Research Article

High-Dimensional D. H. Lehmer Problem over Quarter Intervals

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The high-dimensional D. H. Lehmer problem over quarter intervals is studied. By using the properties of character sum and the estimates of Dirichlet L -function, the previous result is improved to be the best possible in the case of $q = p$, an odd prime with $p \equiv 1 \pmod{4}$, which is shown by studying the mean square value of the error term.

1. Introduction and Main Results

Let $q > 2$ be an odd integer and let a be an integer coprime to q . For each integer b with $1 \leq b < q$ and $(b, q) = 1$, there is a unique integer c with $1 \leq c < q$ such that $bc \equiv a \pmod{q}$. Let $M(a, q)$ denote the number of solutions (b, c) of the congruence equation $bc \equiv a \pmod{q}$ with $1 \leq b, c < q$ such that b, c are of opposite parity. D. H. Lehmer posed the problem to find $M(1, p)$ or at least to say something nontrivial about it (see problem F12 of [1], page 251), where p is an odd prime. Zhang [2] proved that

$$M(1, q) = \frac{\phi(q)}{2} + O(q^{1/2} d^2(q) \ln^2 q), \quad (1)$$

where $\phi(q)$ is the Euler function and $d(q)$ is the Dirichlet divisor function. For the further properties of $M(a, p)$, he [3] studied the mean square value of the error term $M(a, p) - (p-1)/2$ and obtained

$$\sum_{a=1}^{p-1} \left(M(a, p) - \frac{p-1}{2} \right)^2 = \frac{3}{4} p^2 + O\left(p \exp\left(\frac{3 \ln p}{\ln \ln p} \right) \right). \quad (2)$$

For general odd integer q , the similar properties of $M(a, q)$ were studied in [4].

It is interesting to study the D. H. Lehmer problem over short intervals $[1, \lambda q]$ with $0 < \lambda \leq 1$ being a real number. Denote by $N(a, q)$ the number of pairs of integers b, c with

$bc \equiv a \pmod{q}$, $1 \leq b, c \leq (q-1)/2$, and b, c having different parity. In [5], Xu and Zhang studied the mean square value of error term

$$E(a, q) = N(a, q) - \frac{1}{8} \phi(q) \quad (3)$$

in the case of $q = p$ and obtained a sharp asymptotic formula

$$\sum_{a=1}^{p-1} E^2(a, p) = \frac{9}{64} p^2 + O(p^{1+\epsilon}). \quad (4)$$

Let k be a positive integer and n a nonnegative integer; let $0 < \lambda_1, \dots, \lambda_{k+1} \leq 1$ be real numbers and $\mathbf{w} = (\lambda_1, \lambda_2, \dots, \lambda_{k+1})$. Let $q \geq \max\{[1/\lambda_i] : 1 \leq i \leq k+1\}$ be a positive integer and a an integer coprime to q . In [6], Xu and Zhang studied the high-dimensional D. H. Lehmer problem over short intervals as

$$N(a, k, \mathbf{w}, q, n) = \sum_{b_1=1}^{[\lambda_1 q]} \cdots \sum_{b_k=1}^{[\lambda_k q]} \sum_{c=1}^{[\lambda_{k+1} q]} \sum_{\substack{b_1 \cdots b_k c \equiv a \pmod{q} \\ 2 \nmid (b_1 + \cdots + b_k + c)}} (b_1 \cdots b_k - c)^{2n}, \quad (5)$$

and obtained an interesting asymptotic formula

$$\begin{aligned} N(a, k, \mathbf{w}, q, n) &= C(k, \mathbf{w}, n) \phi^k(q) q^{2kn} + O\left(4^n q^{(2n+1)k-1/2} d^2(q) \ln q \right), \end{aligned} \quad (6)$$

where

$$C(k, \mathbf{w}, n) = \begin{cases} \frac{(\lambda_1 \cdots \lambda_k)^{2n+1} \lambda_{k+1}}{2(2n+1)^k}, & \text{if } k \geq 2; \\ \frac{\lambda_1^{2n+2} + \lambda_2^{2n+2} - (\lambda_1 - \lambda_2)^{2n+2}}{4(n+1)(2n+1)}, & \text{if } k = 1. \end{cases} \quad (7)$$

They also improved the result for $N_{1/2}(a, k, q) = N(a, k, \mathbf{w}, q, 0)$ in the case $\mathbf{w} = (1/2, 1/2, \dots, 1/2)$ to be the best possible, by studying the mean square value of the error term $E_{1/2}(a, k, q) = N_{1/2}(a, k, q) - \phi^k(q)/2^{k+2}$.

In this paper, we consider the high-dimensional D. H. Lehmer problem over quarter intervals. Let $N_{1/4}(a, k, q) = N(a, k, \mathbf{w}, q, 0)$ in the case $\mathbf{w} = (1/4, 1/4, \dots, 1/4)$. That is,

$$N_{1/4}(a, k, q) = \sum_{b_1=1}^{(q-1)/4} \sum_{b_2=1}^{(q-1)/4} \cdots \sum_{b_k=1}^{(q-1)/4} \sum_{c=1}^{(q-1)/4} 1. \quad (8)$$

$b_1 b_2 \cdots b_k c \equiv a \pmod{q}$
 $2 \nmid (b_1 + b_2 + \cdots + b_k + c)$

We will use the properties of character sum and the estimates of Dirichlet L -function to obtain a sharper asymptotic formula of $N_{1/4}(a, k, q)$ in the case of $q = p$, an odd prime with $p \equiv 1 \pmod{4}$. In order to show that our result is close to be the best possible, the mean square value of $N_{1/4}(a, k, p) - \phi^k(p)/2^{2k+3}$ is studied too.

In this paper, we will use the following notations:

$\sum_{\chi \bmod q, \chi(-1)=-1}^*$ denotes the summation over all primitive characters modulo q such that $\chi(-1) = -1$;

$J(q)$ denotes the number of all primitive characters modulo q ;

$d_k(n)$ denotes the k -th divisor function (i.e., the number of solutions of the equation $n_1 n_2 \cdots n_k = n$ in positive integers $n_1, n_2, \dots, n_k$);

χ_4 denotes the primitive character modulo 4;

χ_{8_1} denotes the primitive character modulo 8 with $\chi_{8_1}(-1) = -1$, and χ_{8_2} denotes the primitive character modulo 8 with $\chi_{8_2}(-1) = 1$.

The main results are the following.

Theorem 1. Let $p \geq 5$ be an odd prime with $p \equiv 1 \pmod{4}$ and a coprime to p . Then, for any positive integer k with $(p, k(k+1)) = 1$, one has the asymptotic formula that

$$N_{1/4}(a, k, p) = \frac{p^k}{2^{2k+3}} + O\left(2^{k^2+2k+2} k p^{k/2} \ln^{k+1} p\right). \quad (9)$$

Theorem 2. Let $p \geq 5$ be an odd prime with $p \equiv 1 \pmod{4}$. Then, for any positive integer k , one has

$$\sum_{a=1}^{p-1} |E_{1/4}(a, k, p)|^2 = A(k) p^{k+1} + O_k(p^{k+\epsilon}), \quad (10)$$

where

$$\begin{aligned} A(k) = & \left(\frac{1}{8} - 2^{k-1} + (-1)^k 2^{k-1}\right) \sum_{n=1}^{\infty} \frac{d_{k+1}^2(n)}{n^2} \\ & + \sum_{i=0}^{k+1} C_{k+1}^i \sum_{j=0}^i 6^j \sum_{s=0}^{i-j} \sum_{t=0}^{k+1-i} C_{i-j}^s C_{k+1-i}^t \\ & \times \frac{(-1)^{k+1-i}}{2^{|2s+4t+i+j-2k-2|+k+i+5}} \\ & \times \sum_{n=1}^{\infty} \frac{d_{k+1}(2^{|2s+4t+i+j-2k-2|} n) d_{k+1}(n)}{n^2} \\ & + \sum_{j_1=0}^{k+1} \sum_{l_1=0}^{j_1} \sum_{l_2=0}^{k+1} C_{k+1}^{j_1} C_{j_1}^{l_1} C_{k+1}^{l_2} \frac{(-1)^{k+j_1+l_2}}{2^{|2k+2+l_2-j_1-l_1|+k+3-l_1-l_2}} \\ & \times \sum_{n=1}^{\infty} \frac{d_{k+1}(2^{|2k+2+l_2-j_1-l_1|} n) d_{k+1}(n)}{n^2} \\ & + \sum_{j_1=0}^{k+1} \sum_{j_2=1}^k \sum_{l_1=0}^{j_1} \sum_{l_2=0}^{j_2} C_{k+1}^{j_1} C_{k+1}^{j_2} C_{j_1}^{l_1} C_{j_2}^{l_2} \\ & \times (-1)^{k+j_1+l_2} \sqrt{2}^{2j_1+j_2+4l_1-7k-11} \\ & \times \sum_{n=1}^{\infty} \frac{r_{j_2}(2^{3k+3+l_2-j_1-j_2-l_1} n) d_{k+1}(n)}{n^2} \\ & + \sum_{j_2=1}^{k+1} (-1)^{k+j_2} \sqrt{2}^{2k+j_2+2} C_{k+1}^{j_2} \sum_{n=1}^{\infty} \frac{d_{k+1}(n) r_{j_2}(n)}{n^2} \\ & + \sum_{l_1=0}^{k+1} \sum_{l_2=0}^{k+1} C_{k+1}^{l_1} C_{k+1}^{l_2} \frac{(-1)^{k+l_1+l_2}}{2^{|l_1-l_2|+2-l_1-l_2}} \\ & \times \sum_{n=1}^{\infty} \frac{d_{k+1}(2^{|l_1-l_2|} n) d_{k+1}(n)}{n^2} \\ & + \sum_{j_2=0}^k \sum_{l_1=0}^{k+1} \sum_{l_2=0}^{j_2} (-1)^{k+l_1+l_2} C_{k+1}^{j_2} C_{k+1}^{l_1} C_{j_2}^{l_2} \sqrt{2}^{3k+4l_2+1-3j_2} \\ & \times \sum_{j_2-l_2 \geq k+1-l_1} \frac{d_{k+1}(2^{(j_2-l_2)-(k+1-l_1)} n) r_{j_2}(n)}{n^2} \\ & + \sum_{j_2=0}^k \sum_{l_1=0}^{k+1} \sum_{l_2=0}^{j_2} (-1)^{k+l_1+l_2} C_{k+1}^{j_2} C_{k+1}^{l_1} C_{j_2}^{l_2} \sqrt{2}^{4l_1+j_2-k-3} \end{aligned}$$

$$\begin{aligned}
& \times \sum_{\substack{n=1 \\ (n,p)=1}}^{\infty} \frac{d_{k+1}(n) r_{j_2} \left(2^{(k+1-l_1)-(j_2-l_2)} n \right)}{n^2} \\
& + \sum_{j_1=0}^k \sum_{\substack{l_1=0 \\ j_1-l_1 \geq k+1-l_2}}^{j_1} \sum_{l_2=0}^{k+1} (-1)^{k+l_1+l_2} C_{k+1}^{j_1} C_{j_1}^{l_1} C_{k+1}^{l_2} \sqrt{2}^{3k+4l_1+1-3j_1} \\
& \times \sum_{\substack{n=1 \\ (n,p)=1}}^{\infty} \frac{d_{k+1} \left(2^{(j_1-l_1)-(k+1-l_2)} n \right) r_{j_2}(n)}{n^2} \\
& + \sum_{j_1=0}^k \sum_{\substack{l_1=0 \\ j_1-l_1 < k+1-l_2}}^{j_1} \sum_{l_2=0}^{k+1} (-1)^{k+l_1+l_2} C_{k+1}^{j_1} C_{j_1}^{l_1} C_{k+1}^{l_2} \sqrt{2}^{4l_2+j_1-k-3} \\
& \times \sum_{\substack{n=1 \\ (n,p)=1}}^{\infty} \frac{d_{k+1}(n) r_{j_2} \left(2^{(k+1-l_2)-(j_1-l_1)} n \right)}{n^2} \\
& + \sum_{j_1=1}^k \sum_{j_2=1}^k \sum_{\substack{l_1=0 \\ j_1-l_1 \geq j_2-l_2}}^{j_1} \sum_{l_2=0}^{j_2} (-1)^{k+l_1+l_2} \\
& \times C_{k+1}^{j_1} C_{k+1}^{j_2} C_{j_1}^{l_1} C_{j_2}^{l_2} \sqrt{2}^{2k+4l_1+j_2-3j_1} \\
& \times \sum_{\substack{n=1 \\ (n,p)=1}}^{\infty} \frac{r_{j_1-l_1}(n) r_{j_2-l_2} \left(2^{(j_1-l_1)-(j_2-l_2)} n \right)}{n^2} \\
& + \sum_{j_1=1}^k \sum_{j_2=1}^k \sum_{\substack{l_1=0 \\ j_1-l_1 < j_2-l_2}}^{j_1} \sum_{l_2=0}^{j_2} (-1)^{k+l_1+l_2} \\
& \times C_{k+1}^{j_1} C_{k+1}^{j_2} C_{j_1}^{l_1} C_{j_2}^{l_2} \sqrt{2}^{2k+4l_1+j_2-3j_1} \\
& \times \sum_{\substack{n=1 \\ (n,p)=1}}^{\infty} \frac{r_{j_2-l_2}(n) r_{j_1-l_1} \left(2^{(j_2-l_2)-(j_1-l_1)} n \right)}{n^2}, \tag{11}
\end{aligned}$$

and where $C_k^j = k!/j!(k-j)!$, $r_j(n) = \sum_{t|n} d_j(t) d_{k-j}(n/t)$, $\chi_{8_2}(n/t)$, and ϵ is any fixed positive number.

From Theorem 2 we know that $|E_{1/4}(a, k, p)| \gg p^{k/2}$ for some a and thus the bound in Theorem 1 is close to be the best possible.

For general odd number $q \geq 3$, whether there exist similar asymptotic formulae for $N(a, k, \mathbf{w}, q, 0)$ in the cases of $\mathbf{w} = (1/2, 1/2, \dots, 1/2)$ and $\mathbf{w} = (1/4, 1/4, \dots, 1/4)$ are open problems.

2. Several Auxiliary Lemmas

To establish the main results of our theorems, we need the following several auxiliary lemmas.

Lemma 3. Let $q \geq 9$ be an odd integer and let χ be a primitive Dirichlet character modulo q . Then one has

$$\begin{aligned}
\sum_{b=1}^{[q/4]} \chi(b) &= \begin{cases} \frac{\tau(\chi)}{\pi} L(1, \bar{\chi} \chi_4) & \text{if } \chi(-1) = 1; \\ \frac{\tau(\chi) L(1, \bar{\chi})}{2i\pi} \\ \quad \times (2 + \bar{\chi}(2) - \bar{\chi}(4)) & \text{if } \chi(-1) = -1, \end{cases} \\
\sum_{b=1}^{[q/8]} \chi(b) &= \begin{cases} \frac{\tau(\chi)}{2\pi} (\bar{\chi}(2) L(1, \bar{\chi} \chi_4) + \sqrt{2} L(1, \bar{\chi} \chi_{8_1})) & \text{if } \chi(-1) = 1; \\ \frac{\tau(\chi)}{4i\pi} ((4 + \bar{\chi}(4) - \bar{\chi}(8)) L(1, \bar{\chi}) - 2\sqrt{2} L(1, \bar{\chi} \chi_{8_2})) & \text{if } \chi(-1) = -1. \end{cases} \tag{12}
\end{aligned}$$

Proof. See [7] or Lemma 2 in [8]. $\square$

Lemma 4. Let $p \geq 5$ be an odd prime with $p \equiv 1 \pmod{4}$ and a coprime to p . Then, for any positive integer k , one has

$$\begin{aligned}
E_{1/4}(a, k, p) &= \frac{1}{2(2i\pi)^{k+1} \phi(p)} \\
& \times \sum_{\substack{\chi \bmod p \\ \chi(-1)=-1}} \bar{\chi}(a) (2 + \bar{\chi}(2) - \bar{\chi}(4))^{k+1} \tau^{k+1}(\chi) L^{k+1}(1, \bar{\chi}) \\
& + \frac{1}{2\pi^{k+1} \phi(p)} \sum_{\substack{\chi \bmod p \\ \chi(-1)=1 \\ \chi \neq \chi_0}} \bar{\chi}(a) \tau^{k+1}(\chi) L^{k+1}(1, \bar{\chi} \chi_4) \\
& + O\left(\frac{1}{p}\right) - \frac{\sqrt{2}^{k-1}}{\pi^{k+1} \phi(p)} \\
& \times \sum_{\substack{\chi \bmod p \\ \chi(-1)=1 \\ \chi \neq \chi_0}} \bar{\chi}(a) \chi^{k+1}(2) \tau^{k+1}(\chi) L^{k+1}(1, \bar{\chi} \chi_{8_1})
\end{aligned}$$

$$\begin{aligned}
& - \frac{1}{2(i\pi)^{k+1} \phi(p)} \\
& \times \sum_{\substack{\chi \bmod p \\ \chi(-1)=-1}} \bar{\chi}(a) \tau^{k+1}(\chi) \\
& \times \left((2\chi(2) - 1)L(1, \bar{\chi}) - \sqrt{2}\chi(2)L(1, \bar{\chi}\chi_{8_2}) \right)^{k+1}.
\end{aligned} \tag{13}$$

Proof. From the definition of $N_{1/4}(a, k, p)$ we have

$$\begin{aligned}
N_{1/4}(a, k, p) &= \frac{1}{2} \sum_{b_1=1}^{(p-1)/4} \sum_{b_2=1}^{(p-1)/4} \cdots \sum_{b_k=1}^{(p-1)/4} \sum_{c=1}^{(p-1)/4} (1 - (-1)^{b_1+b_2+\cdots+b_k+c}) \\
& \quad b_1 b_2 \cdots b_k c \equiv a \pmod{p} \\
&= \frac{1}{2\phi(p)} \left(\sum_{\chi \bmod p} \bar{\chi}(a) \left(\sum_{b=1}^{(p-1)/4} \chi(b) \right)^{k+1} \right. \\
& \quad \left. - \sum_{\chi \bmod p} \bar{\chi}(a) \left(\sum_{b=1}^{(p-1)/4} (-1)^b \chi(b) \right)^{k+1} \right) \\
&= \frac{\phi^k(p)}{2^{2k+3}} + \frac{1}{2\phi(p)} \sum_{\substack{\chi \bmod p \\ \chi \neq \chi_0}} \bar{\chi}(a) \left(\sum_{b=1}^{(p-1)/4} \chi(b) \right)^{k+1} \\
& \quad - \frac{1}{2\phi(p)} \sum_{\chi \bmod p} \bar{\chi}(a) \left(\sum_{b=1}^{(p-1)/4} (-1)^b \chi(b) \right)^{k+1}.
\end{aligned} \tag{14}$$

Note that $p \equiv 1 \pmod{4}$; we have

$$\begin{aligned}
& \sum_{b=1}^{(p-1)/4} (-1)^b \chi(b) \\
&= 2 \sum_{b=1, 2|b}^{(p-1)/4} \chi(b) - \sum_{b=1}^{(p-1)/4} \chi(b) \\
&= 2\chi(2) \sum_{b=1, 2|b}^{(p-1)/8} \chi(b) - \sum_{b=1}^{(p-1)/4} \chi(b) \\
&= \begin{cases} O(1) & \text{if } \chi = \chi_0 \text{ is a principal character;} \\ \sqrt{2}\chi(2) \frac{\tau(\chi)}{\pi} L(1, \bar{\chi}\chi_{8_1}) & \text{if } \chi(-1) = 1, \chi \neq \chi_0; \\ \frac{\tau(\chi)}{i\pi} \left((2\chi(2) - 1)L(1, \bar{\chi}) - \sqrt{2}\chi(2)L(1, \bar{\chi}\chi_{8_2}) \right) & \text{if } \chi(-1) = -1. \end{cases}
\end{aligned} \tag{15}$$

Combining the above with Lemma 3, we can get

$$\begin{aligned}
N(a, k, p) &= \frac{\phi^k(p)}{2^{2k+3}} + \frac{1}{2(2i\pi)^{k+1} \phi(p)} \\
& \times \sum_{\substack{\chi \bmod p \\ \chi(-1)=-1}} \bar{\chi}(a) (2 + \bar{\chi}(2) - \bar{\chi}(4))^{k+1} \tau^{k+1}(\chi) L^{k+1}(1, \bar{\chi}) \\
& + \frac{1}{2\pi^{k+1} \phi(p)} \\
& \times \sum_{\substack{\chi \bmod p \\ \chi(-1)=1 \\ \chi \neq \chi_0}} \bar{\chi}(a) \tau^{k+1}(\chi) L^{k+1}(1, \bar{\chi}\chi_4) + O\left(\frac{1}{p}\right) \\
& - \frac{\sqrt{2}^{k-1}}{\pi^{k+1} \phi(p)} \sum_{\substack{\chi \bmod p \\ \chi(-1)=1 \\ \chi \neq \chi_0}} \bar{\chi}(a) \chi^{k+1}(2) \tau^{k+1}(\chi) L^{k+1}(1, \bar{\chi}\chi_{8_1}) \\
& - \frac{1}{2(i\pi)^{k+1} \phi(p)} \sum_{\substack{\chi \bmod p \\ \chi(-1)=-1}} \bar{\chi}(a) \tau^{k+1}(\chi) \\
& \times \left((2\chi(2) - 1)L(1, \bar{\chi}) - \sqrt{2}\chi(2)L(1, \bar{\chi}\chi_{8_2}) \right)^{k+1}.
\end{aligned} \tag{16}$$

Noting the definition of $E_{1/4}(a, k, p)$, we can immediately get Lemma 4. $\square$

Lemma 5. Let $q \geq 3$ be an odd integer and a coprime to q . Then, for any positive integer k with $(q, k(k+1)) = 1$, one has

$$\begin{aligned}
& \left| \sum_{\substack{\chi \bmod q \\ \chi(-1)=-1}}^* \bar{\chi}(a) \tau^{k+1}(\chi) L^{k+1}(1, \bar{\chi}) \right| \\
& \ll 2^{k^2+2k+1} q^{k/2} \phi(q) (2k)^{\omega(q)} \ln^{k+1} q, \\
& \left| \sum_{\substack{\chi \bmod q \\ \chi(-1)=1}}^* \bar{\chi}(a) \tau^{k+1}(\chi) L^{k+1}(1, \bar{\chi}\chi_4) \right| \\
& \ll 2^{k^2+2k+1} q^{k/2} \phi(q) (2k)^{\omega(q)} \ln^{k+1} q, \\
& \left| \sum_{\substack{\chi \bmod q \\ \chi(-1)=1}}^* \bar{\chi}(a) \tau^{k+1}(\chi) L^{k+1}(1, \bar{\chi}\chi_{8_1}) \right|
\end{aligned}$$

$$\begin{aligned}
&\ll 2^{k^2+2k+1} q^{k/2} \phi(q) (2k)^{\omega(q)} \ln^{k+1} q, \\
&\left| \sum_{\substack{\chi \bmod q \\ \chi(-1)=-1}}^* \bar{\chi}(a) \tau^{k+1}(\chi) L^{k+1}(1, \bar{\chi} \chi_{8_2}) \right| \\
&\ll 2^{k^2+2k+1} q^{k/2} \phi(q) (2k)^{\omega(q)} \ln^{k+1} q.
\end{aligned} \tag{17}$$

Proof. Using the similar method as proving Lemma 5 of [9], we can obtain these estimates. $\square$

Lemma 6. Let q and r be integers with $q \geq 2$ and $(r, q) = 1$, and let χ be a Dirichlet character modulo q . Then one has the identities

$$\begin{aligned}
\sum_{\chi \bmod q}^* \chi(r) &= \sum_{d|(q, r-1)} \mu\left(\frac{q}{d}\right) \phi(d), \\
J(q) &= \sum_{d|q} \mu(d) \phi\left(\frac{q}{d}\right),
\end{aligned} \tag{18}$$

where $\sum_{\chi \bmod q}^*$ denotes the summation over all primitive characters modulo q and $J(q)$ is the number of primitive characters modulo q .

Proof. This is Lemma 3 of [10]. $\square$

Lemma 7. Let q, m be nonnegative integers with $q \geq 3$ an odd integer, and let χ be a Dirichlet character modulo q . Then, for any positive integer k , one has

$$\begin{aligned}
&\sum_{\chi(-1)=-1}^* \chi(2^m) |L(1, \bar{\chi})|^{2k} \\
&= \frac{J(q)}{2^{m+1}} \sum_{\substack{n=1 \\ (n, q)=1}}^{\infty} \frac{d_k(2^m n) d_k(n)}{n^2} + O(q^\epsilon), \\
&\sum_{\chi(-1)=-1}^* \bar{\chi}(2^m) |L(1, \bar{\chi})|^{2k} \\
&= \frac{J(q)}{2^{m+1}} \sum_{\substack{n=1 \\ (n, q)=1}}^{\infty} \frac{d_k(2^m n) d_k(n)}{n^2} + O(q^\epsilon), \\
&\sum_{\chi(-1)=1}^* |L(1, \bar{\chi} \chi_4)|^{2k} \\
&= \frac{J(q)}{2} \sum_{\substack{n=1 \\ (n, 2q)=1}}^{\infty} \frac{d_k^2(n)}{n^2} + O(q^\epsilon), \\
&\sum_{\chi(-1)=1}^* |L(1, \bar{\chi} \chi_{8_1})|^{2k}
\end{aligned}$$

$$\begin{aligned}
&= J(q) \sum_{\substack{n=1 \\ (n, 2q)=1}}^{\infty} \frac{d_k^2(n)}{n^2} + O(q^\epsilon), \\
&\sum_{\chi(-1)=-1}^* |L(1, \bar{\chi} \chi_{8_2})|^{2k} \\
&= J(q) \sum_{\substack{n=1 \\ (n, 2q)=1}}^{\infty} \frac{d_k^2(n)}{n^2} + O(q^\epsilon).
\end{aligned} \tag{19}$$

Proof. Using the similar method as proving Lemma 4 of [11], we can get the results. $\square$

Lemma 8. Let $q \geq 3$ be an odd integer, and let $k \geq 2$ be a positive integer. Then, for any fixed nonnegative integers m, j such that $1 \leq j \leq k$, one has

$$\begin{aligned}
&\sum_{\chi(-1)=-1}^* \chi(2^m) L^k(1, \bar{\chi} \chi_{8_2}) L^j(1, \chi) L^{k-j}(1, \chi \chi_{8_2}) \\
&= \frac{J(q)}{2^{m+1}} \sum_{\substack{n=1 \\ (n, q)=1}}^{\infty} \frac{\chi_{8_2}(2^m n) d_k(2^m n) r_j(n)}{n^2} + O(q^\epsilon), \\
&\sum_{\chi(-1)=-1}^* \bar{\chi}(2^m) L^k(1, \chi \chi_{8_2}) L^j(1, \bar{\chi}) L^{k-j}(1, \bar{\chi} \chi_{8_2}) \\
&= \frac{J(q)}{2^{m+1}} \sum_{\substack{n=1 \\ (n, q)=1}}^{\infty} \frac{\chi_{8_2}(2^m n) d_k(2^m n) r_j(n)}{n^2} + O(q^\epsilon), \\
&\sum_{\chi(-1)=1}^* \bar{\chi}(2^k) L^k(1, \bar{\chi} \chi_4) L^k(1, \chi \chi_{8_1}) = O(q^\epsilon), \\
&\sum_{\chi(-1)=1}^* \chi(2^k) L^k(1, \chi \chi_4) L^k(1, \bar{\chi} \chi_{8_1}) = O(q^\epsilon), \\
&\sum_{\chi(-1)=-1}^* \bar{\chi}(2^m) L^k(1, \bar{\chi}) L^k(1, \chi \chi_{8_2}) \\
&= \frac{J(q)}{2^{m+1}} \sum_{\substack{n=1 \\ (n, q)=1}}^{\infty} \frac{\chi_{8_2}(2^m n) d_k(2^m n) d_k(n)}{n^2} + O(q^\epsilon), \\
&\sum_{\chi(-1)=-1}^* \chi(2^m) L^k(1, \chi) L^k(1, \bar{\chi} \chi_{8_2}) \\
&= \frac{J(q)}{2^{m+1}} \sum_{\substack{n=1 \\ (n, q)=1}}^{\infty} \frac{\chi_{8_2}(2^m n) d_k(2^m n) d_k(n)}{n^2} + O(q^\epsilon), \\
&\sum_{\chi(-1)=-1}^* \bar{\chi}(2^m) L^k(1, \bar{\chi}) L^j(1, \chi) L^{k-j}(1, \chi \chi_{8_2}) \\
&= \frac{J(q)}{2^{m+1}} \sum_{\substack{n=1 \\ (n, q)=1}}^{\infty} \frac{d_k(n) r_j(2^m n)}{n^2} + O(q^\epsilon),
\end{aligned}$$

$$\begin{aligned}
& \sum_{\chi(-1)=-1}^* \chi(2^m) L^k(1, \chi) L^j(1, \bar{\chi}) L^{k-j}(1, \bar{\chi} \chi_{8_2}) \\
&= \frac{J(q)}{2^{m+1}} \sum_{\substack{n=1 \\ (n,q)=1}}^{\infty} \frac{d_k(n) r_j(2^m n)}{n^2} + O(q^\epsilon), \\
& \sum_{\chi(-1)=-1}^* \chi(2^m) L^k(1, \bar{\chi}) L^j(1, \chi) L^{k-j}(1, \chi \chi_{8_2}) \\
&= \frac{J(q)}{2^{m+1}} \sum_{\substack{n=1 \\ (n,q)=1}}^{\infty} \frac{d_k(2^m n) r_j(n)}{n^2} + O(q^\epsilon), \\
& \sum_{\chi(-1)=-1}^* \bar{\chi}(2^m) L^k(1, \chi) L^j(1, \bar{\chi}) L^{k-j}(1, \bar{\chi} \chi_{8_2}) \\
&= \frac{J(q)}{2^{m+1}} \sum_{\substack{n=1 \\ (n,q)=1}}^{\infty} \frac{d_k(2^m n) r_j(n)}{n^2} + O(q^\epsilon),
\end{aligned} \tag{20}$$

where $r_j(n) = \sum_{t|n} d_j(t) d_{k-j}(n/t) \chi_{8_2}(n/t)$.

Proof. We only prove the first formula; the others can be obtained by the similar method.

For convenience, we put

$$A(y, \chi) = \sum_{N < n \leq y} \chi(n) d_k(n), \quad B(y, \chi) = \sum_{N < n \leq y} \chi(n) r_j(n), \tag{21}$$

where N is a parameter with $q \leq N < q^{2k+1}$. Then from Abel's identity we have

$$\begin{aligned}
& L^j(1, \chi) L^{k-j}(1, \chi \chi_{8_2}) \\
&= \sum_{n=1}^{\infty} \frac{\chi(n) d_j(n_1)}{n_1} \sum_{n_2=1}^{\infty} \frac{\chi \chi_{8_2}(n_2) d_{k-j}(n_2)}{n_2} \\
&= \sum_{n=1}^{\infty} \frac{\chi(n) r_j(n)}{n} = \sum_{1 \leq n \leq N} \frac{\chi(n) r_j(n)}{n} + \int_N^{\infty} \frac{B(y, \chi)}{y^2} dy.
\end{aligned} \tag{22}$$

Hence, we have

$$\begin{aligned}
& \sum_{\chi(-1)=-1}^* \chi(2^m) L^k(1, \bar{\chi} \chi_{8_2}) L^j(1, \chi) L^{k-j}(1, \chi \chi_{8_2}) \\
&= \sum_{\chi(-1)=-1}^* \chi(2^m) \\
&\quad \times \left(\sum_{1 \leq n_1 \leq N} \frac{\bar{\chi} \chi_{8_2}(n_1) d_k(n_1)}{n_1} + \int_N^{\infty} \frac{A(y, \bar{\chi} \chi_{8_2})}{y^2} dy \right) \\
&\quad \times \left(\sum_{1 \leq n_2 \leq N} \frac{\chi(n_2) r_j(n_2)}{n_2} + \int_N^{\infty} \frac{B(y, \chi)}{y^2} dy \right).
\end{aligned} \tag{23}$$

From the proof of Lemma 6 of [11], we know that only the term which does not contain the infinite integral will make contribution to the main term. That is,

$$\begin{aligned}
& \sum_{\chi(-1)=-1}^* \chi(2^m) L^k(1, \bar{\chi} \chi_{8_2}) L^j(1, \chi) L^{k-j}(1, \chi \chi_{8_2}) \\
&= \sum_{\chi(-1)=-1}^* \chi(2^m) \sum_{1 \leq n_1 \leq N} \sum_{1 \leq n_2 \leq N} \frac{\chi(\bar{n}_1 n_2) \chi_{8_2}(n_1) d_k(n_1) r_j(n_2)}{n_1 n_2} \\
&\quad + O(q^\epsilon).
\end{aligned} \tag{24}$$

Then from Lemma 6, we can write

$$\begin{aligned}
& \sum_{\chi(-1)=-1}^* \chi(2^m) \sum_{1 \leq n_1 \leq N} \sum_{1 \leq n_2 \leq N} \frac{\chi(\bar{n}_1 n_2) \chi_{8_2}(n_1) d_k(n_1) r_j(n_2)}{n_1 n_2} \\
&= \frac{1}{2} \sum'_{1 \leq n_1 \leq N} \sum'_{1 \leq n_2 \leq N} \frac{\chi_{8_2}(n_1) d_k(n_1) r_j(n_2)}{n_1 n_2} \\
&\quad \times \sum_{d|(q, \bar{n}_1 2^m n_2 - 1)} \mu(d) \phi\left(\frac{q}{d}\right) \\
&\quad - \frac{1}{2} \sum'_{1 \leq n_1 \leq N} \sum'_{1 \leq n_2 \leq N} \frac{\chi_{8_2}(n_1) d_k(n_1) r_j(n_2)}{n_1 n_2} \\
&\quad \times \sum_{d|(q, \bar{n}_1 2^m n_2 + 1)} \mu(d) \phi\left(\frac{q}{d}\right) \\
&= \frac{1}{2} \sum_{d|q} \mu(d) \phi\left(\frac{q}{d}\right) \sum'_{1 \leq n_1 \leq N} \sum'_{\substack{1 \leq n_2 \leq N \\ \bar{n}_1 2^m n_2 \equiv 1 \pmod{d}}} \frac{\chi_{8_2}(n_1) d_k(n_1) r_j(n_2)}{n_1 n_2} \\
&\quad - \frac{1}{2} \sum_{d|q} \mu(d) \phi\left(\frac{q}{d}\right) \\
&\quad \times \sum'_{1 \leq n_1 \leq N} \sum'_{\substack{1 \leq n_2 \leq N \\ \bar{n}_1 2^m n_2 \equiv -1 \pmod{d}}} \frac{\chi_{8_2}(n_1) d_k(n_1) r_j(n_2)}{n_1 n_2},
\end{aligned} \tag{25}$$

where $\sum'_{1 \leq n \leq N}$ denotes the summation over all integers n with $(n, q) = 1$ and $1 \leq n \leq N$. Now we split the above first sum into the following four cases:

- (i) $d \leq n_1 \leq N$ and $d/2^m \leq n_2 \leq N$;
- (ii) $d \leq n_1 \leq N$ and $1 \leq n_2 \leq d/2^m - 1$;
- (iii) $1 \leq n_1 \leq d - 1$ and $d/2^m \leq n_2 \leq N$;
- (iv) $1 \leq n_1 \leq d - 1$ and $1 \leq n_2 \leq d/2^m - 1$.

By using the similar method as proving Lemma 6 of [11], we know that the main term will be

$$\begin{aligned} & \frac{1}{2} \sum_{d|q} \mu(d) \phi\left(\frac{q}{d}\right) \sum'_{1 \leq n_1 \leq N} \sum'_{1 \leq n_2 \leq N} \frac{\chi_{8_2}(n_1) d_k(n_1) r_j(n_2)}{n_1 n_2} \\ &= \frac{1}{2} \sum_{d|q} \mu(d) \phi\left(\frac{q}{d}\right) \\ & \quad \times \sum_{\substack{n_2=1 \\ (n_2, q)=1}}^{\infty} \frac{\chi_{8_2}(2^m n_2) d_k(2^m n_2) r_j(n_2)}{2^m n_2^2} + O(q^\epsilon). \end{aligned} \quad (26)$$

Hence we have

$$\begin{aligned} & \sum_{\chi(-1)=-1}^* \chi(2^m) L^k(1, \bar{\chi} \chi_{8_2}) L^j(1, \chi) L^{k-j}(1, \chi \chi_{8_2}) \\ &= \frac{J(q)}{2^{m+1}} \sum_{\substack{n=1 \\ (n, q)=1}}^{\infty} \frac{\chi_{8_2}(2^m n) d_k(2^m n) r_j(n)}{n^2} + O(q^\epsilon). \end{aligned} \quad (27)$$

This proves Lemma 8. $\square$

Lemma 9. Let $q \geq 3$ be an odd integer and let $k \geq 2$ be a positive integer. Then, for any fixed nonnegative integers $a, b \leq k$ such that $a \neq 0$ while $b = 0$, and $a \neq k$ while $b = k$, one has

$$\begin{aligned} & \sum_{\substack{\chi \bmod q \\ \chi(-1)=-1}}^* \bar{\chi}(2^a) \chi(2^b) L^a(1, \bar{\chi}) L^b(1, \chi) \\ & \quad \times L^{k-a}(1, \bar{\chi} \chi_{8_2}) L^{k-b}(1, \chi \chi_{8_2}) \\ &= \begin{cases} \frac{J(q)}{2^{b-a+1}} \sum_{\substack{n=1 \\ (n, q)=1}}^{\infty} \frac{r_a(2^{b-a} n) r_b(n)}{n^2} + O(q^\epsilon) & \text{if } a \leq b; \\ \frac{J(q)}{2^{a-b+1}} \sum_{\substack{n=1 \\ (n, q)=1}}^{\infty} \frac{r_a(n) r_b(2^{a-b} n)}{n^2} + O(q^\epsilon) & \text{if } a > b. \end{cases} \end{aligned} \quad (28)$$

Proof. Using the similar method as proving Lemma 6 of [11], we can get the results. $\square$

3. Proof of Theorems

In this section, we will complete the proof of our theorems. From Lemmas 4 and 5, we can immediately get the result of Theorem 1.

Now we come to prove Theorem 2. Noting that

$$\sum_{a=1}^p \chi(a) = 0 \quad (29)$$

if $\chi \neq \chi_0$, from Lemma 4 and the orthogonality of Dirichlet character, we can write

$$\begin{aligned} & \sum_{a=1}^{p-1} |E_{1/4}(a, k, p)|^2 \\ &= \frac{1}{4\pi^{2k+2} \phi(p)} \sum_{\substack{\chi \bmod p \\ \chi(-1)=1}}^* |\tau(\chi)|^{2k+2} |L(1, \bar{\chi} \chi_4)|^{2k+2} \\ & \quad + O\left(\frac{1}{p}\right) - \frac{2^{k-1}}{\pi^{2k+2} \phi(p)} \sum_{\substack{\chi \bmod p \\ \chi(-1)=1}}^* |\tau(\chi)|^{2k+2} \\ & \quad \times |L(1, \bar{\chi} \chi_{8_1})|^{2k+2} \\ & \quad + \frac{1}{2^{2k+4} \pi^{2k+2} \phi(p)} \\ & \quad \times \sum_{\substack{\chi \bmod p \\ \chi(-1)=-1}}^* |2 + \bar{\chi}(2) - \bar{\chi}(4)|^{2k+2} |\tau(\chi)|^{2k+2} |L(1, \bar{\chi})|^{2k+2} \\ & \quad - \frac{(-1)^{k+1}}{4\pi^{2k+2} \phi(p)} \sum_{\substack{\chi \bmod p \\ \chi(-1)=-1}}^* |\tau(\chi)|^{2k+2} \\ & \quad \times |(2\chi(2) - 1)L(1, \bar{\chi}) - \sqrt{2}\chi(2)L(1, \bar{\chi} \chi_{8_2})|^{2k+2} \\ & \quad - \frac{\sqrt{2}^{k-3}}{\pi^{2k+2} \phi(p)} \sum_{\substack{\chi \bmod p \\ \chi(-1)=1}}^* \bar{\chi}(2^{k+1}) |\tau(\chi)|^{2k+2} \\ & \quad \times L^{k+1}(1, \bar{\chi} \chi_4) L^{k+1}(1, \chi \chi_{8_1}) \\ & \quad - \frac{\sqrt{2}^{k-3}}{\pi^{2k+2} \phi(p)} \sum_{\substack{\chi \bmod p \\ \chi(-1)=1}}^* \chi(2^{k+1}) |\tau(\chi)|^{2k+2} \\ & \quad \times L^{k+1}(1, \chi \chi_4) L^{k+1}(1, \bar{\chi} \chi_{8_1}) \\ & \quad - \frac{(-1)^{k+1}}{2^{k+3} \pi^{2k+2} \phi(p)} \sum_{\substack{\chi \bmod p \\ \chi(-1)=-1}}^* |\tau(\chi)|^{2k+2} \\ & \quad \times (2 + \bar{\chi}(2) - \bar{\chi}(4))^{k+1} L^{k+1}(1, \bar{\chi}) \\ & \quad \times ((2\bar{\chi}(2) - 1)L(1, \chi) - \sqrt{2}\bar{\chi}(2)L(1, \chi \chi_{8_2}))^{k+1} \\ & \quad - \frac{(-1)^{k+1}}{2^{k+3} \pi^{2k+2} \phi(p)} \sum_{\substack{\chi \bmod p \\ \chi(-1)=-1}}^* |\tau(\chi)|^{2k+2} \\ & \quad \times (2 + \chi(2) - \chi(4))^{k+1} L^{k+1}(1, \chi) \\ & \quad \times ((2\chi(2) - 1)L(1, \bar{\chi}) - \sqrt{2}\chi(2)L(1, \bar{\chi} \chi_{8_2}))^{k+1} \end{aligned}$$

$$\begin{aligned}
&= \frac{p^{k+1}}{4\pi^{2k+2}\phi(p)} \sum_{\substack{\chi \bmod p \\ \chi(-1)=1}}^* |L(1, \bar{\chi}\chi_4)|^{2k+2} \\
&\quad - \frac{2^{k-1}p^{k+1}}{\pi^{2k+2}\phi(p)} \sum_{\substack{\chi \bmod p \\ \chi(-1)=1}}^* |L(1, \bar{\chi}\chi_{8_1})|^{2k+2} \\
&\quad + \frac{p^{k+1}}{2^{2k+4}\pi^{2k+2}\phi(p)} \sum_{i=0}^{k+1} (-2)^{k+1-i} C_{k+1}^i \\
&\quad \times \sum_{j=0}^i 6^j \sum_{s=0}^{i-j} \sum_{t=0}^{i-j-k+1-i} C_{i-j}^s C_{k+1-i}^t \\
&\quad \times \sum_{\substack{\chi \bmod p \\ \chi(-1)=-1}}^* \chi(2^{2s+4t+i+j-2k-2}) |L(1, \bar{\chi})|^{2k+2} \\
&\quad - \frac{(-p)^{k+1}}{4\pi^{2k+2}\phi(p)} \sum_{j_1=0}^{k+1} \sum_{j_2=0}^{k+1} \sum_{l_1=0}^{j_1} \sum_{l_2=0}^{j_2} (-1)^{l_1+l_2} \sqrt{2}^{2k+2(l_1+l_2)+2-(j_1+j_2)} \\
&\quad \times C_{k+1}^{j_1} C_{k+1}^{j_2} C_{j_1}^{l_1} C_{j_2}^{l_2} \sum_{\substack{\chi \bmod p \\ \chi(-1)=-1}}^* \bar{\chi}(2^{j_1-l_1}) \chi(2^{j_2-l_2}) L^{j_1}(1, \bar{\chi}) \\
&\quad \times L^{j_2}(1, \chi) L^{k+1-j_1}(1, \bar{\chi}\chi_{8_2}) L^{k+1-j_2}(1, \chi\chi_{8_2}) \\
&\quad - \frac{\sqrt{2}^{k-3} p^{k+1}}{\pi^{2k+2}\phi(p)} \sum_{\substack{\chi \bmod p \\ \chi(-1)=1}}^* \bar{\chi}(2^{k+1}) L^{k+1}(1, \bar{\chi}\chi_4) L^{k+1}(1, \chi\chi_{8_1}) \\
&\quad - \frac{\sqrt{2}^{k-3} p^{k+1}}{\pi^{2k+2}\phi(p)} \sum_{\substack{\chi \bmod p \\ \chi(-1)=1}}^* \chi(2^{k+1}) L^{k+1}(1, \chi\chi_4) L^{k+1}(1, \bar{\chi}\chi_{8_1}) \\
&\quad - \frac{(-p)^{k+1}}{2^{k+3}\pi^{2k+2}\phi(p)} \\
&\quad \times \sum_{j_1=0}^{k+1} \sum_{j_2=0}^{k+1} \sum_{l_1=0}^{j_1} \sum_{l_2=0}^{j_2} (-1)^{j_1+l_2} \sqrt{2}^{k+2(l_1+l_2)+1-j_2} C_{k+1}^{j_1} C_{k+1}^{j_2} C_{j_1}^{l_1} C_{j_2}^{l_2} \\
&\quad \times \sum_{\substack{\chi \bmod p \\ \chi(-1)=-1}}^* \bar{\chi}(2^{3k+3+l_2-j_1-j_2-l_1}) \\
&\quad \times L^{k+1}(1, \bar{\chi}) L^{j_2}(1, \chi) L^{k+1-j_2}(1, \chi\chi_{8_2}) \\
&\quad - \frac{(-p)^{k+1}}{2^{k+3}\pi^{2k+2}\phi(p)} \\
&\quad \times \sum_{j_1=0}^{k+1} \sum_{j_2=0}^{k+1} \sum_{l_1=0}^{j_1} \sum_{l_2=0}^{j_2} (-1)^{j_1+l_2} \sqrt{2}^{k+2(l_1+l_2)+1-j_2} C_{k+1}^{j_1} C_{k+1}^{j_2} C_{j_1}^{l_1} C_{j_2}^{l_2}
\end{aligned}$$

$$\begin{aligned}
&\times \sum_{\substack{\chi \bmod p \\ \chi(-1)=-1}}^* \chi(2^{3k+3+l_2-j_1-j_2-l_1}) \\
&\times L^{k+1}(1, \chi) L^{j_2}(1, \bar{\chi}) L^{k+1-j_2}(1, \bar{\chi}\chi_{8_2}).
\end{aligned} \tag{30}$$

And consider that

$$\begin{aligned}
&\sum_{j_1=0}^{k+1} \sum_{j_2=0}^{k+1} \sum_{l_1=0}^{j_1} \sum_{l_2=0}^{j_2} (-1)^{l_1+l_2} \sqrt{2}^{-2k+2(l_1+l_2)+2-(j_1+j_2)} \\
&\quad \times C_{k+1}^{j_1} C_{k+1}^{j_2} C_{j_1}^{l_1} C_{j_2}^{l_2} \\
&\quad \times \sum_{\substack{\chi \bmod p \\ \chi(-1)=-1}}^* \bar{\chi}(2^{j_1-l_1}) \times \chi(2^{j_2-l_2}) L^{j_1}(1, \bar{\chi}) \\
&\quad \times L^{j_2}(1, \chi) L^{k+1-j_1}(1, \bar{\chi}\chi_{8_2}) L^{k+1-j_2}(1, \chi\chi_{8_2}) \\
&= 2^{k+1} \sum_{\substack{\chi \bmod p \\ \chi(-1)=-1}}^* |L(1, \bar{\chi}\chi_{8_2})|^{2k+2} \\
&\quad + \sum_{l_1=0}^{k+1} \sum_{l_2=0}^{k+1} (-2)^{l_1+l_2} C_{k+1}^{l_1} C_{k+1}^{l_2} \sum_{\substack{\chi \bmod p \\ \chi(-1)=-1}}^* \chi(2^{l_1-l_2}) |L(1, \bar{\chi})|^{2k+2} \\
&\quad + \sum_{j_2=1}^{k+1} \sum_{l_2=0}^{j_2} (-1)^{l_2} \sqrt{2}^{-2k+2l_2+2-j_2} C_{k+1}^{j_2} C_{j_2}^{l_2} \\
&\quad \times \sum_{\substack{\chi \bmod p \\ \chi(-1)=-1}}^* \chi(2^{j_2-l_2}) L^{j_2}(1, \chi) \\
&\quad \times L^{k+1}(1, \bar{\chi}\chi_{8_2}) L^{k+1-j_2}(1, \chi\chi_{8_2}) \\
&\quad + \sum_{j_1=1}^{k+1} \sum_{l_1=0}^{j_1} (-1)^{l_1} \sqrt{2}^{-2k+2l_1+2-j_1} C_{k+1}^{j_1} C_{j_1}^{l_1} \\
&\quad \times \sum_{\substack{\chi \bmod p \\ \chi(-1)=-1}}^* \bar{\chi}(2^{j_1-l_1}) L^{j_1}(1, \bar{\chi}) \\
&\quad \times L^{k+1-j_1}(1, \bar{\chi}\chi_{8_2}) L^{k+1}(1, \chi\chi_{8_2}) \\
&\quad + \sum_{j_2=0}^k \sum_{l_1=0}^{k+1} \sum_{l_2=0}^{j_2} (-1)^{l_1+l_2} \sqrt{2}^{k+2(l_1+l_2)+1-j_2} C_{k+1}^{j_2} C_{k+1}^{l_1} C_{j_2}^{l_2} \\
&\quad \times \sum_{\substack{\chi \bmod p \\ \chi(-1)=-1}}^* \chi(2^{(j_2-l_2)-(k+1-l_1)}) L^{k+1}(1, \bar{\chi}) \\
&\quad \times L^{j_2}(1, \chi) L^{k+1-j_2}(1, \chi\chi_{8_2}) \\
&\quad + \sum_{j_1=0}^k \sum_{l_1=0}^{j_1} \sum_{l_2=0}^{k+1} (-1)^{l_1+l_2} \sqrt{2}^{k+2(l_1+l_2)+1-j_1} C_{k+1}^{j_1} C_{j_1}^{l_1} C_{k+1}^{l_2}
\end{aligned}$$

$$\begin{aligned}
& \times \sum_{\substack{\chi \bmod p \\ \chi(-1)=-1}}^* \chi \left(2^{(k+1-l_2)-(j_1-l_1)} \right) L^{j_1} (1, \bar{\chi}) \\
& \times L^{k+1} (1, \chi) L^{k+1-j_1} (1, \bar{\chi} \chi_{8_2}) \\
& + \sum_{j_1=1}^k \sum_{j_2=1}^k \sum_{l_1=0}^{j_1} \sum_{l_2=0}^{j_2} (-1)^{l_1+l_2} \sqrt{2}^{2k+2(l_1+l_2)+2-(j_1+j_2)} \\
& \times C_{k+1}^{j_1} C_{k+1}^{j_2} C_{j_1}^{l_1} C_{j_2}^{l_2} \sum_{\substack{\chi \bmod p \\ \chi(-1)=-1}}^* \bar{\chi} \left(2^{j_1-l_1} \right) \times \chi \left(2^{j_2-l_2} \right) L^{j_1} (1, \bar{\chi}) \\
& \times L^{j_2} (1, \chi) L^{k+1-j_1} (1, \bar{\chi} \chi_{8_2}) L^{k+1-j_2} (1, \chi \chi_{8_2}), \\
& \sum_{j_1=0}^{k+1} \sum_{j_2=0}^{k+1} \sum_{l_1=0}^{j_1} \sum_{l_2=0}^{j_2} (-1)^{j_1+l_2} \sqrt{2}^{k+2(l_1+l_2)+1-j_2} C_{k+1}^{j_1} C_{k+1}^{j_2} C_{j_1}^{l_1} C_{j_2}^{l_2} \\
& \times \sum_{\substack{\chi \bmod p \\ \chi(-1)=-1}}^* \bar{\chi} \left(2^{3k+3+l_2-j_1-j_2-l_1} \right) \\
& \times L^{k+1} (1, \bar{\chi}) L^{j_2} (1, \chi) L^{k+1-j_2} (1, \chi \chi_{8_2}) \\
& = \sum_{j_1=0}^{k+1} \sum_{l_1=0}^{j_1} (-1)^{j_1} \sqrt{2}^{k+2l_1+1} C_{k+1}^{j_1} C_{j_1}^{l_1} \\
& \times \sum_{\substack{\chi \bmod p \\ \chi(-1)=-1}}^* \bar{\chi} \left(2^{3k+3-j_1-l_1} \right) L^{k+1} (1, \bar{\chi}) L^{k+1} (1, \chi \chi_{8_2}) \\
& + \sum_{j_1=0}^{k+1} \sum_{l_1=0}^{j_1} \sum_{l_2=0}^{k+1} (-1)^{j_1+l_2} 2^{l_1+l_2} C_{k+1}^{j_1} C_{j_1}^{l_1} C_{k+1}^{l_2} \\
& \times \sum_{\substack{\chi \bmod p \\ \chi(-1)=-1}}^* \bar{\chi} \left(2^{2k+2+l_2-j_1-l_1} \right) |L(1, \chi)|^{2k+2} \\
& + \sum_{j_1=0}^{k+1} \sum_{j_2=1}^k \sum_{l_1=0}^{j_1} \sum_{l_2=0}^{j_2} (-1)^{j_1+l_2} \sqrt{2}^{k+2(l_1+l_2)+1-j_2} \\
& \times C_{k+1}^{j_1} C_{k+1}^{j_2} C_{j_1}^{l_1} C_{j_2}^{l_2} \sum_{\substack{\chi \bmod p \\ \chi(-1)=-1}}^* \bar{\chi} \left(2^{3k+3+l_2-j_1-j_2-l_1} \right) \\
& \times L^{k+1} (1, \bar{\chi}) L^{j_2} (1, \chi) L^{k+1-j_2} (1, \chi \chi_{8_2}).
\end{aligned} \tag{31}$$

Combining the above with Lemmas 7–9, we have

$$\sum_{a=1}^{p-1} |E_{1/4}(a, k, p)|^2 = \frac{A(k)}{\pi^{2k+2}} p^{k+1} + O_k(p^{k+\epsilon}), \tag{32}$$

where

$$\begin{aligned}
A(k) &= \left(\frac{1}{8} - 2^{k-1} + (-1)^k 2^{k-1} \right) \sum_{\substack{n=1 \\ (n, 2p)=1}}^{\infty} \frac{d_{k+1}^2(n)}{n^2} \\
&+ \sum_{i=0}^{k+1} C_{k+1}^i \sum_{j=0}^i 6^j \sum_{s=0}^{i-j} \sum_{t=0}^{k+1-i} C_{i-j}^s C_{k+1-i}^t \\
&\times \frac{(-1)^{k+1-i}}{2^{|2s+4t+i+j-2k-2|+k+i+5}} \\
&\times \sum_{\substack{n=1 \\ (n, p)=1}}^{\infty} \frac{d_{k+1} \left(2^{|2s+4t+i+j-2k-2|} n \right) d_{k+1}(n)}{n^2} \\
&+ \sum_{j_1=0}^{k+1} \sum_{l_1=0}^{j_1} \sum_{l_2=0}^{k+1} C_{k+1}^{j_1} C_{j_1}^{l_1} C_{k+1}^{l_2} \frac{(-1)^{k+j_1+l_2}}{2^{|2k+2+l_2-j_1-l_1|+k+3-l_1-l_2}} \\
&\times \sum_{\substack{n=1 \\ (n, p)=1}}^{\infty} \frac{d_{k+1} \left(2^{|2k+2+l_2-j_1-l_1|} n \right) d_{k+1}(n)}{n^2} \\
&+ \sum_{j_1=0}^{k+1} \sum_{j_2=1}^k \sum_{l_1=0}^{j_1} \sum_{l_2=0}^{j_2} C_{k+1}^{j_1} C_{k+1}^{j_2} C_{j_1}^{l_1} C_{j_2}^{l_2} \\
&\times (-1)^{k+j_1+l_2} \sqrt{2}^{2j_1+j_2+4l_1-7k-11} \\
&\times \sum_{\substack{n=1 \\ (n, p)=1}}^{\infty} \frac{r_{j_2} \left(2^{3k+3+l_2-j_1-j_2-l_1} n \right) d_{k+1}(n)}{n^2} \\
&+ \sum_{j_2=1}^{k+1} (-1)^{k+j_2} \sqrt{2}^{2k+j_2+2} C_{k+1}^{j_2} \sum_{\substack{n=1 \\ (n, p)=1}}^{\infty} \frac{d_{k+1}(n) r_{j_2}(n)}{n^2} \\
&+ \sum_{l_1=0}^{k+1} \sum_{l_2=0}^{k+1} C_{k+1}^{l_1} C_{k+1}^{l_2} \frac{(-1)^{k+l_1+l_2}}{2^{|l_1-l_2|+2-l_1-l_2}} \\
&\times \sum_{\substack{n=1 \\ (n, p)=1}}^{\infty} \frac{d_{k+1} \left(2^{|l_1-l_2|} n \right) d_{k+1}(n)}{n^2} \\
&+ \sum_{j_2=0}^k \sum_{l_1=0}^{k+1} \sum_{l_2=0}^{j_2} (-1)^{k+l_1+l_2} C_{k+1}^{j_2} C_{k+1}^{l_1} C_{j_2}^{l_2} \sqrt{2}^{3k+4l_2+1-3j_2} \\
&\times \sum_{\substack{n=1 \\ (n, p)=1}}^{\infty} \frac{d_{k+1} \left(2^{(j_2-l_2)-(k+1-l_1)} n \right) r_{j_2}(n)}{n^2}
\end{aligned}$$

$$\begin{aligned}
& + \sum_{j_2=0}^k \sum_{l_1=0}^{k+1} \sum_{\substack{j_2 \\ j_2-l_2 < k+1-l_1}}^{j_2} (-1)^{k+l_1+l_2} C_{k+1}^{j_2} C_{k+1}^{l_1} C_{j_2}^{l_2} \sqrt{2}^{4l_1+j_2-k-3} \\
& \times \sum_{\substack{n=1 \\ (n,p)=1}}^{\infty} \frac{d_{k+1}(n) r_{j_2} \left(2^{(k+1-l_1)-(j_2-l_2)} n \right)}{n^2} \\
& + \sum_{j_1=0}^k \sum_{l_1=0}^{j_1} \sum_{l_2=0}^{k+1} (-1)^{k+l_1+l_2} C_{k+1}^{j_1} C_{j_1}^{l_1} C_{k+1}^{l_2} \sqrt{2}^{3k+4l_1+1-3j_1} \\
& \times \sum_{\substack{n=1 \\ (n,p)=1}}^{\infty} \frac{d_{k+1} \left(2^{(j_1-l_1)-(k+1-l_2)} n \right) r_{j_2}(n)}{n^2} \\
& + \sum_{j_1=0}^k \sum_{l_1=0}^{j_1} \sum_{\substack{l_2=0 \\ j_1-l_1 < k+1-l_2}}^{k+1} (-1)^{k+l_1+l_2} C_{k+1}^{j_1} C_{j_1}^{l_1} C_{k+1}^{l_2} \sqrt{2}^{4l_2+j_1-k-3} \\
& \times \sum_{\substack{n=1 \\ (n,p)=1}}^{\infty} \frac{d_{k+1}(n) r_{j_2} \left(2^{(k+1-l_2)-(j_1-l_1)} n \right)}{n^2} \\
& + \sum_{j_1=1}^k \sum_{j_2=1}^k \sum_{l_1=0}^{j_1} \sum_{\substack{l_2=0 \\ j_1-l_1 \geq j_2-l_2}}^{j_2} (-1)^{k+l_1+l_2} \\
& \times C_{k+1}^{j_1} C_{k+1}^{j_2} C_{j_1}^{l_1} C_{j_2}^{l_2} \sqrt{2}^{2k+4l_1+j_2-3j_1} \\
& \times \sum_{\substack{n=1 \\ (n,p)=1}}^{\infty} \frac{r_{j_1-l_1}(n) r_{j_2-l_2} \left(2^{(j_1-l_1)-(j_2-l_2)} n \right)}{n^2} \\
& + \sum_{j_1=1}^k \sum_{j_2=1}^k \sum_{l_1=0}^{j_1} \sum_{\substack{l_2=0 \\ j_1-l_1 < j_2-l_2}}^{j_2} (-1)^{k+l_1+l_2} \\
& \times C_{k+1}^{j_1} C_{k+1}^{j_2} C_{j_1}^{l_1} C_{j_2}^{l_2} \sqrt{2}^{2k+4l_1+j_2-3j_1} \\
& \times \sum_{\substack{n=1 \\ (n,p)=1}}^{\infty} \frac{r_{j_2-l_2}(n) r_{j_1-l_1} \left(2^{(j_2-l_2)-(j_1-l_1)} n \right)}{n^2}.
\end{aligned}$$

(33)

This proves Theorem 2.

Conflict of Interests

The author declares that there is no conflict of interests regarding the publication of this paper.

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