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Research Article

Noncompact Equilibrium Points for Set-Valued Maps

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We prove a generalized result on the existence of equilibria for a monotone set-valued map defined on noncompact domain and take its values in an order of topological vector space. As consequence, we give a new variational inequality.

1. Introduction

In the literature, the notion of an equilibrium point (or equilibrium problem) has been firstly introduced by Karamardian in [1] and Allen in [2]. By using the well-known KKM principle, they proved that for a real valued function f defined on a product of two sets X and Y, there exists an element $\overline{x}$ of X, which will be called an *equilibrium point*, satisfying for all $y \in X$:

$$f(\overline{x}, y) \geq 0. \tag{1}$$

The classical hypothesis used to prove this type of equilibrium result concerns the convexity and compactness of the domain X, the monotonicity, the convexity, and the continuity of f and all extensions of this result obtained in the literature are about these hypotheses. In a recent work (see [3]), this result was extended to the noncompact case by using a coercivity type condition on a bifunction f. In this context the function f is supposed to take its values in a topological vector space endowed with an order defined by a cone C in the same way that has been used by [4–7]. Note that the result on the existence of equilibrium points proved in [3] was obtained via a result on the existence of what we called weak equilibrium points, that is, a point $\overline{x} \in X$, satisfying the following condition:

$$f(\overline{x}, y) \notin -\operatorname{int} C, \quad \forall y \in X, \tag{2}$$

where $\operatorname{int} C$ denotes the interior of the cone C in Y.

In this paper, we investigate the extension of equilibrium points to set-valued maps F in the same context. Generally, we have many choices to formulate the notion of equilibrium point. In fact, if P is a closed convex cone of a topological vector space Y with nonempty interior, $(P \neq Y)$, and $F : X \times X \to Y$ is a set-valued map, then the equilibrium point for a set-valued map can be extended in several possible ways (see [8, 9]) as follows: $F(x, y) \subseteq P$; $F(x, y) \cap -\operatorname{int} P = \emptyset$; $F(x, y) \nsubseteq -\operatorname{int} P$; $F(x, y) \cap P \neq \emptyset$. In this paper, we select the one that will be more adapted technically to our arguments. We will put a "moving" order on Y by a cone and the notions of convexity and continuity are naturally extended in our setting. We will use the pseudomonotonicity condition on F borrowed from [10]. As an application, we prove a variational inequality. The results obtained in this paper generalize the corresponding one in [9, 10].

2. Preliminaries

We extend the notions of convexity, monotonicity, and continuity given previously to set-valued maps. If X and Y are two sets, a set-valued map $F : X \to 2^Y$, where 2^X denotes the family of all subsets of X, is a map that is assigned to each $x \in X$, a subset $F(x) \subseteq Y$. Note that for the notation of set-valued maps, we will simply write $F : X \to Y$ instead of $F : X \to 2^Y$.

We firstly need to define an order on the codomain of set-valued maps as it has done for single valued maps. If X is a subset of some real topological vector space E, let Y be another real topological vector space, and let $C \subseteq Y$ be a closed convex cone (not necessarily pointed) with nonempty

interior and $C \neq Y$. Then C defines an ordering "$\succeq$" on Y by means of

$$y \succeq 0 \Longleftrightarrow y \in C, \quad y \succ 0 \Longleftrightarrow y \in \operatorname{int} C. \tag{3}$$

We extend this notation to arbitrary subset $S \subseteq Y$ by setting

$$\begin{aligned} S \succeq 0 &\Longleftrightarrow S \subseteq C, \quad S \succ 0 \Longleftrightarrow S \subseteq \operatorname{int} C, \\ S \preceq 0 &\Longleftrightarrow S \subseteq -C, \quad S \prec 0 \Longleftrightarrow S \subseteq -\operatorname{int} C. \end{aligned} \tag{4}$$

By using this order, we naturally extend the notion of convexity for set-valued maps as follows.

Definition 1. Given a set-valued map $F : X \to Y$ defined on a vector space X with values in a vector space Y endowed with an order defined by a convex cone $C \subseteq Y$, we say that F is convex with respect to C if for all $x, y \in X$ and $\alpha \in [0, 1]$:

$$F(\alpha x + (1-\alpha) y) \preceq \alpha F(x) + (1-\alpha) F(y), \tag{5}$$

which means that

$$F(\alpha x + (1-\alpha) y) \subseteq \alpha F(x) + (1-\alpha) F(y) - C. \tag{6}$$

Note that in particular, if $X = Y = \mathbb{R}$ and $C = \mathbb{R}^+$, we obtain the standard definition of convex set-valued maps.

As in the case of single valued maps, we can find many kinds of monotonicity for set-valued maps in the literature. We will use the notion of pseudomonotonicity defined in [10] which in turn extends the corresponding one defined in [7] for single valued maps.

Definition 2. Let E and Y be two real topological vector spaces, let $X \subset E$ be a nonempty closed and convex set, and let $C : X \to Y$ be a set-valued map such that for every $x \in X, C(x)$ is a closed and convex cone in Y with $\operatorname{int} C(x) \neq \emptyset$. Consider a set-valued map $F : X \times X \to Y$. F is said to be pseudomonotone if, for any given $x, y \in X$,

$$F(x, y) \not\subseteq -\operatorname{int} C(x) \Longrightarrow F(y, x) \subseteq -C(y). \tag{7}$$

We recall the classical notions of continuity for set-valued maps as follows.

Definition 3. Given a set-valued map $F : X \to Y$ defined on a vector space X with values in a vector space Y. Then

(1) F is said to be lower semicontinuous (l.s.c) at $x_0 \in X$ if, for every open set $V \subseteq Y$ with $F(x_0) \cap V \neq \emptyset$, there exists a neighborhood $U \subseteq X$ of x with $F(x) \cap V \neq \emptyset$ for all $x \in U$. F is said to be l.s.c. on X if F is l.s.c. at every $x \in X$.

(2) F is said to be upper semicontinuous (u.s.c) at $x_0 \in X$ if, for every open set $V \subseteq Y$ with $F(x_0) \subseteq V$, there exists a neighborhood set $U \subseteq X$ of x with $F(x) \subseteq V$ for all $x \in U$. F is said to be u.s.c. on X if F is u.s.c. at every $x \in X$.

(3) A set-valued map which is both lower and upper semicontinuous is called continuous.

In this paper, we will use the definition of coercing family borrowed from [11].

Definition 4. Consider a subset X of a topological vector space and a topological space Y. A family $\{(C_i, K_i)\}_{i \in I}$ of pair of sets is said to be coercing for a set-valued map $F : X \to Y$ if and only if

(i) for each $i \in I$, C_i is contained in a compact convex subset of X and K_i is a compact subset of Y;

(ii) for each $i, j \in I$, there exists $k \in I$ such that $C_i \bigcup C_j \subseteq C_k$;

(iii) for each $i \in I$, there exists $k \in I$ with $\bigcap_{x \in C_k} F(x) \subset K_i$.

Remark 5. Definition 1 can be reformulated by using the "dual" set-valued map $F^* : Y \to X$ defined for all $y \in Y$ by $F^*(y) = X \setminus F^{-1}(y)$. Indeed, a family $\{(C_i, K_i)\}_{i \in I}$ is coercing for F if and only if it satisfies conditions (i), (ii) of Definition 4, and the following one:

$$\forall i \in I, \quad \exists k \in I, \quad \forall y \in Y \setminus K_i, \quad F^*(y) \cap C_k \neq \emptyset. \tag{8}$$

Note that in the case where the family is reduced to one element, condition (iii) of Definition 4 and in the sense of Remark 5 appeared first in this generality (with two sets K and C) in [12] and generalized condition of Karamardian [1] and Allen [2]. Condition (iii) is also an extension of the coercivity condition given by Fan [13]. For other examples of set-valued maps admitting a coercing family that is not necessarily reduced to one element, see [11].

The following generalization of KKM principle obtained in [11] will be used in the proof of the main result of this paper.

Proposition 6. *Let E be a Hausdorff topological vector space, Y a convex subset of E, X a nonempty subset of Y, and $F : X \to Y$ a KKM map with compactly closed values in Y (i.e., for all $x \in X$, $F(x) \cap C$ is closed for every compact set C of Y). If F admits a coercing family, then $\bigcap_{x \in X} F(x) \neq \phi$.*

3. The Main Result

As it is mentioned in the introduction, at an abstract level all possible extension of equilibria can be handled equally well. But there are great practical differences if we try to replace the resulting abstract conditions by simpler, verifiable hypotheses like convexity or semicontinuity. This is even more so if we admit a "moving" ordering cone $P(x)$ (see [10]). For these reasons we choose to consider here the following generalized equilibrium problem.

Definition 7. Let X be a nonempty convex subset of some real topological vector space E, Y a real topological vector space, and $P : X \to Y$ a set-valued map such that for any $x \in X$, $P(x)$ is a closed convex cone with $\operatorname{int} P(x) \neq \emptyset$ and $P(x) \neq Y$. Let $F : X \times X \to Y$ be a set-valued map. The generalized equilibrium problem is to find $\overline{x} \in X$ such that

$$F(\overline{x}, y) \not\subseteq -\operatorname{int} P(\overline{x}) \quad \forall y \in X; \tag{9}$$

in this case, $\overline{x}$ is said to be an *equilibrium point*.

Theorem 8. *Let E and Y be real topological vector spaces (not necessarily Hausdorff). Let a nonempty, convex set $X \subseteq E$ and three set-valued mappings $F : X \times X \to Y$, $C : X \to Y$, and $D : X \to Y$ be given. Suppose that the following conditions are satisfied.*

(1) *For all $x, y \in X$, $F(x, y) \not\subseteq C(x)$ implies $F(y, x) \subseteq D(y)$ (pseudomonotonicity).*

(2) *For all $y \in X$, $\{x \in X : F(y, x) \subseteq D(y)\}$ is closed in X.*

(3) *For all $x \in X$, $\{y \in X : F(x, y) \subseteq C(x)\}$ is convex.*

(4) *For all $x \in X$, $F(x, x) \not\subseteq C(x)$.*

(5) *There exists a family $\{(C_i, K_i)\}_{i\in I}$ satisfying conditions (i) and (ii) of Definition 4 and the following one: for each $i \in I$, there exists $k \in I$ such that*

$$\{x \in X : F(y, x) \subseteq D(y), \forall y \in C_k\} \subseteq K_i. \tag{10}$$

Then there exists $\overline{x} \in X$ such that $F(y, \overline{x}) \subseteq D(y)$ for all $y \in X$.

Proof. Let us consider a set-valued map $S : X \to Y$ defined for every $y \in X$ by

$$S(y) := \{x \in X : F(y, x) \subseteq D(y)\}. \tag{11}$$

Then we can see firstly that S is a KKM map; that is, for every finite subset $\{y_1, \ldots, y_n\}$ of X there holds

$$\operatorname{co}\{y_1, \ldots, y_n\} \subseteq \bigcup_{i=1}^{n} S(y_i). \tag{12}$$

In fact, let $z \in \operatorname{co}\{y_1, \ldots, y_n\}$ and assume by contradiction that $z \notin \bigcup_{i=1}^{n} S(y_i)$; it means that $z = \sum_{i\in I} \lambda_i y_i$ with $\lambda_i \geq 0$, $\sum_{i\in I} \lambda_i = 1$ and $z \notin S(y_i)$ for all i. Then $F(y_i, z) \not\subseteq D(y_i)$ for all i, hence from condition (1) $F(z, y_i) \subseteq C(z)$ for all i. It follows from condition (3) that $F(z, \sum_{i\in I} \lambda_i y_i) \subseteq C(z)$, and then $F(z, z) \subseteq C(z)$, which contradicts condition (4); thus S is a KKM map.

It is also clear from condition (2) that, for all $y \in X$, $S(y)$ is closed.

In addition, we can verify that condition (5) implies that the family $\{(C_i, K)\}_{i\in I}$ satisfies the following condition: for all $i \in I$ there exists $k \in I$ with

$$\bigcap_{y\in C_k} S(y) \subset K_i. \tag{13}$$

We deduce that S satisfies all hypothesis of Proposition 6, so we have

$$\bigcap_{y\in X} S(y) \neq \emptyset. \tag{14}$$

Therefore there exists $\overline{x} \in X$ such that for any $y \in X$, $\overline{x} \in S(y)$. Hence

$$F(y, \overline{x}) \subseteq D(y), \quad \forall y \in X. \tag{15}$$

□

Theorem 9. *Let E, Y, X, F, C, and D satisfy the assumptions of Theorem 8 and the additional following conditions.*

(6) *For all $x, y \in X$ with $y \neq x$ and $u \in (x, y)$ if $F(u, x) \subseteq D(u)$ and $F(u, y) \subseteq C(u)$, then $F(u, v) \subseteq C(u)$ for all $v \in (x, y)$.*

(7) *For all $x, y \in X$ with $y \neq x$, $\{u \in [x, y] : F(u, y) \subseteq C(u)\}$ is open in $[x, y]$.*

Then there exists $\overline{x} \in X$ such that $F(\overline{x}, y) \not\subseteq C(\overline{x})$ for all $y \in X$.

Proof. By Theorem 8, there exists $\overline{x} \in X$ with $F(y, \overline{x}) \subseteq D(y)$ for all $y \in X$. Assume that $F(\overline{x}, y) \subseteq C(\overline{x})$ for some $y \in X$; then $y \neq \overline{x}$ by (6) and from (7) there exists $u \in (\overline{x}, y)$ such that $F(u, y) \subseteq C(u)$. Since $F(u, \overline{x}) \subseteq D(u)$, we deduce that $F(u, u) \subseteq C(u)$, but this contradicts (6) and the theorem is proved. □

The following result, which corresponds to Theorem 1 in [10], can be deduced from the two previous theorems.

Corollary 10. *Let F, C, D satisfy hypothesis* (1–4) *of Theorem 8,* (6, 7) *of Theorem 9 and the following condition.*

(5′) *There exists a nonempty compact set $A \subseteq X$ and a compact convex set $B \subseteq X$ such that for every $x \in X \setminus A$ there exists $y \in B$ with $F(x, y) \subseteq C(x)$.*

Then there exists $\overline{x} \in A$ such that $F(\overline{x}, y) \not\subseteq C(x)$ for all $y \in X$.

Proof. By taking for all $i \in I$, $C_i = B$, which is convex compact set, and $K_i = A$, which is compact set, and by using hypothesis (5′), we can see that S admits a coercing family in the sense of Remark 5; that is, for all $x \in X \setminus A$, $S^*(x) \cap B \neq \emptyset$. Suppose, per absurdum, that there exists $x_0 \in X \setminus A$ with $S^*(x_0) \cap B = \emptyset$. Hence for all $y \in B$, $y \notin S^*(x_0)$. This means that for all $y \in B$, $y \in S^{-1}(x_0)$ and so $x_0 \in S(y)$. Therefore, there exists $x_0 \in X \setminus A$ such that for all $y \in B$, we have

$$F(y, x_0) \subseteq D(y). \tag{16}$$

Then by Theorem 9, we deduce that there exists $x_0 \in X \setminus A$ such that for all $y \in B$

$$F(x_0, y) \not\subseteq C(x_0), \tag{17}$$

but this contradicts hypothesis (5′). □

Corollary 11. *Let $F : X \times X \to Y$ be a set-valued map satisfy the following conditions.*

(1) *For all $x, y \in X$, $F(x, y) \not\subseteq -\operatorname{int} P(x)$ implies $F(y, x) \subseteq -P(y)$.*

(2) *For all $y \in X$, $F(y, \cdot)$ is lower semicontinuous.*

(3) *For all $x \in X$, $F(x, \cdot)$ is convex with respect to $P(x)$.*

(4) *The map $\operatorname{int} P(x)$ has open graph in $X \times Y$.*

(5) *For all $x, y \in X$, $F(\cdot, y)$ is upper semicontinuous and compact valued on $[x, y]$.*

(6) *For all $x \in X$, $F(x, x) \not\subseteq -\operatorname{int} P(x)$.*

(7) *There exists a family* $\{C_i, K_i\}_{i\in I}$ *satisfying conditions* (*i*) *and* (*ii*) *of Definition 4 coercing and the following one. For each* $i \in I$, *there exists* $k \in I$ *such that*

$$\{x \in X : F(y,x) \subseteq -P(y), \forall y \in C_k\} \subseteq K_i. \quad (18)$$

Then there exists $\overline{x} \in X$ *such that* $F(\overline{x}, y) \not\subseteq -\operatorname{int} P(\overline{x})$ *for all* $y \in X$.

Proof. Following [10], if the map $F(y,\cdot)$ is lower semicontinuous and $D(y)$ is closed, then condition (7) of Theorem 9 is satisfied. Furthermore and also by [10], condition (7) of Theorem 9 is fulfilled, if for all $x \in X$, the map $F(\cdot, x)$ is upper semicontinuous along line segments $[x, y] \subseteq X$ with compact values, and the map $C(\cdot)$ has open graph in $X \times Y$. □

Now let $L : X \to Z$ denote the space of all continuous linear operators $X \to Z$. For $\phi \in L(X,Z)$, we write $\langle \phi, x\rangle : \phi(x)$ and for $\Phi \subseteq L(X,Z)$, we write $\langle \Phi, x\rangle := \{\langle \phi, x\rangle : \phi \in \Phi\}$. The following result is a variational inequality formulation of our main result.

Corollary 12. *Let a map* $\Phi : K \to L(X,Z)$ *be given such that for all* $x \in K$, $\Phi(x)$ *is nonempty. Suppose the following.*

(1) *For all* $x, y \in K$, $\langle \Phi(x), y - x\rangle \not\subseteq -\operatorname{int} P(x)$ *implies* $\langle \Phi(y), x - y\rangle \subseteq -P(y)$.

(2) *The map* $\operatorname{int} P(\cdot)$ *has open graph in* $K \times Z$.

(3) *For all* $x, y \in K$, $\langle \Phi(\cdot), y - x\rangle$ *is upper semicontinuous on* $[x, y]$ *and compact valued.*

(4) *There exists a family* $\{C_i, K_i\}_{i\in I}$ *satisfying conditions* (*i*) *and* (*ii*) *of Definition 4 and the following one: for each* $i \in I$, *there exists* $k \in I$ *such that*

$$\{x \in X : \langle \Phi(y), x - y\rangle \subseteq -P(y), \forall y \in C_k\} \subseteq K_i. \quad (19)$$

Then there exists $\overline{x} \in X$ *such that* $\langle \Phi(\overline{x}), y - \overline{x}\rangle) \not\subseteq -\operatorname{int} P(\overline{x})$ *for all* $y \in K$.

Proof. Take $F(x,y) := \langle \Phi(x), y - x\rangle$, $C(x) := -\operatorname{int} P(x)$, and $D(x) := -P(x)$. Then conditions (1) and (5) of Theorem 9 are clearly satisfied. (2) holds since each member of $\Phi(y)$ is continuous and $D(y)$ is closed. (4) is satisfied since $F(x,x) = \{0\}$ and $P(x) \neq Z$. (3) and (6) hold since for all $\alpha \in [0,1]$:

$$F(x, \alpha y_1 + (1-\alpha) y_2) \subseteq \alpha F(x, y_1) + (1-\alpha) F(x, y_2). \quad (20)$$

To verify hypothesis (7), we have to show that $R = \{u \in [x,y] : \langle \Phi(u), y - u\rangle\}$ is closed in $[x,y]$. Let $\{u_i\}$ be a net in R converging to $u \in [x,y]$; we may assume $u \neq y$, since $y \in R$, and we may assume $u_i \neq y$ for all i as well. Thus $y - u = \lambda(y - x)$ with $\lambda \neq 0$ and $y - u_i = \lambda_i(y - x)$ with $\lambda_i \neq 0$. For every i, there exists $w_i \in \langle \Phi(u_i), y - u_i\rangle$ with $w_i \notin -\operatorname{int} P(u_i)$; then $z_i = \lambda_i^{-1} w_i \in \langle \Phi(u_i), y - x\rangle$. We conclude as above that there is a subnet z_j converging to some $z \in \langle \Phi(u), y - x\rangle$. The corresponding w_j converges to $w = \lambda z \langle \Phi(u), y - u\rangle$, since $-\operatorname{int} P(\cdot)$ has open graph; we obtain $w \notin -\operatorname{int} P(u)$; hence $u \in R$. □

Note that Corollaries 11 and 12 extend, respectively, Corollaries 1 and 2 in [10] obtained in noncompact case since our coercivity condition is more general.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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References

[1] S. Karamardian, "Generalized complementarity problem," *Journal of Optimization Theory and Applications*, vol. 8, pp. 161–168, 1971.

[2] G. Allen, "Variational inequalities, complementarity problems, and duality theorems," *Journal of Mathematical Analysis and Applications*, vol. 58, no. 1, pp. 1–10, 1977.

[3] Z. Al-Rumaih, S. Chebbi, and H. K. Xu, "Non-compact equilibrium point and applications," *Journal of Applied Mathematics*, vol. 2012, Article ID 373462, 9 pages, 2012.

[4] M. Bianchi, N. Hadjisavvas, and S. Schaible, "Vector equilibrium problems with generalized monotone bifunctions," *Journal of Optimization Theory and Applications*, vol. 92, no. 3, pp. 527–542, 1997.

[5] M. Bianchi and S. Schaible, "Generalized monotone bifunctions and equilibrium problems," *Journal of Optimization Theory and Applications*, vol. 90, no. 1, pp. 31–43, 1996.

[6] E. Blum and W. Oettli, "From optimization and variational inequalities to equilibrium problems," *The Mathematics Student*, vol. 63, no. 1–4, pp. 123–145, 1994.

[7] N. X. Tan and P. N. Tinh, "On the existence of equilibrium points of vector functions," *Numerical Functional Analysis and Optimization*, vol. 19, no. 1-2, pp. 141–156, 1998.

[8] G. Y. Chen, "Existence of solutions for a vector variational inequality: an extension of the Hartmann-Stampacchia theorem," *Journal of Optimization Theory and Applications*, vol. 74, no. 3, pp. 445–456, 1992.

[9] Q. H. Ansari, W. Oettli, and D. Schläger, "A generalization of vectorial equilibria," *Mathematical Methods of Operations Research*, vol. 46, no. 2, pp. 147–152, 1997.

[10] W. Oettli and D. Schläger, "Existence of equilibria for monotone multivalued mappings," *Mathematical Methods of Operations Research*, vol. 48, no. 2, pp. 219–228, 1998.

[11] H. Ben-El-Mechaiekh, S. Chebbi, and M. Florenzano, "A generalized KKMF principle," *Journal of Mathematical Analysis and Applications*, vol. 309, no. 2, pp. 583–590, 2005.

[12] H. Ben-El-Mechaiekh, P. Deguire, and A. Granas, "Points fixed et coincidences pour les applications multivoque (application de Ky fan)," *Comptes Rendus des Séances de l'Académie des Sciences I*, vol. 295, no. 3, pp. 257–259, 1982.

[13] K. Fan, "Some properties of convex sets related to fixed point theorems," *Mathematische Annalen*, vol. 266, no. 4, pp. 519–537, 1984.