Some Subordination Results on $q$-Analogue of Ruscheweyh Differential Operator

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We provide some notations and concepts of $q$-calculus used in this paper. All the results can be found in [12–14]. For $n \in \mathbb{N}, 0 < q < 1$, we define

\[
[n]_q = \frac{1 - q^n}{1 - q}, \\
[n]_q! = \begin{cases} (\frac{1 - q^n}{1 - q})(\frac{1 - q^{n-1}}{1 - q})\cdots(\frac{1 - q}{1 - q}), & n=1,2,\ldots; \\ 1, & n=0. \end{cases}
\]

As $q \to 1$, $[n]_q \to n$, and this is the bookmark of a $q$-analogue: the limit as $q \to 1$ recovers the classical object.

For complex parameters $a, b, c, q \in \mathbb{C} \setminus \{0, -1, -2, \ldots, |q| < 1\}$, the $q$-analogue of Gauss’s hypergeometric function $\Phi_1(a, b; c, z)$ is defined by

\[
\Phi_1(a, b; c, q, z) = \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{(c)_k} \frac{z^k}{q^k}, \quad (z \in \mathbb{U}),
\]

where $(n, q)_k$ is the $q$-analogue of Pochhammer symbol defined by

\[
(n, q)_k = \begin{cases} 1, & k = 0; \\ (1 - n)(1 - nq)(1 - nq^2)\cdots(1 - nq^{k-1}), & k \in \mathbb{N}. \end{cases}
\]
The $q$-derivative of a function $h(x)$ is defined by
\[ D_q (h(x)) = \frac{h(qx) - h(x)}{(q-1)x}, \quad q \neq 1, \ x \neq 0, \] (4)
and $D_q(h(0)) = f'(0)$. For a function $h(z) = z^k$ observe that
\[ D_q (h(z)) = D_q (z^k) = \frac{1 - q^k}{1 - q} z^{k-1} = [k]_q z^{k-1}; \] (5)
then $\lim_{q \to 1} D_q (h(z)) = \lim_{q \to 1} [k]_q z^{k-1} = k z^{k-1} = h'(z)$, where $h'(z)$ is the ordinary derivative.

Next, we state the class $\mathcal{A}$ of all functions of the following form:
\[ f(z) = z + \sum_{k=2}^{\infty} a_k z^k, \] (6)
which are analytic in the open unit disk $U = \{ z \in \mathbb{C} : |z| < 1 \}$. If $f$ and $g$ are analytic functions in $U$, we say that $f$ is subordinate to $g$; written $f \prec g$, if there is a function $w$ analytic in $U$, with $w(0) = 0$, $|w(z)| < 1$, for all $z \in U$, such that $f(z) = g(w(z))$ for all $z \in U$. If $g$ is univalent, then $f \prec g$ if and only if $f(0) = g(0)$ and $f(U) \subseteq g(U)$.

For each $A$ and $B$ such that $-1 \leq B < A \leq 1$, we define the function
\[ h(A,B;z) = \frac{1 + Az}{1 + Bz}, \quad (z \in U). \] (7)
It is well known that $h(A,B;z)$ for $-1 \leq B < 1$ is the conformal map of the unit disk onto the disk symmetrical with respect to the real axis having the center $(1-AB)/(1-B^2)$ for $B \neq \pm 1$ and radius $(A-B)/(1-B^2)$. The boundary circle cuts the real axis at the points $(1-A)/(1-B)$ and $(1+A)/(1+B)$.

**Definition 1.** Let $f \in \mathcal{A}$. Denote by $\mathcal{R}_q^A$ the $q$-analogue of Ruscheweyh operator defined by
\[ \mathcal{R}_q^A f(z) = z + \sum_{k=2}^{\infty} \frac{[k + \lambda - 1]_q!}{[k]_q!} a_k z^k, \] (8)
where $[k]_q$ and $[k]_q^!$ are defined in (1).

From the definition we observe that, if $q \to 1$, we have
\[ \lim_{q \to 1} \mathcal{R}_q^A f(z) = z + \sum_{k=2}^{\infty} \frac{(k + \lambda - 1)!}{(\lambda)! (k-1)!} a_k z^k = \mathcal{R}^A f(z), \] (9)
where $\mathcal{R}^A$ is Ruscheweyh differential operator which was defined in [15] and has been studied by many authors, for example [16–18].

It can also be shown that this $q$-operator is hypergeometric in nature as
\[ \mathcal{R}_q^A f(z) = z_2 \Phi_1 \left( \lambda + 1, q, q, q, z \right) \ast f(z), \] (10)
where $\Phi_1$ is the $q$-analogue of Gauss hypergeometric function defined in (2), and the symbol $(\ast)$ stands for the Hadamard product (or convolution).

The following identity is easily verified for the operator $\mathcal{R}_q^A$:
\[ q^k z \left( \mathcal{R}_q^A (f(z)) \right) = \left[ \lambda + 1 \right]_q \mathcal{R}_q^{\lambda + 1} f(z) - \left[ \lambda \right]_q \mathcal{R}_q^1 f(z). \] (11)

**2. Main Results**

Before we obtain our results, we state some known lemmas.

Let $P(\beta)$ be the class of functions of the form
\[ \phi(z) = 1 + c_1 z + c_2 z^2 + \cdots, \] (12)
which are analytic in $U$ and satisfy the following inequality:
\[ \Re \left( \phi(z) \right) > \beta, \quad (0 \leq \beta < 1; z \in U). \] (13)

**Lemma 2** (see [19]). Let $\phi_j \in P(\beta_j)$ be given by (12), where $0 \leq \beta_j < 1; j = 1, 2$; then
\[ (\phi_1 \ast \phi_2) \in P(\beta_3), \quad (\beta_3 = 1 - 2 \left( 1 - \beta_1 \right) (1 - \beta_2)), \] (14)
and the bound $\beta_3$ is the best possible.

**Lemma 3** (see [20]). Let the function $\phi$, given by (12), be in the class $P(\beta)$. Then
\[ \Re \phi(z) > 2 \beta - 1 + \frac{2(1 - \beta)}{1 + |z|}, \quad (0 \leq \beta < 1). \] (15)

**Lemma 4** (see [21]). The function $(1-z)^\gamma \equiv e^{\gamma \log(1-z)}, \gamma \neq 0$, is univalent in $U$ if and only if $\gamma$ is either in the closed disk $|\gamma-1| \leq 1$ or in the closed disk $|\gamma+1| \leq 1$.

We now generalize the lemmas introduced in [22] and [23], respectively, using $q$-derivative.

**Lemma 5.** Let $h(z)$ be analytic and convex univalent in $U$ and $h(0) = 1$ and let $g(z) = 1 + b_1 z + b_2 z^2 + \cdots$ be analytic in $U$. If
\[ g(z) + \frac{z D_q (g(z))}{c} < h(z), \quad (z \in U; c \neq 0), \] (16)
Then, for $\Re(c) \geq 0$,
\[ g(z) < \frac{c}{z^2} \int_0^z t^{\lambda-1} h(t) \, dt. \] (17)

**Proof.** Suppose that $h$ is analytic and convex univalent in $U$ and $g$ is analytic in $U$. Letting $q \to 1$ in (16), we have
\[ g(z) + \frac{z g'(z)}{c} < h(z), \quad (z \in U; c \neq 0). \] (18)
Then, from Lemma in [22], we obtain
\[ g(z) < \frac{c}{z^2} \int_0^z t^{\lambda-1} h(t) \, dt. \] (19)
Lemma 6. Let $q(z)$ be univalent in $U$ and let $\theta(w)$ and $\phi(w)$ be analytic in a domain $D$ containing $q(U)$ with $\phi(w) \neq 0$ when $w \in q(U)$. Set $Q(z) = zD_q(q(z))\phi(q(z)), h(z) = \theta(q(z)) + Q(z)$ and suppose that

1. $Q(z)$ is starlike univalent in $U$;
2. $Re \left( \frac{zD_q(h(z))/Q(z)}{Q(z)} \right) = Re \left( \frac{zD_q(\theta(q(z)))/\phi(q(z))}{(zD_q(Q(z))/Q(z))} \right) > 0$ (for $z \in U$).

If $p(z)$ is analytic in $U$, with $p(0) = q(0), p(U) \subset D$, and

$$\theta(p(z)) + zD_q(p(z))\phi(p(z)) < \theta(q(z)) + zD_q(q(z))\phi(q(z)) = h(z),$$

then $p(z) < q(z)$ and $q(z)$ is the best dominant.

The proof is similar to the proof of Lemma 5.

Theorem 7. Let $0 < \alpha > 0$, and $-1 \leq B < A \leq 1$. If $f \in \mathscr{A}$ satisfies

$$(1-\alpha) \frac{R_{q}^{\lambda} f(z)}{z} + \alpha \frac{R_{q}^{\lambda+1} f(z)}{z} < h(A, B; z),$$

then

$$Re \left( \left( \frac{R_{q}^{\lambda} f(z)}{z} \right)^{1/n} \right) > \left( \frac{[\lambda + 1]_q}{q^{\lambda}} \right)^{1/n} \left( 1 \right)$$

$$(n \geq 1).$$

The result is sharp.

Proof. Let

$$g(z) = \frac{R_{q}^{\lambda} f(z)}{z},$$

for $f \in \mathscr{A}$. Then the function $g(z) = 1 + b_1 z + \cdots$ is analytic in $U$. By using logarithmic $q$-differentiation on both sides of (23) and multiplying by $z$, we have

$$zD_q \left( \frac{g(z)}{z} \right) = \frac{zD_q R_{q}^{\lambda} f(z)}{R_{q}^{\lambda} f(z)} - 1;$$

by making use of identity (11), we obtain

$$zD_q \left( \frac{g(z)}{z} \right) = \left[ \frac{[\lambda + 1]_q}{q^{\lambda}} \right] \frac{R_{q}^{\lambda+1} f(z)}{R_{q}^{\lambda} f(z)} - 1.$$  (24)

Taking into account that $[\lambda + 1]_q = [\lambda]_q + q^{\lambda}$, we obtain

$$\frac{q^{\lambda}}{[\lambda + 1]_q} zD_q \left( \frac{g(z)}{z} \right) + g(z) = \frac{R_{q}^{\lambda+1} f(z)}{z}.$$  (25)

From (11), (23), and (26), we get

$$g(z) + \frac{q^{\lambda}}{[\lambda + 1]_q} zD_q \left( \frac{g(z)}{z} \right) < h(A, B; z).$$  (27)

Now, applying Lemma 5, we have

$$g(z) < \frac{[\lambda + 1]_q}{q^{\lambda}} u^{-[\lambda + 1]_q/q^\lambda} \int_{0}^{1} \left( 1 - Au \right) dt,$$

or by the concept of subordination

$$\frac{R_{q}^{\lambda} f(z)}{z} < \frac{[\lambda + 1]_q}{q^{\lambda}} \int_{0}^{1} u^{-[\lambda + 1]_q/q^\lambda} \left( 1 - Au \right) dt.$$  (28)

In view of $-1 \leq B < A \leq 1$ and $\gamma > 0$, it follows from (29) that

$$Re \left( \left( \frac{R_{q}^{\lambda} f(z)}{z} \right)^{1/n} \right) > \left( \frac{[\lambda + 1]_q}{q^{\lambda}} \right)^{1/n} \left( 1 - Au \right) du.$$  (30)

For this function, we find that

$$(1-\alpha) \frac{R_{q}^{\lambda} f(z)}{z} + \alpha \frac{R_{q}^{\lambda+1} f(z)}{z} = \frac{1 + Az}{1 - Bz},$$

$$\frac{R_{q}^{\lambda} f(z)}{z} < \frac{[\lambda + 1]_q}{q^{\lambda}} \int_{0}^{1} u^{-[\lambda + 1]_q/q^\lambda} \left( 1 - Au \right) du.$$  (32)

This completes the proof.

Corollary 8. Let $A = 2\beta - 1$ and $B = -1$, where $0 \leq \beta < 1$ and $a, \lambda > 1$. If $f \in \mathscr{A}$ satisfies

$$(1-\alpha) \frac{R_{q}^{\lambda} f(z)}{z} + \alpha \frac{R_{q}^{\lambda+1} f(z)}{z} < h(2\beta - 1, -1; z),$$

then

$$Re \left( \left( \frac{R_{q}^{\lambda} f(z)}{z} \right)^{1/n} \right) > \left( 2\beta - 1 \right) u^{-[\lambda + 1]_q/q^\lambda} \int_{0}^{1} \left( 1 + u \right) du^{1/n}$$

$$(n \geq 1).$$  (34)
Proof. Following the same steps as in the proof of Theorem 7 and considering \( g(z) = R_q^{\lambda} f(z)/z \), the differential subordination (27) becomes

\[
g(z) + \frac{q^\lambda \alpha}{[\lambda + 1]_q} z D_q (g(z)) < \frac{1 + (2\beta - 1) z}{1 + z}.
\]

Therefore,

\[
\Re \left( \left( \frac{R_q^{\lambda} f(z)}{z} \right)^{1/n} \right) > \left( \frac{[\lambda + 1]_q}{q^\lambda \alpha} \right)^{1/n} \int_0^1 \frac{[\lambda + 1]_q [q^\lambda \alpha]^{-1}}{1 + u} \left( \frac{1 + (1 - 2\beta) u}{1 + u} \right) du.
\]

\[
\left(\begin{array}{c}
\frac{\lambda + 1}{q^\lambda \alpha} \\
\end{array}\right)^{1/n} \int_0^1 \frac{[\lambda + 1]_q [q^\lambda \alpha]^{-1}}{1 + u} \left( \frac{1 + (1 - 2\beta) u}{1 + u} \right) du.
\]

\[
\left(\begin{array}{c}
2 (\beta - 1) \\
\frac{q^\lambda}{[\lambda + 1]_q} \end{array}\right)^{1/n} \int_0^1 \frac{[\lambda + 1]_q [q^\lambda \alpha]^{-1}}{1 + u} \left( \frac{1 + (1 - 2\beta) u}{1 + u} \right) du.
\]

We now assume that

\[
q(z) = \frac{1}{(1 - z)^{2(1 - \rho)[\lambda + 1]_q / q^\lambda \alpha}}, \quad \theta(u) = 1,
\]

\[
\phi(w) = \frac{q^\lambda}{y[\lambda + 1]_q w};
\]

then \( q(z) \) is univalent by condition of the theorem and Lemma 4. Further, it is easy to show that \( q(z), \theta(u), \) and \( \phi(w) \) satisfy the conditions of Lemma 6. Note that the function

\[
Q(z) = z D_q (q(z)) \phi(q(z)) = \frac{2(1 - \rho) z}{1 - z}
\]

is univalent starlike in \( U \) and

\[
h(z) = \theta(q(z)) + Q(z) = \frac{1 + (1 - 2\rho) z}{1 - z},
\]

Combining (40) and Lemma 6 we get the assertion of Theorem 9.

**Theorem 9.** Let \( \lambda > 0 \) and \( 0 \leq \rho < 1 \). Let \( y \) be a complex number with \( y \neq 0 \) and satisfy either \( 2y(1 - \rho)[(\lambda + 1]_q / q^\lambda \alpha) - 1 \leq 1 \) or \( 2y(1 - \rho)[(\lambda + 1]_q / q^\lambda \alpha) + 1 \leq 1 \). If each of the functions \( f_i \in \mathcal{S} \) satisfies the following subordination condition,

\[
(1 - \alpha) \frac{R_q^{\lambda} f_i(z)}{z} + \alpha \frac{R_q^{\lambda + 1} f_i(z)}{z} < h(A_i, B_i; z),
\]

then,

\[
(1 - \alpha) \frac{R_q^{\lambda} \Theta(z)}{z} + \alpha \frac{R_q^{\lambda + 1} \Theta(z)}{z} < h(1 - 2y, -1; z),
\]

where

\[
\Theta(z) = R_q^{\lambda} (f_1 * f_2)(z),
\]

\[
y = 1 - 4(A_1 - B_1) (A_2 - B_2) (1 - B_1) (1 - B_2)
\]

\[
\times \left(1 - \frac{[\lambda + 1]_q}{q^\lambda \alpha} \int_0^1 \frac{[\lambda + 1]_q [q^\lambda \alpha]^{-1}}{1 + u} \left( \frac{1 + (1 - 2\beta) u}{1 + u} \right) du.\right)
\]

**Proof.** We define the function \( h_i \) by

\[
h_i(z) = (1 - \alpha) \frac{R_q^{\lambda} f_i(z)}{z} + \alpha \frac{R_q^{\lambda + 1} f_i(z)}{z}, \quad (f_i \in \mathcal{S}, i = 1, 2);
\]

we have \( h_i(z) \in P(\beta_i) \), where \( \beta_i = (1 - A_i)/(1 - B_i) \) \((i = 1, 2)\). By making use of (11) and (48), we obtain

\[
R_q^{\lambda} f_i(z) = \frac{[\lambda + 1]_q}{q^\lambda \alpha} \int_0^1 \frac{[\lambda + 1]_q [q^\lambda \alpha]^{-1}}{1 + u} h_i(t) dt, \quad (i = 1, 2),
\]

which, in the light of (46), can show that

\[
R_q^{\lambda} \Theta(z) = \frac{[\lambda + 1]_q}{q^\lambda \alpha} \int_0^1 \frac{[\lambda + 1]_q [q^\lambda \alpha]^{-1}}{1 + u} h_i(t) dt,
\]

(50)
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where, for convenience,
\[ h_0(z) = (1 - \alpha) \frac{R_{\lambda}^q \Theta(z)}{z} + \alpha \frac{R_{\lambda+1}^q \Theta(z)}{z} \]
\[ = \frac{[\lambda + 1]_{q^\alpha}}{q^\alpha} z^{1 - ([\lambda + 1]_{q^\alpha}) - 1} \int_0^1 \frac{t^{([\lambda + 1]_{q^\alpha}) - 1} (h_1 * h_2)(t)}{z^t} dt. \]

Note that, by using Lemma 2, we have \( h_1 * h_2 \in P(\beta_3) \), where \( \beta_3 = 1 - 2(1 - \beta_1)(1 - \beta_2) \).

Now, with an application of Lemma 3, we have
\[
\text{Re} \left( h_0(z) \right) = \frac{[\lambda + 1]_{q^\alpha}}{q^\alpha} \int_0^1 t^{([\lambda + 1]_{q^\alpha}) - 1} \text{Re} \left( (h_1 * h_2)(uz) \right) du \\
\geq \frac{[\lambda + 1]_{q^\alpha}}{q^\alpha} \int_0^1 t^{([\lambda + 1]_{q^\alpha}) - 1} \left( 2\beta_3 - 1 + \frac{2(1 - \beta_1)}{1 + u |z|} \right) du \\
> \frac{[\lambda + 1]_{q^\alpha}}{q^\alpha} \int_0^1 t^{([\lambda + 1]_{q^\alpha}) - 1} \left( 2\beta_3 - 1 + \frac{2(1 - \beta_1)}{1 + u} \right) du \\
= 1 - \frac{4(A_1 - B_1)(A_2 - B_2)}{(1 - B_1)(1 - B_2)} \\
\times \left( 1 - \frac{[\lambda + 1]_{q^\alpha}}{q^\alpha} \int_0^1 \frac{t^{([\lambda + 1]_{q^\alpha}) - 1}}{1 + u} du \right) = \gamma,
\]

which shows that the desired assertion of Theorem 10 holds.

\[ \square \]

Conflict of Interests

The authors declare that they have no competing interests regarding the publication of this paper.

Authors’ Contribution

Huda Aldweby and Maslina Darus read and approved the final manuscript.

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