Research Article

Complexity Analysis of a Master-Slave Oligopoly Model and Chaos Control

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Received 11 April 2014; Revised 11 June 2014; Accepted 18 June 2014; Published 13 August 2014

Academic Editor: Simone Marsiglio

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We establish a master-slave oligopoly game model with an upstream monopoly whose output is considered and two downstream oligopolies whose prices are considered. The existence and the local stable region of the Nash equilibrium point are investigated. The complex dynamic properties, such as bifurcation and chaos, are analyzed using bifurcation diagrams, the largest Lyapunov exponent diagrams, and the strange attractor graph. We further analyze the long-run average profit of the three firms and find that they are all optimal in the stable region. In addition, delay feedback control method and limiter control method are used in nondelayed model to control chaos. Furthermore, a delayed master-slave oligopoly game model is considered, and the three firms' profit in various conditions is analyzed. We find that suitable delayed parameters are important for eliminating chaos and maximizing the profit of the players.

1. Introduction

Oligopoly is a market structure between monopoly and perfect competition. It is characterized by a domination of several firms, which completely control trade. These firms manufacture the same or homogeneous products. They have to consider both the market demand, that is, the behavior of consumers, and the strategies of their competitors; that is, they form expectations concerning how their rivals will act. The most widely used and simultaneously the first formal model of oligopoly market were proposed by Cournot. Cournot model assumed that each company adjusts its quantity of production to that of its rivals and there is no retaliation at all, so that in every step the player perceives the latest move made by the competitors to remain his last. Besides Cournot model, there is another important model: the Bertrand model. The former is under the assumption that producers in an oligopoly decide their policy assuming that other producers will maintain their output at its existing level, while Bertrand model is based on the assumption that producers act on the belief that competitors will maintain their price to maximize profits rather than their output.

Works on Cournot or Bertrand model showed that it has an ample dynamical behavior under different expectations [1–6]. A large number of literatures have been published. Puu considered a triopoly Cournot competition model [7]. Agiza and Elsadany studied the dynamics of a Cournot duopoly with heterogeneous players [8]. Zhang and Ma analyzed Bertrand competition model of four oligopolists with heterogeneous expectations [9]. And there are some articles about Cournot-Bertrand competition [10, 11]. For instance, Ma and Pu studied a Cournot-Bertrand duopoly model with bounded rational expectations [12]. The master-slave Cournot or Bertrand model exists in realistic economy. Xin and Chen studied a master-slave duopoly Bertrand game model in the setting that the upstream firm might be the monopolistic supplier of fresh water and the downstream firm might be the monopolistic supplier of pure distilled water [13]. And the upstream monopolistic firm's output is used as the main factor of production by the downstream monopolistic firm who is a negligible purchaser of the upstream monopolist's output.

In general situations, a system with nonlinear term will not always be stable and sometimes can even be chaotic.

However, the appearance of chaos in the economic system is not expected and even is harmful. Thus, people hope to find some methods to control the chaos of economic system. By controlling the chaotic phenomenon occurring on the market, bifurcation and chaos are delayed or eliminated, and the system is stabilized to the Nash equilibrium; that is, the market goes back to orderly competition. In recent years, scholars put forward many control methods to different chaotic systems such as OGY control, adaptive control, and feedback control.

As the game player makes decisions at time t that depend on past observed variables by means of a prediction feedback and the functional relationships describing the dynamics of the model may depend on both the current state and the past states, a delayed structure in economics models emerges. Yassen and Agiza considered a delayed duopoly game and got some important conclusions [14]. Since then, many experts and scholars also extensively studied delay oligopoly models, such as Peng et al. [15] and Ma and Wu [16]. These studies focus on the changes of stability domain of system in the case of delay or the bifurcation of system with the parameters changing, but the research to the players' profits is less.

Based on the research of experts and scholars on these models, this paper builds a master-slave oligopoly game model in which upstream monopolistic firm competes in output and two downstream oligopolistic firms compete in price. We use two control methods: the delayed feedback control method and the limiter control method to control chaos. In addition, we study the delayed game model. And we have given thorough discussion on profits of players.

The paper is organized as follows. In Section 2, a non-delayed oligopoly model with bounded rational expectations is presented. Equilibrium points and stability are analyzed. And numerical simulations which confirm analytical results are given. Two chaos control methods, the delayed feedback control method and the limiter control method, are shown in Section 3. Delayed system is investigated in Section 4. Section 5 gives the conclusion.

2. The Nondelayed Master-Slave Model

2.1. Model. In an area, there are three firms in the market and firm i produces goods x_i , $i = 1, 2, 3$. Firm 1 represents the upstream monopolistic firm and firms 2 and 3 represent two downstream oligopolistic firms. The output and price of firm i are represented, respectively, as q_i and p_i .

This model is based on the following assumptions.

- (1) The upstream monopolist supplies output (e.g., fresh water) to two downstream firms that compete in a final goods market (e.g., distilled bottled water). The upstream market is monopoly (a single firm: firm 1), and the downstream market is duopoly (two firms: 2 and 3). In addition, the two downstream oligopolistic firms do not cooperate. Then we have $q_1 > q_2 + q_3$, $p_1 < p_2$ and $p_1 < p_3$.
- (2) The cost of the two upstream monopolistic firms is a quadratic function. Because firm 1's price will affect the cost of the downstream firms (assuming 1 unit of

fresh water is used to make 1 unit spring water), we assume the two firms' marginal cost is p_1 .

- (3) In the vertically connected market, the three firms make decisions at the same time. The upstream monopolistic firm's decision variable is its output q_1 while the two downstream firms' decision variables are their prices p_2 and p_3 .

Assume that the inverse demand function of firm 1 and demand functions of firms 2 and 3, respectively, are

$$\begin{aligned} p_1(t) &= a_1 - b_1 q_1(t), \\ q_2(t) &= a_2 - b_2 p_2(t) + d_2 p_3(t), \\ q_3(t) &= a_3 - b_3 p_3(t) + d_3 p_2(t), \end{aligned} \quad (1)$$

where $a_i, b_i > 0$, $i = 1, 2, 3$, and $d_i > 0$, $i = 2, 3$. d_2 and d_3 mean the competition parameters between firms 2 and 3.

The cost functions of the first firm has the following form [17]:

$$C_1(t) = c_1 q_1^2(t), \quad (2)$$

where $2c_1$ is marginal cost and c_1 is positive for any $t > 0$. Firm 1's price will affect the cost of the two downstream firms. That is, $C_j = f(p_1)$, $j = 2, 3$. We assume firm j 's cost function is

$$C_j(t) = p_1(t) q_j(t), \quad (3)$$

where $j = 2, 3$.

The profit functions of the three firms are

$$\begin{aligned} \Pi_1(t) &= q_1(t) p_1(t) - C_1(t) \\ &= q_1(t) (a_1 - b_1 q_1(t)) - c_1 q_1^2(t) \\ &= -b_1 q_1^2(t) - c_1 q_1^2(t) + a_1 q_1(t), \\ \Pi_2(t) &= q_2(t) p_2(t) - C_2(t) \\ &= (a_2 - b_2 p_2(t) + d_2 p_3(t)) p_2(t) - p_1(t) q_2(t) \\ &= a_2 p_2(t) - b_2 p_2^2(t) + d_2 p_2(t) p_3(t) \\ &\quad - (a_1 - b_1 q_1(t)) (a_2 - b_2 p_2(t) + d_2 p_3(t)), \\ \Pi_3(t) &= q_3(t) p_3(t) - C_3(t) \\ &= (a_3 - b_3 p_3(t) + d_3 p_2(t)) p_3(t) - p_1(t) q_3(t) \\ &= a_3 p_3(t) - b_3 p_3^2(t) + d_3 p_2(t) p_3(t) \\ &\quad - (a_1 - b_1 q_1(t)) (a_3 - b_3 p_3(t) + d_3 p_2(t)). \end{aligned} \quad (4)$$

We assume that the three firms do not have a complete knowledge of the market and the other players. In games, players behave adaptively, following a bounded rational adjustment process and they build decisions on the basis of the expected marginal profit; that is, if the marginal profit is positive (negative), they increase (decrease) their production

or price in the next period. The marginal profit functions of the three firms are as follows:

$$\begin{aligned}\frac{\partial \Pi_1(t)}{\partial q_1(t)} &= -2b_1q_1(t) - 2c_1q_1(t) + a_1, \\ \frac{\partial \Pi_2(t)}{\partial p_2(t)} &= a_2 - 2b_2p_2(t) + d_2p_3(t) + b_2(a_1 - b_1q_1(t)) \\ &= a_2 + a_1b_2 - b_1b_2q_1(t) - 2b_2p_2(t) + d_2p_3(t), \\ \frac{\partial \Pi_3(t)}{\partial p_3(t)} &= a_3 - 2b_3p_3(t) + d_3p_2(t) + b_3(a_1 - b_1q_1(t)) \\ &= a_3 + a_1b_3 - b_1b_3q_1(t) + d_3p_2(t) - 2b_3p_3(t).\end{aligned}\quad (5)$$

Under the above assumptions, the dynamic adjustment equation of the master-slave game is

$$\begin{aligned}q_1(t+1) &= q_1(t) + \alpha q_1(t) \frac{\partial \Pi_1(t)}{\partial q_1(t)}, \\ p_2(t+1) &= p_2(t) + \beta p_2(t) \frac{\partial \Pi_2(t)}{\partial p_2(t)}, \\ p_3(t+1) &= p_3(t) + \gamma p_3(t) \frac{\partial \Pi_3(t)}{\partial p_3(t)},\end{aligned}\quad (6)$$

where α, β, γ are the adjustment speeds of the three firms, respectively. And $\alpha, \beta, \gamma > 0$.

Substituting (5) into (6), then the model has the following form:

$$\begin{aligned}q_1(t+1) &= q_1(t) + \alpha q_1(t) \\ &\quad \times (-2b_1q_1(t) - 2c_1q_1(t) + a_1), \\ p_2(t+1) &= p_2(t) + \beta p_2(t) \\ &\quad \times [a_2 + a_1b_2 - b_1b_2q_1(t) - 2b_2p_2(t) + d_2p_3(t)], \\ p_3(t+1) &= p_3(t) + \gamma p_3(t) \\ &\quad \times [a_3 + a_1b_3 - b_1b_3q_1(t) + d_3p_2(t) - 2b_3p_3(t)].\end{aligned}\quad (7)$$

2.2. Equilibrium Points and Local Stability. The system (7) has eight equilibrium points:

$$\begin{aligned}E_0 &= (0, 0, 0), \\ E_1 &= \left(0, 0, \frac{(a_3 + a_1b_3)}{2b_3}\right), \\ E_2 &= \left(0, \frac{(a_2 + a_1b_2)}{2b_2}, 0\right), \\ E_3 &= \left(\frac{a_1}{2(b_1 + c_1)}, 0, 0\right) \\ E_4 &= \left(0, \frac{(2a_2b_3 + 2a_1b_2b_3 + a_3d_2 + a_1b_3d_2)}{(4b_2b_3 - d_2d_3)},\right.\end{aligned}$$

$$\begin{aligned}&\left.\frac{(2a_3b_2 + 2a_1b_2b_3 + a_2d_3 + a_1b_2d_3)}{(4b_2b_3 - d_2d_3)}\right), \\ E_5 &= \left(\frac{a_1}{2(b_1 + c_1)}, 0, \frac{(2a_2b_1 + 2a_3c_1 + a_1b_1b_3 + 2a_1b_3c_1)}{4b_3(b_1 + c_1)}\right), \\ E_6 &= \left(\frac{a_1}{2(b_1 + c_1)}, \frac{(2a_2b_1 + 2a_2c_1 + a_1b_1b_2 + 2a_1b_2c_1)}{4b_2(b_1 + c_1)}, 0\right), \\ E^* &= \left(\frac{a_1}{2(b_1 + c_1)}, \frac{h_2}{g}, \frac{h_3}{g}\right) = (q_1^*, p_2^*, p_3^*),\end{aligned}\quad (8)$$

where

$$\begin{aligned}h_2 &= 4a_2b_1b_3 + 4a_2b_3c_1 + 2a_1b_1b_2b_3 + 4a_1b_2b_3c_1 \\ &\quad + 2a_3b_1d_2 + 2a_3c_1d_2 + a_1b_1b_3d_2 + 2a_1b_3c_1d_2, \\ h_3 &= 4a_3b_1b_2 + 4a_3b_2c_1 + 2a_2b_1d_3 + 2a_2c_1d_3 \\ &\quad + 2a_1b_1b_2b_3 + 4a_1b_2b_3c_1 + a_1b_1b_2d_3 + 2a_1b_2c_1d_3, \\ g &= 2(b_1 + c_1)(4b_2b_3 - d_2d_3).\end{aligned}\quad (9)$$

Since all the equilibrium points should be nonnegative, the parameters satisfy $d_1d_2 < 4b_2b_3$. Since $q_1^* > q_2^* + q_3^*$, we should have

$$\begin{aligned}&2[2b_2b_3(a_2 + a_3) + (a_2b_3d_3 + a_3b_2d_3)](b_1 + c_1) \\ &\quad + a_1b_2b_3[(d_2 + d_3)(b_1 + 2c_1) - 2] \\ &< a_1[1 + (b_1 + 2c_1)(b_2 + b_3)](2b_2b_3 - d_2d_3).\end{aligned}\quad (10)$$

The local stability of equilibrium points can be determined by the nature of the eigenvalues of the Jacobian matrix evaluated at the corresponding equilibrium points. The Jacobian matrix of the system (7) corresponding to the state variables (q_1, p_2, p_3) is

$$J(E) = \begin{pmatrix} w_{11} & 0 & 0 \\ w_{21} & w_{22} & w_{23} \\ w_{31} & w_{32} & w_{33} \end{pmatrix}, \quad (11)$$

where

$$\begin{aligned}w_{11} &= 1 + \alpha(-4b_1q_1 + a_1 - 4c_1q_1), \\ w_{21} &= -b_1b_2\beta p_2, \\ w_{22} &= 1 + \beta(a_2 + a_1b_2 - b_1b_2q_1 - 4b_2p_2 + d_2p_3), \\ w_{23} &= d_2\beta p_2, \\ w_{31} &= -b_1b_3\gamma p_3, \\ w_{32} &= d_3\gamma p_3, \\ w_{33} &= 1 + \gamma(a_3 + a_1b_3 - b_1b_3q_1 + d_3p_2 - 4b_3p_3).\end{aligned}\quad (12)$$

Theorem 1. All the boundary equilibrium points $E_0, E_1, \dots, E_6$ are unstable.

Proof. E_0 , E_1 , E_2 , and E_4 all have the eigenvalue $\lambda_1 = 1 + a_1\alpha$. Since $a_1, \alpha > 0$, then $\lambda_1 > 1$. Hence the equilibrium points E_0 , E_1 , E_2 , and E_4 are unstable equilibrium points [18, 19]. Similarly, E_3 has one eigenvalue $\lambda = 1 + a_2\beta + a_1b_2\beta(b_1 + 2c_1)/2(b_1 + c_1) > 1$; $\lambda = 1 + a_2\beta + \beta(2a_3b_1d_1 + 2a_3c_1d_2 + 2a_1b_1b_2b_3 + 4a_1b_2b_3c_1 + a_1b_1b_3d_2 + 2a_1b_3c_1d_2)/4b_3(b_1 + c_1) > 1$ for E_5 and $\lambda = 1 + a_3\gamma + \gamma(2a_2b_1d_3 + 2a_3c_1d_3 + 2a_2b_1b_2b_3 + 4a_1b_2b_3c_1 + a_1b_1b_2d_3 + 2a_1b_2c_1d_3)/4b_2(b_1 + c_1) > 1$ for E_6 . Then all the boundary equilibrium points are unstable. $\square$

Now we investigate the local stability of Nash equilibrium point E^* . The Jacobian matrix $J(E^*)$ is

$$J(E^*) = \begin{pmatrix} w_{11}^* & 0 & 0 \\ w_{21}^* & w_{22}^* & w_{23}^* \\ w_{31}^* & w_{32}^* & w_{33}^* \end{pmatrix}, \quad (13)$$

where

$$\begin{aligned} w_{11}^* &= 1 + \alpha(-4b_1q_1^* + a_1 - 4c_1q_1^*), \\ w_{21}^* &= -b_1b_2\beta p_2^*, \\ w_{22}^* &= 1 + \beta(a_2 + a_1b_2 - b_1b_2q_1^* - 4b_2p_2^* + d_2p_3^*), \\ w_{23}^* &= d_2\beta p_2^*, \\ w_{31}^* &= -b_1b_3\gamma p_3^*, \\ w_{32}^* &= d_3\gamma p_3^*, \\ w_{33}^* &= 1 + \gamma(a_3 + a_1b_3 - b_1b_3q_1^* + d_3p_2^* - 4b_3p_3^*). \end{aligned} \quad (14)$$

The characteristic polynomial of the Jacobian matrix $J(E^*)$ is

$$f(\lambda) = \lambda^3 + A\lambda^2 + B\lambda + C. \quad (15)$$

And its local stability is given by the Jury conditions [20]:

$$\begin{aligned} \text{(i)} : f(1) &= A + B + C + 1 > 0, \\ \text{(ii)} : f(-1) &= A - B + C - 1 < 0, \\ \text{(iii)} : C^2 - 1 &< 0, \\ \text{(iv)} : (1 - C^2)^2 - (B - AC)^2 &> 0. \end{aligned} \quad (16)$$

In order to analyze the stability of Nash equilibrium point, we perform some numerical simulations.

2.3. Numerical Simulations. In this section, we will show the complex behaviors of the system (7) including bifurcation and strange attractor. In order to further analyze long-run profit of the three firms with parameters changing, the long-run average profit figures are given. It is convenient to take the parameters values as follows: $a_1 = 2$, $a_2 = 4$, $a_3 = 3$, $b_1 = 0.2$, $b_2 = 2$, $b_3 = 1.5$, $c_1 = 0.3$, $d_2 = 0.5$, and $d_3 = 0.6$. The initial values are chosen as $(q_1(0), p_2(0), p_3(0)) = (1.8, 2.1, 2.2)$. Through (7), the Nash equilibrium point is $(2, 2.0769, 2.2154)$. Then its Jacobian matrix is

$$J(E^*) = \begin{pmatrix} 1 - 2\alpha & 0 & 0 \\ -0.8308\beta & 1 - 8.3077\beta & 1.0385\beta \\ -0.6646\gamma & 1.3292\gamma & 1 - 6.6462\gamma \end{pmatrix}, \quad (17)$$

FIGURE 1: The stable region of the Nash equilibrium point E^* .

where $A = 2\alpha + 8.3077\beta + 6.6462\gamma - 3$, $B = (6.6462\gamma - 1)(2\alpha + 8.3077\beta - 2) - 1.3804\beta\gamma + (2\alpha - 1)(8.3077\beta - 1)$, and $C = -1.3804\beta\gamma(2\alpha + 8.3077\beta - 2) + 1.3804\beta\gamma(8.3077\beta - 1) + (2\alpha - 1)(8.3077\beta - 1)(6.6462\gamma - 1)$.

Figure 1 gives the stable region of the Nash equilibrium point E^* . We can see that the stable region is $\alpha < 1$, $\beta < 0.24$, $\gamma < 0.3$ approximately. From the figure, we can conclude that the stability region is asymmetric and the higher adjustment speeds will push the system out of the stable region.

Figure 2 displays the bifurcation diagram and the largest Lyapunov exponent with respect to the parameter α which is the adjustment speed of the upstream monopoly when $\beta = 0.15$ and $\gamma = 0.2$. By comparing the largest Lyapunov exponent diagram, one can have a better understanding of the particular properties of the system. In Figure 2, the system (7) converges to the Nash equilibrium point for $0 < \alpha < 0.99$. If α increases, that is, $\alpha > 0.99$, the system turns unstable and complex dynamic behavior is observed. At $\alpha = 0.99$, a flip bifurcation arises, which is followed by further flips and the largest Lyapunov exponent increases to zero for the first time; hence the system enters a period doubling routes to chaos. When $\alpha \geq 1.29$, the largest Lyapunov exponent is positive and chaos emerges.

Figure 3 is the bifurcation diagram with respect to the parameter β , when given $\alpha = 0.5$ and $\gamma = 0.2$. In Figure 3, the output $q_1(t)$ of upstream firm is always stable which illustrates that the adjustment speed β has little effect on the output $q_1(t)$ of the upstream firm, while the prices $p_2(t)$, $p_3(t)$ of the two downstream firms generate bifurcation behaviors at $\beta = 0.2312$. And when $\beta \geq 0.306$, the largest Lyapunov exponent is positive; then the system is in a state of chaos.

Similar to Figure 3, Figure 4 gives the bifurcation diagram with respect to the parameter γ , when given $\alpha = 0.5$ and $\beta = 0.15$. In Figure 4, the output $q_1(t)$ is also stable. And when

FIGURE 2: Bifurcation diagram and the Largest Lyapunov exponent of the discrete dynamic system (7) for $\beta = 0.15$, $\gamma = 0.2$, and α varying from 0 to 1.5.

FIGURE 4: Bifurcation diagram and the Largest Lyapunov exponent of the discrete dynamic system (7) for $\alpha = 0.5$, $\beta = 0.15$, and γ varying from 0 to 0.44.

FIGURE 3: Bifurcation diagram and the Largest Lyapunov exponent of the discrete dynamic system (7) for $\alpha = 0.5$, $\gamma = 0.2$, and β varying from 0 to 0.34.

FIGURE 5: Strange attractors for $\alpha = 1.35$, $\beta = 0.3$, and $\gamma = 0.35$.

$\gamma > 0.2856$ the system turns unstable and enters chaos when $\gamma \geq 0.3822$.

From the above analysis, it can be seen that the adjustment parameter α has an important influence on system (7). That is to say the behavior of upstream monopolist has a decisive influence on the market in economics. And it is harmful for the development of the two downstream firms if the changes of adjustment parameters α , β , γ are too big.

Figure 5 represents the graph of a strange attractors of the dynamical system (7) for the adjustment parameter values $\beta = 0.3$, $\gamma = 0.35$, and $\alpha = 1.35$, which exhibits fractal structure of the system.

Then we analyse the long-run average profit of the three firms. The results are shown in Figures 6, 7, and 8.

From these figures, we can see that the long-run average profit of firm 1 is larger than the other firms. Figure 6 shows

FIGURE 6: The long-run average profits of the players with α for system (7).

FIGURE 7: The long-run average profits of the players with β for system (7).

FIGURE 8: The long-run average profits of the players with γ for system (7).

that the long-run average profit of the three firms fluctuates and is lower than that of the stable state with the production adjustment speed increasing when the system (7) enters into bifurcation and chaotic states. When $\alpha \geq 0.99$, the profit of the three firms begins to decrease. From Figures 7 and 8, we can see that the average profit of the upstream firm is a constant value, so we deduce that the adjustment speeds β and γ have little effect on the profit of firm 1. In Figure 7, the average profit of firms 1 and 3 is always higher than firm 2 and the long-run average profit of the downstream firms decreases when the bifurcation begins. When $\beta = 0.255$, the profit of firm 2 is negative. In Figure 8, the long-run average profit of the downstream firms decreases when the bifurcation begins and when $\gamma = 0.3024$, they are the same. From then on, the profit of firm 2 is higher than firm 3 and when $\gamma = 0.336$, the profit of firm 3 turns to negative.

Our study finds that the overall profit of the system will decrease in bifurcation and chaotic states with adjustment speeds increasing.

3. Chaos Control of Nondelayed Master-Slave Model

3.1. Delay Feedback Control Method. In this part, we first take the delay feedback control method [21, 22] to control chaos. The method is based on the difference between the T -time delayed state and the current state, where T denotes a period of the stabilized orbits. The controlled system is

$$x(t+1) = f(x(t), u(t)), \quad (18)$$

where $u(t)$ is the input signal and $x(t)$ is the state. Consider

$$u(t) = k(x(t+1-T) - x(t+1)), \quad t > T, \quad (19)$$

where T is the time delay and k is the controlling factor.

Because the behavior of firm 1 has great effect on the system, we add the function in the first equation from the system (7). Then the controlled system is

$$\begin{aligned} q_1(t+1) &= q_1(t) + \alpha q_1(t) (-2b_1 q_1(t) - 2c_1 q_1(t) + a_1) \\ &\quad + k(q_1(t+1-T) - q_1(t+1)), \\ p_2(t+1) &= p_2(t) + \beta p_2(t) \\ &\quad \times [a_2 + a_1 b_2 - b_1 b_2 q_1(t) - 2b_2 p_2(t) + d_2 p_3(t)], \\ p_3(t+1) &= p_3(t) + \gamma p_3(t) \\ &\quad \times [a_3 + a_1 b_3 - b_1 b_3 q_1(t) + d_3 p_2(t) - 2b_3 p_3(t)]. \end{aligned} \quad (20)$$

By choosing $T = 1$, the system becomes

$$\begin{aligned} q_1(t+1) &= q_1(t) + \frac{\alpha q_1(t) (-2b_1 q_1(t) - 2c_1 q_1(t) + a_1)}{(1+k)} \\ p_2(t+1) &= p_2(t) + \beta p_2(t) \\ &\quad \times [a_2 + a_1 b_2 - b_1 b_2 q_1(t) - 2b_2 p_2(t) + d_2 p_3(t)], \\ p_3(t+1) &= p_3(t) + \gamma p_3(t) \\ &\quad \times [a_3 + a_1 b_3 - b_1 b_3 q_1(t) + d_3 p_2(t) - 2b_3 p_3(t)]. \end{aligned} \quad (21)$$

It can be seen from Figure 9, ($\alpha = 1.35$, $\beta = 0.15$, $\gamma = 0.2$), that the chaotic system was gradually controlled with controlling parameter k increasing. When $k = 0$, it turns uncontrolled system (7) which is at chaotic state. And when $k \geq 0.36$, the system is controlled. Taking $k = 0.8$, we can see that the stable region of α expands to 1.775 from Figure 10, which indicates that chaos is delayed or eliminated completely. Figure 11 gives the long-run average profit of the players for the controlled system (21). Comparing Figure 11 with Figure 6, we see that in Figure 6 the three firms' profit starts to fall in $\alpha = 0.99$ while in Figure 11 this phenomenon does not happen.

3.2. Limiter Control Method. We use limiter control method [23, 24] which is better for firms 2 and 3. This method only

FIGURE 9: The trend of system with k increasing for $\alpha = 1.35$, $\beta = 0.15$, and $\gamma = 0.2$.

FIGURE 12: The bifurcation diagram of the controlled system (22) for $\beta = 0.15$ and $\gamma = 0.2$.

FIGURE 10: The bifurcation diagram of the controlled system (21) for $k = 0.8$, $\beta = 0.15$, $\gamma = 0.2$, and α varying from 0 to 2.5.

FIGURE 11: The long-run average profits of the players with α for the controlled system (21).

requires the player who wants to improve his performance to take measures without the other players' cooperation. We impose lower limiters on the price of firms 2 and 3, noted $p_2^{\min}$, $p_3^{\min}$, and it has no effect on the behavior of firm 1.

The controlled system is

$$\begin{aligned} q_1(t+1) &= q_1(t) + \alpha q_1(t) (-2b_1 q_1(t) - 2c_1 q_1(t) + a_1), \\ p_2(t+1) &= \text{Max} [f_2(q_1(t), p_2(t), p_3(t)), p_2^{\min}], \\ p_3(t+1) &= \text{Max} [f_3(q_1(t), p_2(t), p_3(t)), p_3^{\min}], \end{aligned} \quad (22)$$

where $p_2^{\min}, p_3^{\min} > 0$, and

$$\begin{aligned} f_2(q_1(t), p_2(t), p_3(t)) &= p_2(t) + \beta p_2(t) \\ &\quad \times [a_2 + a_1 b_2 - b_1 b_2 q_1(t) - 2b_2 p_2(t) + d_2 p_3(t)], \\ f_3(q_1(t), p_2(t), p_3(t)) &= p_3(t) + \gamma p_3(t) \\ &\quad \times [a_3 + a_1 b_3 - b_1 b_3 q_1(t) + d_3 p_2(t) - 2b_3 p_3(t)]. \end{aligned} \quad (23)$$

By choosing $p_2^{\min} = 2.0907$ and $p_3^{\min} = 2.2264$, Figures 12 and 13 give bifurcation diagram and long-run average profit of the three firms with the α changing. They show that the behaviors of firm 1 are the same with the original system (7). Figure 12 shows that the bifurcation and chaotic behaviors of firms 2 and 3 have been controlled. In Figure 13, the decreasing speed of π_2 and π_3 is under control.

4. The Delayed Master-Slave Model

4.1. Model. The primary reason for the occurrence of such a delayed structure in economic models is that (a) decisions

FIGURE 13: The long-run average profits of the players with α for the controlled system (22).

made by economic agents at time t depend on past observed variables by means of a prediction feedback and (b) the functional relationships describing the dynamics of the model may not only depend on the current state of the firm but also, in a nontrivial manner, on past states. Considering these reasons, we introduce the delayed model and compare the three firms' profits in various cases.

Then the bounded rationality dynamical model evolved from system (7) with one step delayed is given by

$$\begin{aligned} q_1(t+1) &= q_1(t) + \alpha q_1(t) (-2b_1((1-w_1)q_1(t) + w_1q_1(t-1)) \\ &\quad - 2c_1((1-w_1)q_1(t) + w_1q_1(t-1)) + a_1), \end{aligned}$$

$$\begin{aligned} p_2(t+1) &= p_2(t) + \beta p_2(t) [a_2 + a_1b_2 - b_1b_2 \\ &\quad \times ((1-w_1)q_1(t) + w_1q_1(t-1)) \\ &\quad - 2b_2((1-w_2)p_2(t) + w_2p_2(t-1)) \\ &\quad + d_2((1-w_3)p_3(t) + w_3p_3(t-1))], \end{aligned}$$

$$\begin{aligned} p_3(t+1) &= p_3(t) + \gamma p_3(t) [a_3 + a_1b_3 - b_1b_3 \\ &\quad \times ((1-w_1)q_1(t) + w_1q_1(t-1)) \\ &\quad + d_3((1-w_2)p_2(t) + w_2p_2(t-1)) \\ &\quad - 2b_3((1-w_3)p_3(t) + w_3p_3(t-1))]. \end{aligned} \quad (24)$$

w_i represent the weights given to previous production and prices, and $0 \leq w_1, w_2, w_3 \leq 1$.

It is convenient to take $x(t) = q_1(t-1)$, $y(t) = p_2(t-1)$, and $z(t) = p_3(t-1)$; then system (24) becomes

$$\begin{aligned} q_1(t+1) &= q_1(t) + \alpha q_1(t) (-2b_1((1-w_1)q_1(t) + w_1x(t)) \\ &\quad - 2c_1((1-w_1)q_1(t) + w_1x(t)) + a_1), \\ p_2(t+1) &= p_2(t) + \beta p_2(t) [a_2 + a_1b_2 - b_1b_2 \\ &\quad \times ((1-w_1)q_1(t) + w_1x(t)) \\ &\quad - 2b_2((1-w_2)p_2(t) + w_2y(t)) \\ &\quad + d_2((1-w_3)p_3(t) + w_3z(t))], \\ p_3(t+1) &= p_3(t) + \gamma p_3(t) [a_3 + a_1b_3 - b_1b_3 \\ &\quad \times ((1-w_1)q_1(t) + w_1x(t)) \\ &\quad + d_3((1-w_2)p_2(t) + w_2y(t)) \\ &\quad - 2b_3((1-w_3)p_3(t) + w_3z(t))], \\ x(t+1) &= q_1(t), \\ y(t+1) &= p_2(t), \\ z(t+1) &= p_3(t). \end{aligned} \quad (25)$$

We can get eight equilibrium points, denoted by $E_i = (q_1^i, p_2^i, p_3^i, x^i, y^i, z^i)$. Consider $(q_1^i, p_2^i, p_3^i) = (x^i, y^i, z^i)$, and (q_1^i, p_2^i, p_3^i) ($i = 0, 1, \dots, 7$) have the same values as equilibrium points of system (7). The Jacobi matrix for the system (25) is

$$J(E) = \begin{pmatrix} J_{11} & 0 & 0 & J_{14} & 0 & 0 \\ J_{21} & J_{22} & J_{23} & J_{24} & J_{25} & J_{26} \\ J_{31} & J_{32} & J_{33} & J_{34} & J_{35} & J_{36} \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}, \quad (26)$$

where $J_{11} = w_{11} + 2\alpha q_1 w_1(b_1 + c_1)$, $J_{14} = -2w_1\alpha q_1(b_1 + c_1)$, $J_{21} = w_{31}(1-w_1)$, $J_{22} = w_{22} + 2\beta w_2 p_2 b_2$, $J_{23} = w_{23}(1-w_3)$, $J_{24} = w_1 w_{21}$, $J_{25} = -2b_2 w_2 \beta p_2$, $J_{26} = w_3 w_{23}$, $J_{31} = w_{31}(1-w_1)$, $J_{32} = w_{32}(1-w_2)$, $J_{33} = w_{33} + 2\gamma w_3 p_3 b_3$, $J_{34} = w_2 w_{32}$, $J_{35} = w_2 w_{32}$, and $J_{36} = -2b_3 \gamma p_3 w_3$.

FIGURE 14: The bifurcation diagram and the corresponding largest Lyapunov exponent of the delayed system (25).

$E_0, \dots, E_6$ are boundary equilibrium points; E_7 is the unique Nash equilibrium point. The Jury conditions are

- (i) : $F(1) = \delta_0 + \delta_1 + \delta_2 + \delta_3 + \delta_4 + \delta_5 + \delta_6 > 0$,
- (ii) : $F(-1) = \delta_0 - \delta_1 + \delta_2 - \delta_3 + \delta_4 - \delta_5 + \delta_6 > 0$,
- (iii) : $|\delta_0| < \delta_6$,
- (iv) : $|f_1(\delta_0, \dots, \delta_6)| > |\varphi_1(\delta_0, \dots, \delta_6)|$,
- (v) : $|f_2(\delta_0, \dots, \delta_6)| > |\varphi_2(\delta_0, \dots, \delta_6)|$,
- (vi) : $|f_3(\delta_0, \dots, \delta_6)| > |\varphi_3(\delta_0, \dots, \delta_6)|$,
- (vii) : $|f_4(\delta_0, \dots, \delta_6)| > |\varphi_4(\delta_0, \dots, \delta_6)|$,

(27)

where $F(\lambda) = \delta_0 + \delta_1\lambda + \delta_2\lambda^2 + \dots + \delta_6\lambda^6$ is the characteristic polynomial, and f_i, φ_i ($i = 1, 2, 3, 4$) are the functions of δ_i ($i = 1, 2, 3, 4, 5, 6$).

4.2. Numerical Simulations. In order to discuss the complexity of delayed system (25), we first take the α, β, γ as constant values and then analyze behaviors with w_1, w_2, w_3 changing.

Figure 14 gives the bifurcation diagrams with respect to the parameters w_i , ($i = 1, 2, 3$) when $\alpha = 1.36, \beta = 0.15, \gamma = 0.2$ in (a), (b), and (c). In the nondelayed system, when α, β, γ take the values as above, as we see from Figure 2, system (7) is at the state of chaos while in Figure 14(a), when $w_2 = w_3 = 0$, system (25) from chaotic state changes to stable state, then from stable state to chaotic state with the changing of w_1 . And when $w_1 = 0, w_i = 0$, as w_j ($i, j = 2, 3, i \neq j$) changing, the system (25) is always in the chaotic state. We conclude that w_1 plays an important role in changing system (7) from chaos to stable state with the rest parameters fixed.

Similarly, taking $\alpha = 0.5, \beta = 0.32$, and $\gamma = 0.2$, we know from Figure 4 that the system is in chaotic state. In Figure 15, the bifurcation diagram (b) shows that when $w_1 = w_3 = 0$, the system (25) is also from the chaotic state to stable state, then from stable state to unstable state with w_2 changing.

FIGURE 15: The bifurcation diagram and the corresponding largest Lyapunov exponent of the delayed system (25).

Figure 16 shows the bifurcation diagrams and the largest Lyapunov exponent diagram taking $\alpha = 0.5$, $\beta = 0.15$, $\gamma = 0.41$. When $w_1 = w_2 = 0$, system (25) also experiences the process from chaos to stability then again to the chaos as the changes of the w_3 .

From the analysis of the Figures 14–16, we conclude that the appropriate delay parameters play an important role for controlling the system from chaos to stability in system (7).

Tables 1, 2, and 3 show three firms' profit and the total profit when the systems are nondelayed, delay feedback controlled, limiter controlled, and delayed. We conclude the following.

- (a) The profit of upstream firm is apparently higher than the other two downstream firms.
- (b) In Table 1, although the value of α goes beyond the stable region of system (7), when $w_1 = 0.2$, the three firms achieve maximal profit: $\pi_1 = 60$, $\pi_2 = 13.6473$,

$\pi_3 = 17.0414$, total profit $\sum \pi = 90.6888$. In Table 2, when the value of β goes beyond the stable region, the three firms achieve maximal profit with $w_2 = 0.2$. While in Table 3, when the value of γ goes beyond the stable region, the three firms achieve maximal profit with $w_3 = 0.2$. It can be seen that the effect of selecting appropriate delay parameters is the same as applying a control on the system, which can make profit of the system maximization.

- (c) Taking the appropriate value of k can reach the best control effect for this case where α goes beyond the stable region. This proves that the delay feedback control method is advantageous to the upstream firm.
- (d) According to the comparison of the three tables, limiter control method is also effective in controlling chaos. And it has great significance when requiring

FIGURE 16: The bifurcation diagram and the corresponding largest Lyapunov exponent of the delayed system (25).

the player who wants to improve his performance to take measures without the other players' cooperation.

5. Conclusions

In this paper, we investigate the dynamics of a bounded rational oligopoly model which contains three players: one is the upstream monopoly who chooses his production to compete and the other two downstream oligopolies adjust their prices to maximize their expected profit. The existence and stability of equilibria, bifurcation, and chaotic behavior are analyzed in this game. In addition, the largest Lyapunov exponent and strange attractors are also applied to display the bifurcation and chaotic behavior of this system. We show that if the adjustment speeds of players are too high, then they will change the stability of equilibrium and cause a market structure to behave chaotically. Furthermore, we give long-run average profit of the three firms, which demonstrates that the equilibrium state is satisfactory to the three firms.

We adopt two kinds of control methods and consider the delayed system. Then the following conclusions are obtained.

- (1) Delay feedback control method and limiter control method both can control chaos and make profit increase.
- (2) From the perspective of profits, the delay feedback control method is advantageous to the upstream firm. The limiter control method is effective in preventing and controlling the two downstream firms' profit decline.
- (3) The effect of selecting appropriate delay parameters is the same as applying a control on the system.

TABLE 1: Comparative analysis of oligarchs' profits when $\alpha = 1.36$, $\beta = 0.15$, and $\gamma = 0.2$.

Parameters	Nondelayed	Delay feedback control	Limiter control	Delayed	Delayed	Delayed
w_1	0	0	0	0.2	0	0
w_2	0	0	0	0	0.2	0
w_3	0	0	0	0	0	0.2
k	0	0.8	0	0	0	0
$p_2^{\min}$	0	0	2.0842	0	0	0
$p_3^{\min}$	0	0	2.2213	0	0	0
π_1	53.9601	60	53.9601	60	53.9601	53.9601
π_2	12.1514	13.6473	12.6415	13.6473	12.4463	12.7958
π_3	15.8208	17.0414	16.3254	17.0414	15.5874	15.3587
$\sum \pi_i$	81.9322	90.6888	82.9271	90.6888	82.9938	82.1146

TABLE 2: Comparative analysis of oligarchs' profits when $\alpha = 0.5$, $\beta = 0.32$, and $\gamma = 0.2$.

Parameters	Nondelayed	Limiter control	Delayed	Delayed	Delayed
w_1	0	0	0.2	0	0
w_2	0	0	0	0.2	0
w_3	0	0	0	0	0.2
$p_2^{\min}$	0	2.0842	0	0	0
$p_3^{\min}$	0	2.2213	0	0	0
π_1	60	60	60	60	60
π_2	-16.7885	9.9633	-15.6247	13.6473	-13.0483
π_3	10.9234	16.5478	10.6893	17.0414	12.4387
$\sum \pi_i$	54.1349	86.5111	55.0646	90.6888	59.3904

TABLE 3: Comparative analysis of oligarchs' profits when $\alpha = 0.5$, $\beta = 0.15$, and $\gamma = 0.41$.

Parameters	Nondelayed	Limiter control	Delayed	Delayed	Delayed
w_1	0	0	0.2	0	0
w_2	0	0	0	0.2	0
w_3	0	0	0	0	0.2
$p_2^{\min}$	0	2.0842	0	0	0
$p_3^{\min}$	0	2.2213	0	0	0
π_1	60	60	60	60	60
π_2	9.4274	12.6782	8.8646	10.2317	13.6473
π_3	-13.2909	13.9479	-19.2747	-10.3930	17.0414
$\sum \pi_i$	56.1364	86.6260	49.5898	59.8387	90.6888

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

Acknowledgments

The authors would like to thank the reviewers for their careful reading and for providing some pertinent suggestions. The research was supported by the National Natural Science Foundation of China (no. 61273231) and Doctoral Fund of Ministry of Education of China (Grant no. 20130032110073), and it was supported by Tianjin University Innovation Fund.

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