDE) method has several features. It makes MDE an attractive method. From one part it is sometimes easy to calculate.
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On the other side this estimator is known to be consistent (see [12]) under some general conditions. Millar [13] studied a general framework for (MDE) of Hilbertian type and showed that the MDE is efficient and asymptotically normal in some situations. Furthermore, the MDE is a class of estimators that is automatically robust in the same sense (for more details see [14]), which is generically optimal according to some quantitative measure of robustness.

For the SDEs driven by Brownian motions, Kutoyants [15] and Kutoyants and Pilibossian [16] proved that ε−1(θ* − θ0) converges in probability to the random variable ζ when L1, L2 or supremum norm and he also proved that ζ is asymptotically normal when T → +∞ and θ0 > 0. Hénaff [17] established the same results in the general case of a norm in some Banach space of functions on [0, T]. For the SDEs driven by fBm, Prakasa Rao [18] studied the minimum L1-norm estimator θ* of the drift parameter for a class of nonlinear scalar SDEs driven by fBm and showed that the MDE is efficient and asymptotically normal in some general framework for (MDE) of Hilbertiantype and showed that the MDE is efficient and asymptotically normal in some situations. Furthermore, the MDE is a class of estimators that is automatically robust in the same sense (for more details see [23], Page 5, Definition 1.11).

Let us take a and b which are two real constants such that (a, b) ≠ (0, 0). By Patrick [10], we introduce the following.

**Definition 1.** A mixed fractional Brownian motion of parameters a, b, and H is a process \( M_H := \{M^H_t(a, b); t \geq 0\} \), defined on the probability space \((\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})\) by

\[
M^H_t = M^H_0 + abH_t, \quad \forall t \in \mathbb{R}_+.
\]

**Remark 2.** From Zili [20], we know that mfBm is a mixed-self-similar process.

\[
M^H_{a,b} = \left\{ M^H_t \left( ah^{1/2}, bh^{1/2} \right) \right\},
\]

where \( h > 0 \) is a constant. Furthermore it follows for the supremum process \( M^* \) that

\[
M^H_{a,b} = \left\{ M^H_t \left( ah^{1/2}, bh^{1/2} \right) \right\}.
\]

By using the self-similarity of mfBm, we obtain the following lemma.

**Lemma 3.** Let \( T > 0 \) be a constant and \( M^H_{a,b} \) an mfBm with parameter \( a, b, H \); then for every \( p > 0 \),

\[
E(\sup_{s \leq T} |aT^{1/2}b + bTH_{a,b}|^p) = E(\sup_{s \leq t} |aT^{1/2}b + bTH_{a,b}|^p).
\]

The value of (6) is not known. However it is fortunate that we have the following two lemmas which give the bounds for the standard Bm and fBm, respectively.

**Lemma 4 (Burkholder-Davis-Gundy inequalities).** For any stopping time \( \tau \) with respect to the filtration generated by the Bm \( B_t \), one has

\[
c(p) E\left( \tau^{p/2} \right) \leq E\left( (B^p_t) \right) \leq C(p) E\left( \tau^{p/2} \right), \quad p > 0,
\]

where the constants \( c(p) \) and \( C(p) > 0 \) depend only upon the parameter \( p \).
B-D-G have a long history and we cite only some works in this area. Maybe the first general results were due to Novikov ($p > 1/2$) and Burkholder (see [24, 25]).

**Lemma 5** (see [26]). Let $\tau$ be a stopping time with respect to the filtration generated by the BM $B_t^H$. Then, for $H \in (1/2, 1)$, one has
\[
c(p, H) E\left(\tau^{pH}\right) \leq E\left((B^{pH}_\tau)^p\right) \leq C(p, H) E\left(\tau^{pH}\right),
\]
for every $\delta > 0$, $P(\epsilon) \theta_0 \{\| \theta \| > \delta\} \leq 2p C_p T_{\epsilon^p} (g(\delta))^{-p} + C_2(p) T^{p/2}
\]
and, for $H \in (0, 1/2)$, one has
\[
c(p, H) E\left(\tau^{pH}\right) \leq E\left((B^{pH}_\tau)^p\right), \quad \forall p > 0,
\]
where the constants $c(p, H)$ and $C(p, H) > 0$ depend only upon the parameters $p, H$.

**Lemma 6** (see [16, page 120]). Let $Z_\epsilon(u)$, $\epsilon > 0$, $u \in \mathbb{R}$, be a sequence of continuous functions and $Z_0(u)$ a convex function which admits a unique minimum $\xi \in \mathbb{R}$. Let $L_{\epsilon, \xi}$, $\epsilon > 0$, be a sequence of positive numbers such that $L_{\epsilon, \xi} \to +\infty$ as $\epsilon \to 0$. We suppose that
\[
\lim \sup_{\epsilon \to 0} |Z_\epsilon(u) - Z_0(u)| = 0.
\]
Then
\[
\lim_{\epsilon \to 0} \arg \min_{|u| \leq L_{\epsilon, \xi}} Z_\epsilon(u) = \xi,
\]
where if there are several minima of $Z_\epsilon$, we choose the arbitrary one.

Now we consider the parameter estimation problem for a class of nonlinear scalar mixed SDE in the following framework:
\[
\frac{dx}{\epsilon} = S_x(\theta, x) dt + \epsilon dM^H_t, \quad x(0) = x_0, \quad 0 \leq t \leq T,
\]
where $S_x(\theta, x)$ is a known measurable functional, the unknown drift parameter $\theta \in \Theta \subset \mathbb{R}$, and $\epsilon > 0$.

Denote $\theta_0$ by the true parameter of $\theta$. Let $\mathbb{P}_{\theta_0}$ be the probability measure induced by the process $\{X_t\}$ and $x(\theta)$ the solution of the differential equation (12) with $\epsilon = 0$.

Assume that the trend functional of the above equation has the following form:
\[
S_x(\theta, x) = V(\theta, t, x) + \int_0^t K_x(\theta, t, s, x) ds,
\]
where $V(\theta, t, x)$ and $K(\theta, t, s, x)$ are two measurable functions.

The function $S_x(\theta, x)$ is measurable with respect to $(t, \theta)$ and, for any $\delta > 0$, one has
\[
g(\delta) = \inf_{|\theta - \theta_0| \leq \delta} \|x(\theta) - x(\theta_0)\|_T.
\]
Note that $g(\delta) > 0$ for any $\delta > 0$.

The $L_{p, T}$ or supremum norm can be denoted by $\| x \|_T$ and the MDE $\theta^*_\epsilon$ (see [15]) is defined by
\[
\theta^*_\epsilon := \arg \min_{\theta \in \Theta} \| x - x(\theta) \|_T.
\]

We also need the following additional conditions. ($A_f$) The functional $S(\cdot, \cdot, \cdot)$ is measurable and nonanticipative and satisfies the following inequalities: for all $t \in [0,T]$ and $X, Y \in \mathbb{C}^\epsilon(0,T]$,
\[
|S_t(\theta, X) - S_t(\theta, Y)| \leq L_1 \int_0^t |X_s - Y_s| ds + L_2 |X_t - Y_t|,
\]
\[
|S_t(\theta, X)| \leq L_1 \int_0^t (1 + |X_s|) ds + L_2 (1 + |X_t|),
\]
where $L_1, L_2$ are positive constants.
($A_f$) The measurable functions $V(\theta, t, X)$ and $K(\theta, t, s, x)$ are continuously twice differentiable in $\theta$ and $x$.

($A_f$) Suppose that
\[
\inf_{\theta \in \Theta} \| \dot{x}(\theta) \|_T > 0
\]
and define by the random variable $\xi_T = \xi_T(\theta_0)$,
\[
\xi_T := \arg \min_{x \in \mathbb{R}} \| x(1) - \dot{x}(\theta_0) \|_T.
\]

Denote $x(\theta)$ by the derivatives of $x(\theta)$ with respect to $\theta$ and introduce a Gaussian process $x^{(1)} = x^{(1)}(\theta)$ which satisfies the equation
\[
dx^{(1)} = \left[ V_x(\theta, t, x) x^{(1)} + \int_0^t K_x(\theta, t, s, x) x_s^{(1)} ds \right] dt + dM^H_t,
\]
where $V_x$ and $K_x$ are the derivatives of $V(\theta, t, x)$ and $K(\theta, t, s, x)$ with respect to $x$ and $x^{(1)}$ is a derivative with probability one of $X_t$ with respect to $x$ as $\epsilon = 0$.

In the paper, we will use $C$ to denote a generic constant which may vary from place to place.

3. Consistency

**Theorem 7.** If the above condition ($A_f$) is satisfied, then, for any $p > 0$, there exist constants $C_1(p, H), C_2(p) > 0$ such that, for every $\delta > 0$,
\[
P_{\theta_0} \left( | \theta^*_\epsilon - \theta_0 | > \delta \right) \leq 2p C_1 C_p T_{\epsilon^p} (g(\delta))^{-p} (C_1(p, H) T^{2p} + C_2 (p) T^{p/2})
\]
\[
= O \left( (g(\delta))^2 \epsilon^p \right).
\]
Proof. Condition (A1) ensures the existence and uniqueness of a strong solution of (12). It is obvious that, with \( P^{(c)} \) probability one,
\[
\|X_t - x_t(\theta_0)\|_{\infty} \leq C\varepsilon \sup_{0 \leq t \leq T} |M^H_t| .
\]  
(21)
For any \( \delta > 0 \),
\[
P^{(c)}_{\theta_0} \left( |\theta^*_\varepsilon - \theta_0| > \delta \right) \leq P^{(c)}_{\theta_0} \left( \|X - x(\theta)\|_T > g(\delta) / 2\varepsilon C_T \right)
\]
\[\leq P^{(c)}_{\theta_0} \left( \sup_{\theta \in \mathcal{S}} |M^H_0| > g(\delta) / 2\varepsilon C_T \right)
\]
\[= P^{(c)}_{\theta_0} \left( M^H T > g(\delta) / 2\varepsilon C_T \right)
\]
(23)
By (21), we have
\[
P^{(c)}_{\theta_0} \left( |\theta^*_\varepsilon - \theta_0| > \delta \right) \leq \frac{2pC^p C^p \varepsilon^p (g(\delta))^{-p} (C_1 p H T^{2p} + C_2 p T^{p/2})}{g(\delta) / 2\varepsilon C_T}
\]
\[= O \left( (g(\delta))^{-p} \varepsilon^p \right).
\]  
(24)
\[\square\]

Remark 8. As a consequence of the above theorem, we obtain that \( \theta^*_\varepsilon \) converges in probability to \( \theta_0 \) under \( P^{(c)}_{\theta_0} \)-measure as \( \varepsilon \to 0 \). Furthermore, the rate of convergence is of order \( O(\varepsilon^p) \) for every \( p > 0 \).

4. Asymptotic Distribution

**Theorem 9.** Under conditions (A2) and (A3), we have that the random variable \( \varepsilon^{-1}(\theta^*_\varepsilon - \theta) \) converges in probability to a random variable whose probability distribution is the same as that of \( \xi_T \) under \( P^{(c)}_{\theta_0} \).
\[
+ \int_0^t \int_0^V \left[ \frac{K(\theta_0, v, h, X_h) - K(\theta_0, v, h, X_h)}{\epsilon} \right] dh \, dv
- K'(\theta_0, v, h, x_h) x_h^{(1)} h \right| dh \, dv
\]

By (33) and (34),
\[
\sup_{|u| < \lambda} |Z_\epsilon (u) - Z_0 (u)|
\]
\[
= \sup_{|u| < \lambda} \left\| \frac{1}{\epsilon} (X - x(\theta_0 + \epsilon u)) \right\|_T - \left\| x^{(1)} - u \dot{x}(\theta_0) \right\|_T
\]
\[
\leq C_T \sup_{|u| < \lambda} \left\| (X - x(\theta_0 + \epsilon u)) - (x^{(1)} - u \dot{x}(\theta_0)) \right\|_{\infty}
\]
\[
\leq C_T \sup_{|u| < \lambda} \left\| (X - x(\theta_0 + \epsilon u)) - (x^{(1)} - u \dot{x}(\theta_0)) \right\|_{\infty}
\]
\[
\leq C_T \left( \sup_{0 \leq T} |q_1| + \sup_{0 \leq T} \sup_{|u| < \lambda} |r_1| \right)
\]
\[
\leq C \sup_{0 \leq T} \left\| M_T^{H^2} \right\| + C \epsilon \lambda^2
\]

(36)

Therefore, if we choose \( \lambda \) such that \( \epsilon \lambda^2 \to 0 \) when \( \epsilon \to 0 \),
then, with probability one, we have
\[
\sup_{|u| < \lambda} |Z_\epsilon (u) - Z_0 (u)| \to 0.
\]

(37)

Using Lemma 6, we obtain the result. The proof completes. \( \square \)

**Conflict of Interests**

The authors declare that there is no conflict of interests regarding the publication of this paper.

**References**


