Research Article

Approximation by Genuine q -Bernstein-Durrmeyer Polynomials in Compact Disks in the Case $q > 1$

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This paper deals with approximating properties of the newly defined q -generalization of the genuine Bernstein-Durrmeyer polynomials in the case $q > 1$, which are no longer positive linear operators on $C[0, 1]$. Quantitative estimates of the convergence, the Voronovskaja-type theorem, and saturation of convergence for complex genuine q -Bernstein-Durrmeyer polynomials attached to analytic functions in compact disks are given. In particular, it is proved that, for functions analytic in $\{z \in \mathbb{C} : |z| < R\}$, $R > q$, the rate of approximation by the genuine q -Bernstein-Durrmeyer polynomials ($q > 1$) is of order q^{-n} versus $1/n$ for the classical genuine Bernstein-Durrmeyer polynomials. We give explicit formulas of Voronovskaja type for the genuine q -Bernstein-Durrmeyer for $q > 1$. This paper represents an answer to the open problem initiated by Gal in (2013, page 115).

1. Introduction

In several recent papers, convergence properties of complex q -Bernstein polynomials, proposed by Phillips [1], attached to an analytic function f in closed disks, were intensively studied. Ostrovska [2, 3] and Wang and Wu [4, 5] have investigated convergence properties of $B_{n,q}$ in the case $q > 1$. In the case $q > 1$, the q -Bernstein polynomials are no longer positive operators; however, for a function analytic in a disc $\mathbb{D}_R := \{z \in \mathbb{C} : |z| < R\}$, $R > q$, it was proved in [2] that the rate of convergence of $\{B_{n,q}(f; z)\}$ to $f(z)$ has the order q^{-n} (versus $1/n$ for the classical Bernstein polynomials). Moreover, Ostrovska [3] obtained Voronovskaja-type theorem for monomials. If $q \geq 1$, then qualitative Voronovskaja-type theorem and saturation results for complex q -Bernstein polynomials were obtained by Wang and Wu [4]. Wu [5] studied saturation of convergence on the interval $[0, 1]$ for the q -Bernstein polynomials of a continuous function f for arbitrary fixed $q > 1$.

Genuine Bernstein-Durrmeyer operators were first considered by Chen [6] and Goodman and Sharma [7] around 1987. In recent years, the genuine Bernstein-Durrmeyer operators have been investigated intensively by a number of authors. Among the many papers written on the genuine

Bernstein-Durrmeyer operators, we mention here only the ones by Gonska et al. [8], Parvanov and Popov [9], Sauer [10], Waldron [11], and the book of Păltănea [12].

On the other hand, Gal [13] obtained quantitative estimates of the convergence and of the Voronovskaja-type theorem in compact disks, for the complex genuine Bernstein-Durrmeyer polynomials attached to analytic functions. Besides, in other very recent papers, similar studies were done for complex Bernstein-Durrmeyer operators in Anastassiou and Gal [14], for complex Bernstein-Durrmeyer operators based on Jacobi weights in Gal [15], for complex genuine q -Bernstein-Durrmeyer operators ($0 < q < 1$) by Mahmudov [16], and for other kinds of complex Durrmeyer operators in Mahmudov [17] and Gal et al. [18]. It should be stressed out that study of q -Durrmeyer-type operators ($0 < q < 1$) in the real case was first initiated by Derriennic [19].

Also, for the case $q > 1$, exact quantitative estimates and quantitative Voronovskaja-type results for complex q -Lorentz polynomials, q -Stancu polynomials [20], q -Stancu-Faber polynomials, q -Bernstein-Faber polynomials, q -Kantorovich polynomials [21], q -Szász-Mirakjan operators [22] obtained by different researchers are collected in the recent book of Gal [23]. In this book the definition and study of complex q -Durrmeyer-kind operators for $q > 1$ presented an open

problem. This paper presents a positive solution to this problem.

In this paper we define the genuine q -Bernstein-Durrmeyer polynomials for $q > 1$. Note that similar to the q -Bernstein operators the genuine q -Bernstein-Durrmeyer operators in the case $q > 1$ are not positive operators on $C[0, 1]$. The lack of positivity makes the investigation of convergence in the case $q > 1$ essentially more difficult than that for $0 < q < 1$. We present upper estimates in approximation and we prove the Voronovskaja-type convergence theorem in compact disks in $\mathbb{C}$, centered at origin, with quantitative estimate of this convergence. These results allow us to obtain the exact degrees of approximation by complex genuine q -Bernstein-Durrmeyer polynomials. Our results show that approximation properties of the complex genuine q -Bernstein-Durrmeyer polynomials are better than approximation properties of the complex Bernstein-Durrmeyer polynomials considered in [13].

2. Main Results

We begin with some notations and definitions of q -calculus; see, for example, [24, 25]. Let $q > 0$. For any $n \in \mathbb{N} \cup \{0\}$, the q -integer $[n]_q$ is defined by

$$[n]_q := 1 + q + \cdots + q^{n-1}, \quad [0]_q := 0; \quad (1)$$

and the q -factorial $[n]_q!$ is defined by

$$[n]_q! := [1]_q [2]_q \cdots [n]_q, \quad [0]_q! := 1. \quad (2)$$

For integers $0 \leq k \leq n$, the q -binomial is defined by

$$\begin{bmatrix} n \\ k \end{bmatrix}_q := \frac{[n]_q!}{[k]_q! [n-k]_q!}. \quad (3)$$

For $q = 1$ we obviously get $[n]_q = n$, $[n]_q! = n!$, and $\begin{bmatrix} n \\ k \end{bmatrix}_q = \binom{n}{k}$. Moreover

$$(1-z)_q^n := \prod_{s=0}^{n-1} (1-q^s z), \quad (4)$$

$$p_{n,k}(q; z) := \begin{bmatrix} n \\ k \end{bmatrix}_q z^k (1-z)_q^{n-k}, \quad z \in \mathbb{C}.$$

For fixed $q > 0$, $q \neq 1$, we denote the q -derivative $D_q f(z)$ of f by

$$D_q f(z) = \begin{cases} \frac{f(qz) - f(z)}{(q-1)z}, & z \neq 0, \\ f'(0), & z = 0. \end{cases} \quad (5)$$

The q -analogue of integration in the interval $[0, A]$ (see [24]) is defined by

$$\int_0^A f(t) d_q t := A(1-q) \sum_{n=0}^{\infty} f(Aq^n) q^n, \quad 0 < q < 1. \quad (6)$$

Let $\mathbb{D}_R$ be a disc $\mathbb{D}_R := \{z \in \mathbb{C} : |z| < R\}$ in the complex plane $\mathbb{C}$. Denote by $H(\mathbb{D}_R)$ the space of all analytic functions on $\mathbb{D}_R$. For $f \in H(\mathbb{D}_R)$ we assume that $f(z) = \sum_{m=0}^{\infty} a_m z^m$ for all $z \in \mathbb{D}_R$. The norm $\|f\|_r := \max\{|f(z)| : |z| \leq r\}$. We denote $e_m(z) = z^m$ for all $m \in \mathbb{N} \cup \{0\}$.

Definition 1. For $f : [0, 1] \rightarrow \mathbb{C}$, the genuine q -Bernstein-Durrmeyer operator is defined as follows:

$$U_{n,q}(f; z) := \begin{cases} f(0) p_{n,0}(q; z) + f(1) p_{n,n}(q; z) \\ \quad + [n-1]_q \sum_{k=1}^{n-1} q^{1-k} p_{n,k}(q; z) \\ \quad \times \int_0^1 p_{n-2,k-1}(q; qt) f(t) d_q t, & 0 < q < 1, \\ f(0) p_{n,0}(z) + f(1) p_{n,n}(z) \\ \quad + (n-1) \sum_{k=1}^{n-1} p_{n,k}(z) \\ \quad \times \int_0^1 p_{n-2,k-1}(t) f(t) dt, & q = 1, \\ f(0) p_{n,0}(q; z) + f(1) p_{n,n}(q; z) \\ \quad + [n-1]_{q^{-1}} \sum_{k=1}^{n-1} q^{k-1} p_{n,k}(q; z) \\ \quad \times \int_0^1 p_{n-2,k-1}(q^{-1}; q^{-1}t) f(q^{-n}t) d_{q^{-1}} t, & q > 1, \end{cases} \quad (7)$$

where for $n = 1$ the sum is empty; that is, it is equal to 0.

$U_{n,q}(f; z)$ are linear operators reproducing linear functions and interpolating every function $f \in C[0, 1]$ at 0 and 1. The genuine q -Bernstein-Durrmeyer operators are positive operators on $C[0, 1]$ for $0 < q \leq 1$, and they are not positive for $q > 1$. As a consequence, the cases $0 < q \leq 1$ and $q > 1$ are not similar to each other regarding the convergence. For $q \rightarrow 1^-$ and $q \rightarrow 1^+$ we recapture the classical ($q = 1$) genuine Bernstein-Durrmeyer polynomials.

We start with the following quantitative estimates of the convergence for complex q -Bernstein-Durrmeyer polynomials attached to an analytic function in a disk of radius $R > 1$ and center 0.

Theorem 2. Let $f \in H(\mathbb{D}_R)$, $1 \leq r < R/q$, and $q > 1$. Then for all $|z| \leq r$ one has

$$|U_{n,q}(f; z) - f(z)| \leq \frac{r(1+r)}{[n+1]_q} \sum_{m=2}^{\infty} |a_m| m(m-1) q^{m-2} r^{m-2}. \quad (8)$$

Theorem 2 says that, for functions analytic in $\mathbb{D}_R$, $R > q$, the rate of approximation by the genuine q -Bernstein-Durrmeyer polynomials ($q > 1$) is of order q^{-n} versus $1/n$ for the classical genuine Bernstein-Durrmeyer polynomials; see [13].

The Voronovskaja theorem for the real case with a quantitative estimate is obtained by Gonska et al. [26] in the following form:

$$\left| U_n(f; x) - f(x) - \frac{x(1-x)}{n+1} f''(x) \right| \leq \frac{x(1-x)}{n+1} \omega\left(f''; \frac{2}{3\sqrt{n+3}}\right), \quad (9)$$

and, for all $n \in \mathbb{N}$, $0 \leq x \leq 1$. For the complex genuine q -Bernstein-Durrmeyer (0 < $q \leq 1$) a quantitative estimate is obtained by Gal [13] ($q = 1$) and Mahmudov [16] (0 < $q < 1$) in the following form:

$$\left| U_{n,q}(f; z) - f(z) - \frac{z(1-z)}{[n+1]_q} f''(z) \right| \leq \frac{M_{r,f}}{[n]_q^2}, \quad 0 < q \leq 1, \quad (10)$$

and, for all $n \in \mathbb{N}$, $|z| \leq r$.

To formulate and prove the Voronovskaja-type theorem with a quantitative estimate in the case $q > 1$ we introduce a function $L_q(f; z)$.

Let $R > q \geq 1$ and let $f \in H(\mathbb{D}_R)$. For $|z| < R/q^2$, we define

$$L_q(f; z) := \frac{(1-z)q(D_q f(z) - D_{q^{-1}} f(z))}{q-1} \quad \text{for } q > 1. \quad (11)$$

And, for $0 < q \leq 1$,

$$L_q(f; z) = L_1(f; z) := f''(z)z(1-z). \quad (12)$$

The next theorem gives Voronovskaja-type result in compact disks; for complex q -Bernstein-Durrmeyer polynomials attached to an analytic function in $\mathbb{D}_R$, $R > q^2 > 1$ and center 0 in terms of the function $L_q(f; z)$.

Theorem 3. Let $f \in H(\mathbb{D}_R)$, $1 \leq r < R/q^2$, and $q > 1$. The following Voronovskaja-type result holds:

$$\left| U_{n,q}(f; z) - f(z) - \frac{1}{[n+1]_q} L_q(f; z) \right| \leq \frac{4r^2(1+r)^2}{[n+1]_q^2} \sum_{m=3}^{\infty} |a_m| (m-1)^2 (m-2)^2 (q^2 r)^{m-2}. \quad (13)$$

For all $n \in \mathbb{N}$, $|z| \leq r$.

Now we are in position to prove that the order of approximation in Theorem 2 is exactly q^{-n} versus $1/n$ for the classical genuine Bernstein-Durrmeyer polynomials; see [13].

Theorem 4. Let $1 < q < R$, $1 \leq r < R/q^2$, and $f \in H(\mathbb{D}_R)$. If f is not a polynomial of degree $\leq l$, the estimate,

$$\|U_{n,q}(f) - f\|_r \geq \frac{1}{[n+1]_q} C_{r,q}(f), \quad n \in \mathbb{N}, \quad (14)$$

holds, where the constant $C_{r,q}(f)$ depends on f , q , and r but is independent of n .

From Theorem 3 we conclude that, for $q > 1$, $[n+1]_q (U_{n,q}(f; z) - f(z)) \rightarrow L_q(f; z)$ in $H(\mathbb{D}_{R/q^2})$ and therefore $L_q(f; z) \in H(\mathbb{D}_{R/q^2})$. Furthermore, we have the following saturation of convergence for the genuine q -Bernstein-Durrmeyer polynomials for fixed $q > 1$.

Theorem 5. Let $1 < q < R$, $1 \leq r < R/q^2$. If a function f is analytic in the disc $\mathbb{D}_{R/q^2}$, then $|U_{n,q}(f; z) - f(z)| = o(q^{-n})$ for infinite number of points having an accumulation point on $\mathbb{D}_{R/q^2}$ if and only if f is linear.

The next theorem shows that $L_q(f; z)$, $q \geq 1$, is continuous in the parameter q for $f \in H(\mathbb{D}_R)$, $R > 1$.

Theorem 6. Let $R > 1$ and $f \in H(\mathbb{D}_R)$. Then, for any r , $0 < r < R$,

$$\lim_{q \rightarrow 1^+} L_q(f; z) = L_1(f; z) \quad (15)$$

uniformly on $\mathbb{D}_R$.

3. Auxiliary Results

The q -analogue of beta function for $0 < q < 1$ (see [24]) is defined as

$$B_q(m, n) = \int_0^1 t^{m-1} (1-qt)_q^{n-1} d_q t, \quad m, n > 0, \quad 0 < q < 1. \quad (16)$$

Since we consider the case $q > 1$, we need to use $B_{q^{-1}}(m, n)$ as follows:

$$B_{q^{-1}}(m, n) = \int_0^1 t^{m-1} (1-q^{-1}t)_{q^{-1}}^{n-1} d_{q^{-1}} t, \quad m, n > 0, \quad 0 < q^{-1} < 1. \quad (17)$$

Also, it is known that

$$B_{q^{-1}}(m, n) = \frac{[m-1]_{q^{-1}}! [n-1]_{q^{-1}}!}{[m+n-1]_{q^{-1}}!}, \quad 0 < q^{-1} < 1. \quad (18)$$

For $m = 0, 1, \dots$, we have

$$\begin{aligned} & [n-1]_{q^{-1}} q^{k-1} \int_0^1 t^m p_{n-2, k-1}(q^{-1}; q^{-1}t) d_{q^{-1}} t \\ &= [n-1]_{q^{-1}} \left[\begin{matrix} n-2 \\ k-1 \end{matrix} \right]_{q^{-1}} q^{m(k-n)} \\ & \times \int_0^1 t^{k+m-1} (1-q^{-1}t)_{q^{-1}}^{n-k-1} d_{q^{-1}} t \\ &= q^{m(k-n)} \frac{[n-1]_{q^{-1}}!}{[k-1]_{q^{-1}}! [n-k-1]_{q^{-1}}!} B_{q^{-1}}(k+m, n-k) \end{aligned}$$

$$\begin{aligned}
&= q^{m(k-n)} \frac{[n-1]_{q^{-1}}!}{[k-1]_{q^{-1}}! [n-k-1]_{q^{-1}}!} \\
&\quad \times \frac{[k+m-1]_{q^{-1}}! [n-k-1]_{q^{-1}}!}{[k+m+n-k-1]_{q^{-1}}!} \\
&= \frac{[n-1]_q! [k+m-1]_q!}{[k-1]_q! [n+m-1]_q!} = \frac{[k+m-1]_q \cdots [k]_q}{[n+m-1]_q \cdots [n]_q}.
\end{aligned} \tag{19}$$

Thus, we get the following formula for $U_{n,q}(e_m; z)$:

$$\begin{aligned}
U_{n,q}(e_m; z) &= f(0) p_{n,0}(q; z) + f(1) p_{n,n}(q; z) \\
&\quad + [n-1]_{q^{-1}} \sum_{k=1}^{n-1} p_{n,k}(q; z) \\
&\quad \times \int_0^1 p_{n-2,k-1}(q^{-1}; q^{-1}t) f(q^{k-n}t) d_{q^{-1}}t \\
&= z^n + \sum_{k=1}^{n-1} p_{n,k}(q; z) \frac{[k+m-1]_q \cdots [k]_q}{[n+m-1]_q \cdots [n]_q}.
\end{aligned} \tag{20}$$

Note that, for $m = 0, 1, 2$, we have

$$\begin{aligned}
U_{n,q}(e_0; z) &= 1, \quad U_{n,q}(e_1; z) = z, \\
U_{n,q}(e_2; z) &= z^2 + \frac{(1+q)z(1-z)}{[n+1]}.
\end{aligned} \tag{21}$$

Lemma 7. $U_{n,q}(e_m; z)$ is a polynomial of degree less than or equal to $\min(m, n)$ and

$$U_{n,q}(e_m; z) = \frac{[n-1]_q!}{[n+m-1]_q!} \sum_{s=1}^m S_q(m, s) [n]_q^s B_{n,q}(e_s; z). \tag{22}$$

Proof. From (20) it follows that

$$\begin{aligned}
&U_{n,q}(e_m; z) \\
&= \sum_{k=1}^n p_{n,k}(q; z) \frac{[k+m-1]_q \cdots [k]_q}{[n+m-1]_q \cdots [n]_q} \\
&= \frac{[n-1]_q!}{[n+m-1]_q!} \sum_{k=1}^n [k]_q [k+1]_q \cdots [k+m-1]_q p_{n,k}(q; z).
\end{aligned} \tag{23}$$

Now using

$$\begin{aligned}
&[k]_q [k+1]_q \cdots [k+m-1]_q \\
&= \prod_{s=0}^{m-1} (q^s [k]_q + [s]_q) = \sum_{s=1}^m S_q(m, s) [k]_q^s,
\end{aligned} \tag{24}$$

where $S_q(m, s) > 0$, $s = 1, 2, \dots, m$, are the constants independent of k , we get

$$\begin{aligned}
U_{n,q}(e_m; z) &= \frac{[n-1]_q!}{[n+m-1]_q!} \sum_{k=0}^n \sum_{s=1}^m S_q(m, s) [k]_q^s p_{n,k}(q; z) \\
&= \frac{[n-1]_q!}{[n+m-1]_q!} \sum_{s=1}^m S_q(m, s) [n]_q^s B_{n,q}(e_s; z).
\end{aligned} \tag{25}$$

Since $B_{n,q}(e_s; z)$ is a polynomial of degree less than or equal to $\min(s, n)$ and $S_q(m, s) > 0$, $s = 1, 2, \dots, m$, it follows that $U_{n,q}(e_m; z)$ is a polynomial of degree less than or equal to $\min(m, n)$. $\square$

Lemma 8. The numbers $S_q(m, s)$, $(m, s) \in (\mathbb{N} \cup \{0\}) \times (\mathbb{N} \cup \{0\})$, given by (24), enjoy the following properties:

$$\begin{aligned}
S_q(0, 0) &= 1, \quad S_q(m, 0) = 0, \quad m \in \mathbb{N}, \\
S_q(m+1, s) &= [m]_q S_q(m, s) + q^m S_q(m, s-1), \\
&\quad m \in \mathbb{N}_0, \quad s \in \mathbb{N}, \\
S_q(m+1, m+1) &= q^m S_q(m, m), \\
S_q(m, s) &= 0 \quad \text{for } s > m.
\end{aligned} \tag{26}$$

Also, the following lemma holds.

Lemma 9. For all $m, n \in \mathbb{N}$ the identity,

$$\frac{[n-1]_q!}{[n+m-1]_q!} \sum_{s=1}^m S_q(m, s) [n]_q^s = 1, \tag{27}$$

holds.

Proof. It follows from end points interpolation property of $U_{n,q}(e_m; z)$ and $B_{n,q}(e_s; z)$. Indeed

$$\begin{aligned}
1 &= U_{n,q}(e_m; 1) = \frac{[n-1]_q!}{[n+m-1]_q!} \sum_{s=1}^m S_q(m, s) [n]_q^s B_{n,q}(e_s; 1) \\
&= \frac{[n-1]_q!}{[n+m-1]_q!} \sum_{s=1}^m S_q(m, s) [n]_q^s.
\end{aligned} \tag{28}$$

Lemma 9 implies that for all $m, n \in \mathbb{N}$ and $|z| \leq r$ we have

$$\begin{aligned}
&|U_{n,q}(e_m; z)| \\
&\leq \frac{[n-1]_q!}{[n+m-1]_q!} \sum_{s=1}^m S_q(m, s) [n]_q^s |B_{n,q}(e_s; z)| \\
&\leq \frac{[n-1]_q!}{[n+m-1]_q!} \sum_{s=1}^m S_q(m, s) [n]_q^s r^s \leq r^m.
\end{aligned} \tag{29}$$

For our purpose first we need a recurrence formula for $U_{n,q}(e_m; z)$.

Lemma 10. For all $m, n \in \mathbb{N} \cup \{0\}$ and $z \in \mathbb{C}$ one has

$$\begin{aligned} U_{n,q}(e_{m+1}; z) &= \frac{q^m z (1-z)}{[n+m]_q} D_q U_{n,q}(e_m; z) \\ &\quad + \frac{q^m [n]_q z + [m]_q}{[n+m]_q} U_{n,q}(e_m; z). \end{aligned} \quad (30)$$

Proof. By simple calculation we obtain (see [27])

$$\begin{aligned} z(1-z) D_q(p_{n,k}(q; z)) &= ([k]_q - [n]_q z) p_{n,k}(q; z), \\ U_{n,q}(e_m; z) &= z^n + \sum_{k=1}^{n-1} p_{n,k}(q; z) \frac{[k+m-1]_q \cdots [k]_q}{[n+m-1]_q \cdots [n]_q} \\ &= z^n + \sum_{k=1}^{n-1} p_{n,k}(q; z) I_{k,m}, \\ I_{k,m} &:= \frac{[k+m-1]_q \cdots [k]_q}{[n+m-1]_q \cdots [n]_q}. \end{aligned} \quad (31)$$

It follows that

$$\begin{aligned} &z(1-z) D_q U_{n,q}(e_m; z) \\ &= [n]_q z (1-z) z^{n-1} + \sum_{k=1}^{n-1} ([k]_q - [n]_q z) p_{n,k}(q; z) I_{k,m} \\ &= [n]_q z^n + \sum_{k=1}^{n-1} [k]_q p_{n,k}(q; z) I_{k,m} \\ &\quad - [n]_q z \sum_{k=1}^{n-1} p_{n,k}(q; z) I_{k,m} - [n]_q z^{n+1} \\ &= [n]_q z^n + \sum_{k=1}^{n-1} [k]_q p_{n,k}(q; z) I_{k,m} \\ &\quad - z[n]_q U_{n,q}(e_m; z) \\ &= [n]_q z^n + q^{-m} \sum_{k=1}^{n-1} p_{n,k}(q; z) (q^m [k]_q + [m]_q - [m]_q) I_{k,m} \\ &\quad - z[n]_q U_{n,q}(e_m; z) \\ &= [n]_q z^n + q^{-m} \sum_{k=1}^{n-1} p_{n,k}(q; z) (q^m [k]_q + [m]_q - [m]_q) I_{k,m} \\ &\quad - z[n]_q U_{n,q}(e_m; z) \\ &= q^{-m} (q^m [n]_q + [m]_q - [m]_q) z^n \\ &\quad + q^{-m} [n+m]_q \sum_{k=1}^{n-1} p_{n,k}(q; z) I_{k,m+1} \\ &\quad - q^{-m} [m]_q \sum_{k=1}^{n-1} p_{n,k}(q; z) I_{k,m} - z[n]_q U_{n,q}(e_m; z) \end{aligned}$$

$$\begin{aligned} &= q^{-m} [n+m]_q U_{n,q}(e_{m+1}; z) - q^{-m} [m]_q U_{n,q}(e_m; z) \\ &\quad - z[n]_q U_{n,q}(e_m; z), \end{aligned} \quad (32)$$

which implies the recurrence in the statement. $\square$

Let

$$\begin{aligned} \Theta_{n,m}(q; z) &:= U_{n,q}(e_m; z) - z^m - \frac{1}{[n+1]_q} \\ &\quad \times \left(q \sum_{i=1}^{m-1} [i]_q + \sum_{i=1}^{m-1} [i]_{q^{-1}} \right) z^{m-1} (1-z). \end{aligned} \quad (33)$$

Using the recurrence formula (30) we prove two more recurrence formulas.

Lemma 11. For all $m, n \in \mathbb{N}$ and $z \in \mathbb{C}$ one has

$$\begin{aligned} &U_{n,q}(e_m; z) - z^m \\ &= \frac{q^{m-1} z (1-z)}{[n+m-1]_q} D_q U_{n,q}(e_{m-1}; z) \\ &\quad + \frac{q^{m-1} [n]_q z + [m-1]_q}{[n+m-1]_q} (U_{n,q}(e_{m-1}; z) - z^{m-1}) \\ &\quad + \frac{[m-1]_q}{[n+m-1]_q} (1-z) z^{m-1}, \end{aligned} \quad (34)$$

$$\begin{aligned} &\Theta_{n,m}(q; z) \\ &= \frac{q^{m-1} z (1-z)}{[n+m-1]_q} D_q (U_{n,q}(e_{m-1}; z) - z^{m-1}) \\ &\quad + \frac{q^{m-1} [n]_q z + [m-1]_q}{[n+m-1]_q} \Theta_{n,m-1}(q; z) + R_{n,m}(q; z), \end{aligned} \quad (35)$$

where

$$\begin{aligned} &R_{n,m}(q; z) \\ &= \frac{[m-1]_q}{[n+m-1]_q [n+1]_q} \\ &\quad \times \left[(1+q^{m-1}) + \left(q \sum_{i=1}^{m-2} [i]_q + \sum_{i=1}^{m-2} [i]_{q^{-1}} \right) (z+1) \right] \\ &\quad \times z^{m-2} (1-z). \end{aligned} \quad (36)$$

Proof. From the recurrence formula in Lemma 10, for all $m \geq 2$, we get

$$\begin{aligned}
& U_{n,q}(e_m; z) - z^m \\
&= \frac{q^{m-1}z(1-z)}{[n+m-1]_q} D_q U_{n,q}(e_{m-1}; z) \\
&+ \frac{q^{m-1}[n]z + [m-1]_q}{[n+m-1]_q} (U_{n,q}(e_{m-1}; z) - z^{m-1}) \\
&+ \frac{q^{m-1}[n]z + [m-1]_q}{[n+m-1]_q} z^{m-1} - z^m \\
&= \frac{q^{m-1}z(1-z)}{[n+m-1]_q} D_q U_{n,q}(e_{m-1}; z) \\
&+ \frac{q^{m-1}[n]z + [m-1]_q}{[n+m-1]_q} (U_{n,q}(e_{m-1}; z) - z^{m-1}) \\
&+ \frac{[m-1]_q}{[n+m-1]_q} (1-z) z^{m-1}, \\
&U_{n,q}(e_m; z) - z^m \\
&- \frac{1}{[n+1]_q} \left(q \sum_{i=1}^{m-1} [i]_q + \sum_{i=1}^{m-1} [i]_{q^{-1}} \right) z^{m-1} (1-z) \\
&= \frac{q^{m-1}z(1-z)}{[n+m-1]_q} D_q (U_{n,q}(e_{m-1}; z) - z^{m-1}) \\
&+ \frac{q^{m-1}[n]z + [m-1]_q}{[n+m-1]_q} \\
&\times \left(U_{n,q}(e_m; z) - z^{m-1} - \frac{1}{[n+1]_q} \right. \\
&\quad \times \left. \left(q \sum_{i=1}^{m-2} [i]_q + \sum_{i=1}^{m-2} [i]_{q^{-1}} \right) z^{m-2} (1-z) \right) \\
&+ R_{n,m}(q; z), \tag{37}
\end{aligned}$$

where

$$\begin{aligned}
& R_{n,m}(q; z) \\
&= \frac{[m-1]_q}{[n+m-1]_q} (1-z) z^{m-1} \\
&- \frac{1}{[n+1]_q} \left(q \sum_{i=1}^{m-1} [i]_q + \sum_{i=1}^{m-1} [i]_{q^{-1}} \right) z^{m-1} (1-z) \\
&+ \frac{q^{m-1}[m-1]_q}{[n+m-1]_q} (1-z) z^{m-1}
\end{aligned}$$

$$\begin{aligned}
& + \frac{q^{m-1}[n]z + [m-1]_q}{[n+m-1]_q} \frac{1}{[n+1]_q} \\
& \times \left(q \sum_{i=1}^{m-2} [i]_q + \sum_{i=1}^{m-2} [i]_{q^{-1}} \right) z^{m-2} (1-z) \\
& := T_{n',m}(q) z^{m-1} (1-z) + \frac{[m-1]_q}{[n+m-1]_q [n+1]_q} \\
& \times \left(q \sum_{i=1}^{m-2} [i]_q + \sum_{i=1}^{m-2} [i]_{q^{-1}} \right) z^{m-2} (1-z). \tag{38}
\end{aligned}$$

Again by simple calculation we obtain

$$\begin{aligned}
& T_{n,m}(q) \\
&= \frac{[m-1]_q}{[n+m-1]_q} - \frac{1}{[n+1]_q} \left(q \sum_{i=1}^{m-1} [i]_q + \sum_{i=1}^{m-1} [i]_{q^{-1}} \right) \\
&+ \frac{q^{m-1}[m-1]_q}{[n+m-1]_q} + \frac{q^{m-1}[n]_q}{[n+m-1]_q} \frac{1}{[n+1]_q} \\
&\times \left(q \sum_{i=1}^{m-1} [i]_q + \sum_{i=1}^{m-1} [i]_{q^{-1}} \right) \\
&- \frac{q^{m-1}[n]_q}{[n+m-1]_q} \frac{1}{[n+1]_q} (q[m-1]_q + [m-1]_{q^{-1}}) \\
&= \left(\frac{[m-1]_q}{[n+m-1]_q} + \frac{q^{m-1}[m-1]_q}{[n+m-1]_q} - \frac{q^{m-1}[n]_q}{[n+m-1]_q} \right. \\
&\quad \times \frac{1}{[n+1]_q} (q[m-1]_q + [m-1]_{q^{-1}}) \Big) \\
&+ \left(\frac{q^{m-1}[n]_q}{q^{m-1}[n]_q + [m-1]_q} - 1 \right) \frac{1}{[n+1]_q} \\
&\times \left(q \sum_{i=1}^{m-1} [i]_q + \sum_{i=1}^{m-1} [i]_{q^{-1}} \right) \\
&:= T_{n,m}^1(q) + T_{n,m}^2(q), \tag{39}
\end{aligned}$$

where $T_{n,m}^1(q)$ and $T_{n,m}^2(q)$ can be simplified as follows:

$$\begin{aligned}
& T_{n,m}^2(q) = \left(1 - \frac{q^{m-1}[n]_q}{[n+m-1]_q} \right) \frac{1}{[n+1]_q} \\
&\times \left(q \sum_{i=1}^{m-2} [i]_q + \sum_{i=1}^{m-2} [i]_{q^{-1}} \right) \\
&= \frac{[m-1]_q}{[n+m-1]_q [n+1]_q} \left(q \sum_{i=1}^{m-2} [i]_q + \sum_{i=1}^{m-2} [i]_{q^{-1}} \right),
\end{aligned}$$

$$\begin{aligned}
T_{n,m}^1(q) &= \frac{[m-1]_q}{[n+m-1]_q} + \frac{q^{m-1}[m-1]_q}{[n+m-1]_q} \\
&\quad - \frac{q^{m-1}[n]_q}{[n+m-1]_q} \frac{1}{[n+1]_q} \\
&\quad \times (q[m-1]_q + [m-1]_{q^{-1}}) \\
&= [m-1]_q \left(\frac{1}{[n+m-1]_q} - \frac{q}{[n+1]_q} \frac{q^{m-1}[n]_q}{[n+m-1]_q} \right) \\
&\quad + [m-1]_q \left(\frac{q^{m-1}}{[n+m-1]_q} - \frac{1}{[n+1]_q} \frac{q[n]_q}{[n+m-1]_q} \right) \\
&= [m-1]_q \frac{[n+1]_q - q^m[n]_q}{[n+m-1]_q[n+1]_q} \\
&\quad + [m-1]_q \frac{q^{m-1}[n+1]_q - q[n]_q}{[n+m-1]_q[n+1]_q} \\
&= [m-1]_q \frac{(1+q^{m-1})[n+1]_q - (1+q^{m-1})q[n]_q}{[n+m-1]_q[n+1]_q} \\
&= \frac{[m-1]_q(1+q^{m-1})}{[n+m-1]_q[n+1]_q}.
\end{aligned} \tag{40}$$

Lemma 12. Let $q > 1$ and $f \in H(\mathbb{D}_R)$. The function $L_q(f; z)$ has the following representation:

$$\begin{aligned}
L_q(f; z) &= \sum_{m=2}^{\infty} a_m \left(q \sum_{i=1}^{m-1} [i]_q + \sum_{i=1}^{m-1} [i]_{q^{-1}} \right) z^{m-1} (1-z), \\
z &\in \mathbb{D}_R.
\end{aligned} \tag{41}$$

Proof. Using the following identity:

$$\begin{aligned}
[m]_q - m &= 1 + q + q^2 + \cdots + q^{m-1} - m \\
&= (1-1) + (q-1) + (q^2-1) + \cdots + (q^{m-1}-1) \\
&= (q-1)[1]_q + (q-1)[2]_q + \cdots + (q-1)[m-1]_q \\
&= (q-1)([1]_q + \cdots + [m-1]_q) = (q-1) \sum_{i=1}^{m-1} [i]_q,
\end{aligned} \tag{42}$$

we get

$$\begin{aligned}
L_q(f; z) &= \sum_{m=2}^{\infty} a_m \left(\frac{q([m]_q - [m]_{q^{-1}})}{q-1} \right) z^{m-1} (1-z) \\
&= \sum_{m=2}^{\infty} a_m \left(\frac{q([m]_q - m)}{q-1} + \frac{[m]_{q^{-1}} - m}{q^{-1} - 1} \right) z^{m-1} (1-z) \\
&= \sum_{m=2}^{\infty} a_m \left(q \sum_{i=1}^{m-1} [i]_q + \sum_{i=1}^{m-1} [i]_{q^{-1}} \right) z^{m-1} (1-z),
\end{aligned} \tag{43}$$

where $f(z) = \sum_{m=0}^{\infty} a_m z^m$. $\square$

4. Proofs of the Main Results

Firstly we prove that $U_{n,q}(f; z) = \sum_{m=0}^{\infty} a_m U_{n,q}(e_m, z)$. Indeed denoting $f_k(z) = \sum_{j=0}^k a_j z^j$, $|z| \leq r$ with $m \in \mathbb{N}$, by the linearity of $U_{n,q}$, we have

$$U_{n,q}(f_k, z) = \sum_{m=0}^k a_m U_{n,q}(e_m, z), \tag{44}$$

and it is sufficient to show that, for any fixed $n \in \mathbb{N}$ and $|z| \leq r$ with $r \geq 1$, we have $\lim_{k \rightarrow \infty} U_{n,q}(f_k, z) = U_{n,q}(f; z)$. But this is immediate from $\lim_{k \rightarrow \infty} \|f_k - f\|_r = 0$, the norm being defined as $\|f\|_r = \max\{|f(z)| : |z| \leq r\}$, and from the inequality

$$\begin{aligned}
&|U_{n,q}(f_k, z) - U_{n,q}(f, z)| \\
&\leq |f_k(0) - f(0)| \cdot |(1-z)^n| + |f_k(1) - f(1)| \cdot |z^n| \\
&\quad + [n+1]_{q^{-1}} \sum_{j=1}^{n-1} |p_{n,j}(q; z)| q^{j-1} \\
&\quad \times \int_0^1 p_{n-2,j-1}(q^{-1}, q^{-1}t) |f_k(t) - f(t)| d_{q^{-1}}t \\
&\leq C_{r,n} \|f_k - f\|_r,
\end{aligned} \tag{45}$$

valid for all $|z| \leq r$, where

$$\begin{aligned}
C_{r,n} &= (1+r)^n + r^n + [n+1]_{q^{-1}} \\
&\quad \times \sum_{j=1}^{n-1} \begin{bmatrix} n \\ j \end{bmatrix}_q (1+r)^{n-j} r^j q^{j-1} \\
&\quad \times \int_0^1 p_{n-2,j-1}(q^{-1}, q^{-1}t) d_{q^{-1}}t \\
&= (1+r)^n + r^n \\
&\quad + \sum_{j=1}^{n-1} \begin{bmatrix} n \\ j \end{bmatrix}_q (1+q^{n-j}r)^{n-j} r^j q^{j-1}.
\end{aligned} \tag{46}$$

Therefore we get

$$\begin{aligned} & |U_{n,q}(f; z) - f(z)| \\ & \leq \sum_{m=0}^{\infty} |a_m| |U_{n,q}(e_m; z) - e_m(z)| = \sum_{m=2}^{\infty} |a_m| \\ & \quad \times |U_{n,q}(e_m; z) - e_m(z)|, \end{aligned} \quad (47)$$

as $U_{n,q}(e_0; z) = e_0(z)$ and $U_{n,q}(e_1; z) = e_1(z)$.

Proof of Theorem 2. From the recurrence formula (34) and the inequality (29) for $m \geq 2$ we get

$$\begin{aligned} & |U_{n,q}(e_m; z) - z^m| \\ & \leq \frac{q^{m-1}z(1-z)}{q^{m-2}[n+1]_q + [m-2]_q} |D_q U_{n,q}(e_{m-1}; z)| \\ & \quad + \frac{q^{m-1}[n]z + [m-1]_q}{q^{m-1}[n]_q + [m-1]_q} \\ & \quad \times |U_{n,q}(e_{m-1}; z) - z^{m-1}| \\ & \quad + \frac{[m-1]_q}{q^{m-2}[n+1]_q + [m-2]_q} |1-z| |z|^{m-1}. \end{aligned} \quad (48)$$

It is known that, by a linear transformation, the Bernstein inequality in the closed unit disk becomes

$$|P'_k(z)| \leq \frac{k}{qr_1} \|P_k\|_{qr}, \quad \forall |z| \leq qr, \quad r \geq 1, \quad (49)$$

which, combined with the mean value theorem in complex analysis, implies

$$|D_q(P_k; z)| \leq \|P'_k\|_{qr} \leq \frac{k}{qr} \|P_k\|_{qr}, \quad (50)$$

for all $|z| \leq qr$, where $P_k(z)$ is a complex polynomial of degree $\leq k$. It follows that

$$\begin{aligned} & |U_{n,q}(e_m; z) - z^m| \\ & \leq \frac{q^{m-1}r(1+r)}{q^{m-2}[n+1]_q + [m-2]_q} \frac{m-1}{qr} \|U_{n,q}(e_{m-1})\|_{qr} \\ & \quad + r |U_{n,q}(e_{m-1}; z) - z^{m-1}| + \frac{[m-1]_{1/q}}{[n+1]_q} (1+r) r^{m-1} \\ & \leq \frac{(m-1)}{[n+1]_q} (1+r) q^{m-1} r^{m-1} \\ & \quad + r |U_{n,q}(e_{m-1}; z) - z^{m-1}| + \frac{[m-1]_{1/q}}{[n+1]_q} (1+r) r^{m-1} \\ & \leq 2q(m-1) \frac{r(1+r)}{[n+1]_q} (qr)^{m-2} \\ & \quad + r |U_{n,q}(e_{m-1}; z) - z^{m-1}|. \end{aligned} \quad (51)$$

By writing the last inequality for $m = 2, 3, \dots$, we easily obtain, step by step, the following:

$$\begin{aligned} & |U_{n,q}(e_m; z) - z^m| \\ & \leq r \left(r |U_{n,q}(e_{m-2}; z) - z^{m-2}| + 2 \frac{(m-2)}{[n+1]_q} r(1+r) (qr)^{m-3} \right) \\ & \quad + 2 \frac{(m-1)}{[n+1]_q} r(1+r) (qr)^{m-2} \\ & = r^2 |U_{n,q}(e_{m-2}; z) - z^{m-2}| \\ & \quad + 2 \frac{r(1+r)}{[n+1]_q} r^{m-2} (m-1+m-2) \\ & \leq \dots \leq \frac{r(1+r)}{[n+1]_q} m(m-1) q^{m-2} r^{m-2}. \end{aligned} \quad (52)$$

It follows that

$$\begin{aligned} |U_{n,q}(f; z) - f(z)| & \leq \sum_{m=2}^{\infty} |a_m| |U_{n,q}(e_m; z) - z^m| \\ & \leq \frac{r(1+r)}{[n+1]_q} \sum_{m=2}^{\infty} |a_m| m(m-1) q^{m-2} r^{m-2}. \end{aligned} \quad (53)$$

□

The second main result of the paper is the Voronovskaja-type theorem with a quantitative estimate for the complex version of genuine q -Bernstein-Durrmeyer polynomials.

Proof of Theorem 3. By Lemma 11 we have

$$\begin{aligned} & \Theta_{n,m}(q; z) \\ & = \frac{q^{m-1}z(1-z)}{[n+m-1]_q} D_q(U_{n,q}(e_{m-1}; z) - z^{m-1}) \\ & \quad + \frac{q^{m-1}[n]z + [m-1]_q}{[n+m-1]_q} \Theta_{n,m-1}(q; z) + R_{n,m}(q; z), \end{aligned} \quad (54)$$

where

$$\begin{aligned} & R_{n,m}(q; z) \\ & = \frac{[m-1]_q}{[n+m-1]_q [n+1]_q} \\ & \quad \times \left[(1+q^{m-1}) + \left(q \sum_{i=1}^{m-2} [i]_q + \sum_{i=1}^{m-2} [i]_{q^{-1}} \right) (z+1) \right] \\ & \quad \times z^{m-2} (1-z). \end{aligned} \quad (55)$$

It follows that

$$\begin{aligned}
 & |R_{n,m}(q; z)| \\
 & \leq \frac{[m-1]_q}{[n+1]_q^2} \\
 & \times \left((1+q^{m-1})r + \left(q \sum_{i=1}^{m-2} [i]_q + \sum_{i=1}^{m-2} [i]_{q^{-1}} \right) (1+r) \right) \\
 & \times (1+r) r^{m-2} \\
 & \leq \frac{[m-1]_q}{[n+1]_q^2} \\
 & \times \left((1+q^{m-1}) + (q(m-2)[m-2]_q + (m-2)^2) \right) \\
 & \times (1+r)^2 r^{m-2} \\
 & = \frac{q^{m-2}[m-1]_{q^{-1}}}{[n+1]_q^2} q^{m-2} \\
 & \times \left(\left(\frac{1}{q^{m-2}} + q \right) + (m-2)[m-2]_{q^{-1}} + \frac{1}{q^{m-2}}(m-2)^2 \right) \\
 & \times (1+r)^2 r^{m-2} \\
 & \leq \frac{3}{[n+1]_q^2} (m-1)(m-2)^2(1+r)^2 (q^2 r)^{m-2}
 \end{aligned} \tag{56}$$

for all $m \geq 2$, $n \in \mathbb{N}$, and $z \in \mathbb{C}$. Equation (54) implies that for $|z| \leq r$

$$\begin{aligned}
 & |\Theta_{n,m}(q; z)| \\
 & \leq r |\Theta_{n,m-1}(q; z)| + \frac{q^{m-1}r(1+r)m-1}{q^{m-2}[n+1]_q qr} \\
 & \times \|U_{n,q}(e_{m-1}) - e_{m-1}\|_{qr} \\
 & + \frac{3}{[n+1]_q^2} (m-1)(m-2)^2(1+r)^2 (q^2 r)^{m-2} \\
 & \leq r |\Theta_{n,m-1}(q; z)| + \frac{r^2(1+r)^2}{[n+1]_q^2} \\
 & \times (m-1)^2(m-2)(q^2 r)^{m-3} \\
 & + \frac{3}{[n+1]_q^2} (m-1)(m-2)^2(1+r)^2 (q^2 r)^{m-2} \\
 & \leq r |\Theta_{n,m-1}(q; z)| + \frac{4r^2(1+r)^2}{[n+1]_q^2} \\
 & \times (m-1)^2(m-2)(q^2 r)^{m-2}.
 \end{aligned} \tag{57}$$

By writing the last inequality for $m = 3, 4, \dots$, we easily obtain, step by step, the following:

$$\begin{aligned}
 & \left| U_{n,q}(f; z) - f(z) - \frac{1}{[n+1]_q} L_q(f; z) \right| \\
 & \leq \frac{4r^2(1+r)^2}{[n+1]_q^2} \sum_{m=2}^{\infty} |a_m| (q^2 r)^{m-2} \\
 & \times \sum_{j=2}^m (j-1)^2 (j-2) \leq \frac{4r^2(1+r)^2}{[n+1]_q^2} \\
 & \times \sum_{m=2}^{\infty} |a_m| (m-1)^2(m-2)^2 (q^2 r)^{m-2}. \quad \square
 \end{aligned} \tag{58}$$

Proof of Theorem 4. For all $z \in \mathbb{D}_R$ and $n \in \mathbb{N}$ we get

$$\begin{aligned}
 & U_{n,q}(f; z) - f(z) \\
 & = \frac{1}{[n+1]_q} \left\{ L_q(f; z) + [n+1]_q \right. \\
 & \quad \times \left. \left(U_{n,q}(f; z) - f(z) - \frac{1}{[n+1]_q} L_q(f; z) \right) \right\}.
 \end{aligned} \tag{59}$$

It follows that

$$\begin{aligned}
 & \|U_{n,q}(f) - f\|_r \\
 & \geq \frac{1}{[n+1]_q} \left\{ \|L_q(f; z)\|_r - [n+1]_q \right. \\
 & \quad \times \left. \|U_{n,q}(f) - f - \frac{1}{[n+1]_q} L_q(f; z)\|_r \right\}.
 \end{aligned} \tag{60}$$

Because by hypothesis f is not a polynomial of degree ≤ 1 in $\mathbb{D}_R$, it follows $\|L_q(f; z)\|_r > 0$. Indeed, assuming the contrary it follows that $L_q(f; z) = 0$ for all $z \in \overline{\mathbb{D}_R}$; that is, $D_q f(z) = D_{q^{-1}} f(z)$ for all $z \in \overline{\mathbb{D}_R}$. Thus $a_m = 0$, $m = 2, 3, \dots$ and f is linear, which is a contradiction with the hypothesis.

Now, by Theorem 3, we have

$$\begin{aligned}
 & [n+1]_q \left| U_{n,q}(f; z) - f(z) - \frac{1}{[n+1]_q} L_q(f; z) \right| \\
 & \leq \frac{4r^2(1+r)^2}{[n+1]_q} \sum_{m=3}^{\infty} |a_m| (m-1)^2(m-2)^2 (q^2 r)^{m-2} \\
 & \longrightarrow 0 \quad \text{as } n \longrightarrow \infty.
 \end{aligned} \tag{61}$$

Consequently, there exists n_1 (depending only on f and r) such that for all $n \geq n_1$ we have

$$\begin{aligned}
 & \|L_q(f; z)\|_r - [n+1]_q \|U_{n,q}(f) - f - \frac{1}{[n+1]_q} L_q(f; z)\|_r \\
 & \geq \frac{1}{2} \|L_q(f; z)\|_r,
 \end{aligned} \tag{62}$$

which implies that

$$\|U_{n,q}(f) - f\|_r \geq \frac{1}{2[n+1]_q} \|L_q(f; z)\|_r, \quad \forall n \geq n_1. \quad (63)$$

For $1 \leq n \leq n_1 - 1$ we have

$$\begin{aligned} \|U_{n,q}(f) - f\|_r &\geq \frac{1}{[n+1]_q} ([n+1]_q \|U_{n,q}(f) - f\|_r) \\ &= \frac{1}{[n+1]_q} M_{r,n,q}(f) > 0, \end{aligned} \quad (64)$$

which finally implies that

$$\|U_{n,q}(f) - f\|_r \geq \frac{1}{[n+1]_q} C_{r,q}(f), \quad (65)$$

for all n , with $C_{r,q}(f) = \min\{M_{r,1,q}(f), \dots, M_{r,n_1-1,q}(f), (1/2)\|L_q(f; z)\|_r\}$, which ends the proof. $\square$

Proof of Theorem 6. Let $1 \leq r < R$, $1 < q_0 < R/r$ be fixed. Then, by Lemma 12 for any $1 \leq q \leq q_0$ and $|z| \leq r$, we have

$$\begin{aligned} L_q(f; z) &= \sum_{m=2}^{\infty} a_m \left(q \sum_{i=1}^{m-1} [i]_q + \sum_{i=1}^{m-1} [i]_{q^{-1}} \right) z^{m-1} (1-z), \\ L_1(f; z) &= \sum_{m=2}^{\infty} a_m m(m-1) z^{m-1} (1-z). \end{aligned} \quad (66)$$

Using the inequality

$$\begin{aligned} &\left| q \sum_{i=1}^{m-1} [i]_q - \frac{m(m-1)}{2} \right| \\ &= q \sum_{i=2}^{m-1} ([i]_q - i) + (q-1) \frac{m(m-1)}{2} \\ &= q(q-1) \sum_{i=2}^{m-1} \sum_{j=1}^i [j]_q + (q-1) \frac{m(m-1)}{2} \\ &\leq q(q-1) [m-1]_q \frac{m(m-1)}{2} + (q-1) \frac{m(m-1)}{2} \\ &= (q-1) \frac{m(m-1)}{2} (q[m-1]_q + 1) \\ &\leq (q-1) q^{m-1} \frac{m^2(m-1)}{2}, \\ &\left| \sum_{i=1}^{m-1} [i]_{q^{-1}} - \frac{m(m-1)}{2} \right| = \sum_{i=2}^{m-1} (i - [i]_{q^{-1}}) \\ &= (1 - q^{-1}) \sum_{i=2}^{m-1} \sum_{j=1}^i [j]_{q^{-1}} \\ &\leq (1 - q^{-1}) \frac{m(m-1)^2}{2}, \end{aligned} \quad (67)$$

we get, for $1 \leq q \leq q_0$ and $|z| \leq r$,

$$\begin{aligned} &|L_q(f; z) - L_1(f; z)| \\ &\leq \sum_{m=2}^{N-1} |a_m| \left| q \sum_{i=1}^{m-1} [i]_q - \frac{m(m-1)}{2} \right| |z^{m-1} - z^m| \\ &\quad + \sum_{m=N}^{\infty} |a_m| \left| q \sum_{i=1}^{m-1} [i]_q - \frac{m(m-1)}{2} \right| |z^{m-1} - z^m| \\ &\quad + \sum_{m=2}^{N-1} |a_m| \left| \sum_{i=1}^{m-1} [i]_{q^{-1}} - \frac{m(m-1)}{2} \right| |z^{m-1} - z^m| \\ &\quad + \sum_{m=N}^{\infty} |a_m| \left| \sum_{i=1}^{m-1} [i]_{q^{-1}} - \frac{m(m-1)}{2} \right| |z^{m-1} - z^m| \quad (68) \\ &\leq (q-1) \sum_{m=2}^{N-1} |a_m| m^2 (m-1) q_0^{m-1} r^m \\ &\quad + 4 \sum_{m=N}^{\infty} |a_m| (m-1)^2 q_0^m r^m \\ &\quad + (1 - q^{-1}) \sum_{m=2}^{N-1} |a_m| m(m-1)^2 r^m \\ &\quad + 2 \sum_{m=N}^{\infty} |a_m| m(m-1) r^m. \end{aligned}$$

Since $f \in H(\mathbb{D}_R)$, we can find that $N = N_\varepsilon \in \mathbb{N}$ such that

$$4 \sum_{m=N}^{\infty} |a_m| (m-1)^2 q_0^m r^m + 2 \sum_{m=N}^{\infty} |a_m| m(m-1) r^m < \frac{\varepsilon}{2}. \quad (69)$$

Thus, for q sufficiently close to 1 from the right, we conclude that

$$\lim_{q \rightarrow 1^+} L_q(f; z) = L_1(f; z) \quad (70)$$

uniformly on $\mathbb{D}_r$. The proof is finished. $\square$

Proof of Theorem 5. Then, by Theorem 3, we get $L_q(f; z) = \lim_{n \rightarrow \infty} [n+1]_q (U_{n,q}(f; z) - f(z)) = 0$ for infinite number of points having an accumulation point on $\mathbb{D}_{R/q^2}$. Since $L_q(f; z) \in H(\mathbb{D}_{R/q^2})$, by the unicity Theorem for analytic functions, we get $L_q(f; z) = 0$ in $\mathbb{D}_{R/q^2}$, and, therefore, by (11), $a_m = 0$, $m = 2, 3, \dots$. Thus, f is linear. Theorem 5 is proved. $\square$

Conflict of Interests

The author declares that there is no conflict of interests regarding the publication of this paper.

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