Research Article

Cyclic ($\phi$)-Contractions in Uniform Spaces and Related Fixed Point Results

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First, we define cyclic ($\phi$)-contractions of different types in a uniform space. Then, we apply these concepts of cyclic ($\phi$)-contractions to establish certain fixed and common point theorems on a Hausdorff uniform space. Some more general results are obtained as corollaries. Moreover, some examples are provided to demonstrate the usability of the proved theorems.

1. Introduction

Let $X$ be a nonempty set. A nonempty family, $\vartheta$, of subsets of $X \times X$ is called the uniform structure of $X$ if it satisfies the following properties:

(i) if $G$ is in $\vartheta$, then $G$ contains the diagonal $\{(x, x) \mid x \in X\}$;
(ii) if $G$ is in $\vartheta$ and $H$ is a subset of $X \times X$ which contains $G$, then $H$ is in $\vartheta$;
(iii) if $G$ and $H$ are in $\vartheta$, then $G \cap H$ is in $\vartheta$;
(iv) if $G$ is in $\vartheta$, then there exists $H$ in $\vartheta$, such that, whenever $(x, y)$ and $(y, z)$ are in $H$, then $(x, z)$ is in $G$;
(v) if $G$ is in $\vartheta$, then $\{(y, x) \mid (x, y) \in G\}$ is also in $\vartheta$.

The pair $(X, \vartheta)$ is called a uniform space and the element of $\vartheta$ is called entourage or neighbourhood or surrounding. The pair $(X, \vartheta)$ is called a quasi-uniform space (see, e.g., [1, 2]) if property (v) is omitted.

Existence and uniqueness of fixed points for various contractive mappings in the setting of uniform spaces have been investigated by several authors; see, for example, [3–12] and the references therein.

Recently, an interesting and remarkable notion of cyclic mapping was introduced and studied by Kirk et al. [13]. Following this paper, a number of authors introduced contractive mapping via the cyclic mappings and reported certain fixed point results in the setting of different type of spaces; see, for example, [13–17].

In this paper, we will give the characterization of cyclic mapping in the context of uniform spaces and, further, prove the existence and uniqueness of fixed and common fixed points of such mappings via $A$-distance and $E$-distance, introduced by Aamri and El Moutawakil [18].

For the sake of completeness, we recollect some basic definitions and fundamental results. Let $\Delta = \{(x, x) \mid x \in X\}$ be the diagonal of a nonempty set $X$. For $V, W \in X \times X$, we will use the following setting in the sequel:

\[ V \circ W = \{(x, y) \mid \text{there exists } z \in X : (x, z) \in W, (z, y) \in V\}, \]
\[ V^{-1} = \{(x, y) \mid (y, x) \in V\}. \]  

(1)

For subset $V \in \vartheta$, a pair of points $x$ and $y$ are said to be $V$-close if $(x, y) \in V$ and $(y, x) \in V$. Moreover, a sequence $\{x_n\}$
in $X$ is called a Cauchy sequence for $\partial$, if for any $V \in \partial$ there exists $N \geq 1$ such that $x_n$ and $x_m$ are $V$-close for $n, m \geq N$. For $(X, \partial)$, there is a unique topology $\tau(\partial)$ on $X$ generated by $V(x) = \{y \in X \mid (x, y) \in V\}$, where $V \in \partial$.

A sequence $\{x_n\}$ in $X$ is convergent to $x$ for $\partial$, denoted by $\lim_{n \to \infty} x_n = x$, if for any $V \in \partial$ there exists $n_0 \in \mathbb{N}$ such that $x_n \in V(x)$ for every $n \geq n_0$. A uniform space $(X, \partial)$ is called Hausdorff if the intersection of all the $V \in \partial$ is equal to $\Delta$ of $X$, that is, if $(x, y) \in V$ for all $V \in \partial$ implies $x = y$. If $V = V^{-1}$, then we say that a subset $V \in \partial$ is symmetrical. Throughout the paper, we assume that each $V \in \partial$ is symmetrical. For more details, see, for example, [1, 18–21].

Now, we recall the notions of $A$-distance and $E$-distance.

**Definition 1** (see, e.g., [18, 19]). Let $(X, \partial)$ be a uniform space. A function $p : X \times X \to [0, \infty)$ is said to be an $A$-distance if for any $V \in \partial$ there exists $\delta > 0$ such that if $p(x, y) = \delta$ and $p(z, y) \leq \delta$ for some $z \in X$, then $(x, y) \in V$.

**Definition 2** (see, e.g., [18, 19]). Let $(X, \partial)$ be a uniform space. A function $p : X \times X \to [0, \infty)$ is said to be a $E$-distance if

$$(p_1) \quad p$ is an $A$-distance,

$$(p_2) \quad p(x, y) \leq p(x, z) + p(z, y), \quad \forall x, y, z \in X.$$
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(i) $A_i$, $i = 1, 2, \ldots, m$, are nonempty sets;
(ii) $T(A_1) \subset A_2, \ldots, T(A_{m-1}) \subset A_m, T(A_m) \subset A_1$.

Definition 12. Let $(X, d)$ be a metric space, $m$ a positive integer, $A_1, A_2, \ldots, A_m$ nonempty subsets of $X$, and $X = \bigcup_{i=1}^{m} A_i$. An operator $T : X \to X$ is a cyclic $(\phi)$-contraction if

(i) $X = \bigcup_{i=1}^{m} A_i$ is a cyclic representation of $X$ with respect to $T$,
(ii) $d(Tx, Ty) \leq \phi(d(x, y))$, for any $x \in A_i$, $y \in A_{i+1}$, $i = 1, 2, \ldots, m$, where $A_{m+1} = A_1$ and $\phi \in \mathcal{C}$.

The main result of [14] is the following.

Theorem 13 (Theorem 6 of [14]). Let $(X, d)$ be a complete metric space, $m$ a positive integer, $A_1, A_2, \ldots, A_m$ nonempty subsets of $X$, and $X = \bigcup_{i=1}^{m} A_i$. Let $T : X \to X$ be a cyclic $(\phi-\psi)$-contraction with $\phi, \psi \in \mathcal{C}$. Then, $T$ has a unique fixed point $x \in \bigcap_{i=1}^{m} A_i$.

The main aim of this paper is to prove results similar to the abovementioned theorems in uniform spaces and to present modifications of Theorem 2.1 [16], Theorems 3.1-3.2 in [18], and other related results.

2. Main Result

First, we present the following definition.

Definition 14. Let $(X, \mathcal{C})$ be a uniform space, $m$ a positive integer, $A_1, A_2, \ldots, A_m$ nonempty subsets of $X$, and $X = \bigcup_{i=1}^{m} A_i$. An operator $T : X \to X$ is a cyclic $(\phi)$-contraction if

(i) $X = \bigcup_{i=1}^{m} A_i$ is a cyclic representation of $X$ with respect to $T$,
(ii) for any $x \in A_i$, $y \in A_{i+1}$, $i = 1, 2, \ldots, m$,
\[ p(Tx, Ty) \leq \phi(p(x, y)), \]
where $A_{m+1} = A_1$ and $\phi \in \mathcal{C}$.

Our main result is the following.

Theorem 15. Let $(X, \mathcal{C})$ be an $S$-complete Hausdorff uniform space such that $p$ is an $E$-distance on $X$, $m$ a positive integer, $A_1, A_2, \ldots, A_m$ nonempty closed subsets of $X$ with respect to the topological space $(X, \tau(\mathcal{C}))$, and $X = \bigcup_{i=1}^{m} A_i$. Let $T : X \to X$ be a cyclic $(\phi)$-contraction. Then, $T$ has a unique fixed point $x \in \bigcap_{i=1}^{m} A_i$.

Proof. We first show that the fixed point of $T$ is unique (if it exists). Suppose, on the contrary, that $y, z \in X$ with $y \neq z$ are fixed points of $T$. The cyclic character of $T$ and the fact that $y, z \in X$ are fixed points of $T$ imply that $y, z \in \bigcap_{i=1}^{m} A_i$. Using the contractive condition, we obtain
\[ p(y, z) = p(Ty, Tz) \leq \phi(p(y, z)) < p(y, z) \]
and from the last inequality
\[ p(y, z) = 0. \]

Similarly, we can show that $p(y, y) = 0$ and, consequently, $y = z$.

Now, we prove the existence of a fixed point. Note that $p$ is not symmetric. To show that the sequence $\{x_n\}$ is Cauchy, we will show that both $\lim_{n \to \infty} p(x_n, x_{n+q}) = 0$ and $\lim_{n \to \infty} p(x_{n+q}, x_n) = 0$, for any $q > 1$.

For this aim, take $x_0 \in X$ and consider the sequence given by
\[ x_{n+1} = Tx_n, \quad n = 0, 1, 2, \ldots. \]

If there exists $n_0 \in \mathbb{N}$ such that $x_{n_0+1} = x_{n_0}$, then the proof is completed. In this case, $x_{n_0}$ is the required fixed point of $T$. Throughout the proof, we assume that
\[ x_{n+1} \neq x_n \quad \text{for any } n = 0, 1, 2, \ldots. \]

Notice that for any $n > 0$ there exists $i_n \in \{1, 2, \ldots, m\}$ such that $x_{n-1} \in A_{i_n}$ and $x_n \in A_{i_n+1}$, since $X = \bigcup_{i=1}^{m} A_i$. Due to the fact that $T$ is a cyclic $(\phi)$-contraction, we have
\[ p(x_n, x_{n+1}) = p(Tx_{n-1}, Tx_n) \leq \phi(p(x_{n-1}, x_n)), \]
by taking $x = x_n$ and $y = x_{n+1}$ in (3). From (8) and taking the monotonicity of $\phi$ into account, we derive by induction that
\[ p(x_n, x_{n+1}) \leq \phi^n(p(x_0, x_1)) \quad \text{for any } n = 1, 2, \ldots. \]

As $p$ is an $E$-distance, we obtain that
\[ p(x_n, x_m) \leq p(x_n, x_{n+1}) + \cdots + p(x_m, x_m), \]
so for $q \geq 1$ we have that
\[ p(x_n, x_{n+q}) \leq \phi(p(x_0, x_1)) + \cdots + \phi^{n-1}(p(x_0, x_1)). \]

In the sequel, we will prove that $\{x_n\}$ is a $p$-Cauchy sequence. Denoting
\[ S_n = \sum_{k=0}^{n} \phi^k(p(x_0, x_1)), \quad n \geq 0, \]
implies that
\[ p(x_{n+q}, x_{n+1}) \leq S_{n+q-1} - S_{n-1}. \]

As $\phi$ is a $(c)$-comparison function, supposing $p(x_0, x_1) > 0$, by Lemma 10, (iv), it follows that
\[ \sum_{k=0}^{\infty} \phi^k(p(x_0, x_1)) < \infty, \]
so there is $S \in [0, \infty)$ such that
\[ \lim_{n \to \infty} S_n = S. \]
Then, by (13) we obtain that
\[ \lim_{n \to \infty} p(x_n, x_{n+q}) = 0. \] (16)

By repeating the same arguments in the proof of (16), we conclude that
\[ \lim_{n \to \infty} p(x_n, x_n) = 0. \] (17)

Consequently, we get that the sequence \( \{x_n\}_{n=0} \) is a \( p \)-Cauchy in the \( S \)-complete space \( X = \bigcup_{i=1}^{m} A_i \). Thus, there exists \( x \in X \) such that \( \lim_{n \to \infty} x_n = x \). In what follows we prove that \( x \) is a fixed point of \( T \). In fact, since \( \lim_{n \to \infty} x_n = x \), as \( X = \bigcup_{i=1}^{m} A_i \) is a cyclic representation of \( X \) with respect to \( T \), the sequence \( \{x_n\} \) has infinite terms in each \( A_i \) for \( i \in \{1, 2, \ldots, m\} \).

Since \( A_i \) is closed for every \( i \), it follows that \( x \in \bigcap_{i=1}^{m} A_i \); thus we take a subsequence \( x_{n_i} \) of \( \{x_n\} \) with \( x_{n_i} \in A_{i-1} \). Using the contractive condition, we can obtain
\[ p(Tx, Ty) \leq p(x, x_{n_i}) + p(x_{n_i}, Ty) \]
\[ = p(x, x_{n_i}) + p(Ty, TTy) \]
\[ \leq p(x, x_{n_i}) + \phi(p(x, x)) \] (18)

and since \( x_{n_i} \to x \) and \( \phi \) belong to \( \mathcal{G} \), letting \( k \to \infty \) in the last inequality, we have \( p(x, Tx) = 0 \). Analogously, we can derive that \( p(x, x) = 0 \) and, therefore, \( x \) is a fixed point of \( T \). This finishes the proof.

**Corollary 16.** Let \((X, \tau)\) be an \( S \)-complete Hausdorff uniform space such that \( p \) is an \( E \)-distance on \( X \), and \( m \) a positive integer, \( A_1, A_2, \ldots, A_m \) nonempty closed subsets of \( X \) with respect to the topological space \((X, \tau(\emptyset))\), and \( X = \bigcup_{i=1}^{m} A_i \). Let operator \( T : X \to X \) satisfy

(i) \( p(Tx, Ty) \leq kp(x, y) \), for any \( x, y \in A_i \), \( i = 1, 2, \ldots, m \), where \( A_{m+1} = A_1 \) and \( 0 < k < 1 \).

Then, \( T \) has a unique fixed point \( z \in \bigcap_{i=1}^{m} A_i \).

**Proof.** By Theorem 15, it is enough to set \( \phi(t) = kt \).

**Corollary 17** (cf. [16]). Let \((X, d)\) be a complete metric space, \( m \) a positive integer, \( A_1, A_2, \ldots, A_m \) nonempty closed subsets of \( X \), and \( X = \bigcup_{i=1}^{m} A_i \). Let \( T : X \to X \) be a cyclic \((\phi)\)-contraction. Then, \( T \) has a unique fixed point \( z \in \bigcap_{i=1}^{m} A_i \).

**Proof.** By Theorem 15, it is enough to set \( \theta = \{E \mid E > 0\} \).

**Corollary 18** (cf. [13]). Let \((X, d)\) be a complete metric space, \( m \) a positive integer, \( A_1, A_2, \ldots, A_m \) nonempty closed subsets of \( X \), and \( X = \bigcup_{i=1}^{m} A_i \), a cyclic representation of \( X \) with respect to \( T \). Let \( T : X \to X \) satisfy
\[ p(Tx, Ty) \leq kp(x, y), \] (19)
for any \( x \in A_i \), \( y \in A_{i+1} \), \( i = 1, 2, \ldots, m \), where \( k \in (0, 1) \) and \( A_{m+1} = A_1 \). Then, \( T \) has a unique fixed point \( z \in \bigcap_{i=1}^{m} A_i \).

**Definition 19.** Let \((X, \tau)\) be a uniform space, \( m \) a positive integer, \( A_1, A_2, \ldots, A_m \) nonempty subsets of \( X \), and \( T, g : X \to X \) self-mappings. An operator \( T \) is a cyclic \((\phi)\)-\( g \)-contraction if

(i) \( gX = \bigcup_{i=1}^{m} A_i \) is a cyclic representation of \( X \) with respect to \( T \),

(ii) \( p(Tx, Ty) \leq \phi(p(gx, gy)) \), for any \( x \in A_i \), \( y \in A_{i+1} \), \( i = 1, 2, \ldots, m \), where \( A_{m+1} = A_1 \) and \( \phi \in \mathcal{G} \).

Inspired by [28], now we prove a common fixed point theorem as an application of our Theorem 15.

**Theorem 20.** Let \((X, \tau)\) be a uniform space, \( T, g : X \to X \) self-maps such that \( T \) is cyclic \((\phi)\)-\( g \)-contraction, and \( gX \)-complete Hausdorff uniform space together with \( p \) being an \( E \)-distance on \( X \). Suppose that \( gA_1, gA_2, \ldots, gA_m \) are nonempty closed subsets of \( gX \) with respect to the uniform topology and \( TX \subset gX = \bigcup_{i=1}^{m} gA_i \). Then, \( T \) and \( g \) have a unique coincidence point. Moreover, if \( T \) and \( g \) are weakly compatible, then they have a unique common fixed point \( z \in \bigcap_{i=1}^{m} gA_i \).

**Proof.** As \( g : X \to X \), since there exists \( E \in X \) such that \( gE = gX \) and \( g : E \to X \) is one-to-one. Now, since \( TX \subset gX \), we define mappings \( h : gE \to gE \) by \( h(gx) = Tx \). Since \( g \) is one-to-one on \( E \), so \( h \) is well defined. As \( T \) is cyclic \((\phi)\)-\( g \)-contraction, so
\[ p(Tx, Ty) \leq \phi(p(gx, gy)), \] (20)
for any \( gx \in gA_1 \), \( gy \in gA_{i+1} \), \( i = 1, 2, \ldots, m \). Thus,
\[ p(h(gx), h(gy)) = p(Tx, Ty) \leq \phi(p(gx, gy)), \] (21)
for any \( gx \in gA_1 \), \( gy \in gA_{i+1} \), \( i = 1, 2, \ldots, m \), which implies that \( h \) is cyclic \((\phi)\)-contraction on \( gX \). Hence, all the conditions of Theorem 15 are satisfied by \( h \), so \( h \) has a unique fixed point \( z = gx \) in \( gX \). That is, \( gx = z \) is \( h(gx) = Tx \), so \( T \) and \( g \) have a unique coincidence point as required. Moreover, if \( T \) and \( g \) are weakly compatible, then they have a unique common fixed point.

**Corollary 21** (cf. Theorem 3.2 [18]). Let \((X, \tau)\) be a uniform space, \( T, g : X \to X \) self-maps such that \( T \) is \((\phi)\)-\( g \)-contraction, and \( gX \)-complete Hausdorff uniform space together with \( p \) being an \( E \)-distance on \( X \). Suppose that \( TX \subset gX \) and \( T \) and \( g \) are commuting. Then, \( T \) and \( g \) have a unique common fixed point \( z \in X \).

**Proof.** Take \( A_1 = X \) for all \( i = 1, \ldots, m \) in Theorem 20.
by \(T(0) = T(1) = 0\) and \(T(1/n) = 1/(4n+1)\), then for every \(x, y \neq 0, 1\) we have
\[
d(Tx, Ty) = d \left( \frac{1}{4n+1}, \frac{1}{4m+1} \right)
= \frac{|4(m-n)|}{4n+1} \leq \frac{|m-n|}{4n \cdot m}
\leq \frac{1}{4} \left( \frac{1}{n} - \frac{1}{m} \right) = \frac{1}{4} d(x, y).
\]

Also, for \(x, y = 0, 1\) the above inequality obviously holds. This shows that the contractive condition of Corollary 16 is satisfied and 0 is fixed point \(T\).

**Definition 23.** Let \((X, \theta)\) be a uniform space, let \(f, g : X \to X\) be two mappings, and let \(A\) and \(B\) be nonempty closed subsets of \(X\). The \(X = A \cup B\) be said to be a cyclic representation of \(X\) with respect to the pair \((f, g)\) if \(f(A) \subset B\) and \(g(B) \subset A\).

**Definition 24.** Let \((X, \theta)\) be a uniform space, \(A, B\) nonempty subsets of \(X\), and \(X = A \cup B\). Two self-maps \(f, g : X \to X\) are called cyclic \((\phi)\)-contraction pair if

(i) \(X = A \cup B\) is a cyclic representation of \(X\) with respect to the pair \((f, g)\),

(ii) \(\max\{p(fx, gy), p(gy, fx)\} \leq \phi(\max\{p(x, fx), p(y, gy)\})\), for any \(x \in A\), \(y \in B\), where \(\phi \in \mathcal{C}\).

**Theorem 25.** Let \((X, \theta)\) be an \(S\)-complete Hausdorff uniform space such that \(p\) is an \(E\)-distance on \(X\) and let \(A, B\) be nonempty closed subsets of \(X\) with respect to the topological space \((X, r(\theta))\) and \(X = A \cup B\). Suppose that \(f, g : X \to X\) are cyclic \((\phi)\)-contraction pair. Then, \(f\) and \(g\) have a unique common fixed point \(x \in A \cap B\).

**Proof.** Take \(x_0 \in X\) and consider the sequence given by
\[
f_{2n} = x_{2n+1}, \quad g_{2n+1} = x_{2n+2}, \quad n = 0, 1, 2, \ldots
\]

Since \(X = A \cup B\), for any \(n > 0\), \(x_{2n} \in A\), and \(x_{2n+1} \in B\), and \((f, g)\) are cyclic \((\phi)\)-contraction pair, we have
\[
p(x_{2n+1}, x_{2n+2}) \leq \max\{p(f_{2n}, g_{2n+1}), p(g_{2n+1}, f_{2n+2})\}
= \phi(\max\{p(x_{2n}, f_{2n}), p(x_{2n+1}, g_{2n+1})\})
= \phi(\max\{p(x_{2n}, x_{2n+1}), p(x_{2n+1}, x_{2n+2})\}).
\]

Hence,
\[
lim_{n \to \infty} p(x_{2n}, x_{2n+1}) = \phi(\lim_{n \to \infty} p(x_{2n}, x_{2n+1})).
\]

Similarly, we have
\[
p(x_{2n}, x_{2n+1}) = \max\{p(x_{2n}, g_{2n+1}), p(g_{2n+1}, f_{2n+2})\}
\leq \phi(\max\{p(x_{2n}, x_{2n+1}), p(x_{2n+1}, x_{2n+2})\}).
\]

Hence,
\[
lim_{n \to \infty} p(x_{2n}, x_{2n+1}) = \phi(\lim_{n \to \infty} p(x_{2n}, x_{2n+1})).
\]

From inequalities (25) and (27) and taking into account the monotonicity of \(\phi\), we get by induction that
\[
p(x_n, x_{n+1}) \leq \phi^n(p(x_0, x_1)) \quad \text{for any } n = 1, 2, \ldots
\]

Since \(p\) is an \(E\)-distance, we find that
\[
p(x_n, x_m) \leq p(x_n, x_{n+1}) + p(x_{n+1}, x_{n+2}) + \cdots + p(x_{m-1}, x_m),
\]
so for \(q \geq 1\) we have that
\[
p(x_n, x_{n+q}) \leq \phi^q(p(x_0, x_1)) + \cdots + \phi^{q-1}(p(x_{n}, x_1)).
\]

In the sequel, we will prove that \(\{x_n\}\) is a \(p\)-Cauchy sequence. Denote
\[
S_n = \sum_{k=0}^{n} \phi^k(p(x_0, x_1)), \quad n \geq 0.
\]

By relation (31), we have
\[
p(x_n, x_{n+q}) \leq S_{n+q-1} - S_{n-1}.
\]

From inequalities (25) and (27) and taking into account the monotonicity of \(\phi\), we get by induction that
\[
p(x_n, x_{n+1}) \leq \phi(p(x_0, x_1)) \quad \text{for any } n = 1, 2, \ldots
\]

Since \(p(x_0, x_1) > 0\), Thus, there is \(S \in (0, \infty)\) such that
\[
lim_{n \to \infty} S_n = S.
\]

Then, by (32) we obtain that
\[
p(x_n, x_{n+q}) \to 0 \quad \text{as } n \to \infty.
\]

In an analogous way, we derive that
\[
p(x_n, x_{n+q}) \to 0 \quad \text{as } n \to \infty.
\]

Hence, we get that \(\{x_n\}_{n \geq 0}\) is a \(p\)-Cauchy sequence in the \(S\)-complete space \(X = A \cup B\). So there exists \(x \in X\) such that \(\lim_{n \to \infty} f_{2n} = \lim_{n \to \infty} g_{2n+1} = x\). In what follows, we prove that \(x\) is a fixed point of \(f, g\). In fact, since
\[
\lim_{n \to \infty} f_{2n} = \lim_{n \to \infty} g_{2n+1} = x
\]
and as \(X = A \cup B\) is a
cyclic representation of $X$ with respect to $f, g$, the sequence \{$x_n$\} has infinite terms in each $A, B$.

Since $A, B$ are closed, it follows that $x \in A \cap B$; thus we take subsequences $x_{2n}$, $x_{2n+1}$ of $\{x_n\}$ with $x_{2n} \in A$ and $x_{2n+1} \in B$. Using the contractive condition, we can obtain

$$p(x, fx) \leq p(x, x_{2n+1}) + \max \{ p(x_{2n+1}, fx), p(fx, gx_{2n+1}) \}$$

and since $x_n \rightarrow x$ and $\phi$ belong to $\mathcal{C}$, letting $n \rightarrow \infty$ in the last inequality, we have $p(x, fx) \leq \phi(p(x, fx)) < p(x, fx)$; hence $p(x, fx) = 0$. Similarly, we can show that $p(x, x) = 0$ and, therefore, $x$ is a fixed point of $f$. Similarly, we can show that $x$ is a fixed point of $g$. Finally, in order to prove the uniqueness of the fixed point, we have $y, z \in X$ with $y$ and $z$ fixed points of $f, g$. The cyclic character of $f, g$ and the fact that $y, z \in X$ are fixed points of $f, g$ imply that $y, z \in A \cap B$. Using the contractive condition, we obtain

$$p(y, z) = p(fy, gz) \leq \max \{ p(fy, gz), p(gz, fy) \}$$

$$\leq \phi(\max \{ p(y, fy), p(z, gz) \}) = 0,$$

and from the last inequality we get

$$p(y, z) = 0.$$  \hspace{1cm} (37)

Using the same arguments above, we can show that $p(y, y) = 0$ and, consequently, $y = z$. This finishes the proof. \hfill \square

**Corollary 26.** Let $(X, d)$ be a complete metric space and $A, B$ nonempty closed subsets of $X$ and $\bar{X} = A \cup B$. Let $f, g : X \rightarrow X$ be cyclic $(\phi)$-contraction pair. Then, $f$ and $g$ have a unique common fixed point $z \in A \cap B$.

**Proof.** By Theorem 25, it is enough to set $\mathcal{C} = \{U_{\epsilon} \mid \epsilon > 0\}$. \hfill \square

**Corollary 27.** Let $(X, \sigma)$ be an $S$-complete Hausdorff uniform space such that $p$ is an $E$-distance on $X$ and let $A, B$ be nonempty closed subsets of $X$ with respect to the topological space $(X, \tau(\sigma))$ and $X = A \cup B$. Suppose that the maps $f, g : X \rightarrow X$ satisfy the following inequality:

$$(i) \max \{ p(fy, gy), p(gy, fx) \} \leq k \max \{ p(x, fx), p(y, gy) \}, \text{ for any } x \in A, y \in B,$$

where $0 < k < 1$.

Then, $f, g$ have a unique common fixed point $z \in A \cap B$.

**Proof.** By Theorem 25, it is enough to set $\phi(t) = kt$. \hfill \square

**Example 28.** Let $(X, p)$ be a partial metric space, where $X = \{1/n \cup \{0, 1\}$ and $p(x, y) = \max \{x, y\}$. Set $A = \{1/2n \cup \{0, 1\}$ and $B = \{1/(2n + 1) \cup \{0, 1\}$. Define $\mathcal{C} = \{U_{\epsilon} \mid \epsilon > 0\}$, where $U_{\epsilon} = \{ (x, y) \in X^2 : p(x, y) < p(x, x) + \epsilon \}$. It is easy to see that $(X, \sigma)$ is a uniform space. If we define $\phi : [0, \infty) \rightarrow [0, \infty)$ by $\phi(t) = kt$ for $0 < k < 1$ and $f : X \rightarrow X$ by $f((1/2n) = 1/(4n + 1)$, $f((0) = f(1) = 0$, and $g(1/(2n + 1) = 1/(4n + 2)$, $g(0) = g(1) = 0$. Then, for every $x, y \neq 0, 1$ we have

$$\max \{ p(fx, gy), p(gy, fx) \}$$

$$= \max \{ p(1/(4n + 1), 1/(4m + 2)), p(1/(4n + 2), 1/(4m + 1)) \}$$

$$\leq \frac{1}{2} \max \{ \frac{1}{4n + 1}, \frac{1}{4m + 2} \}$$

$$\leq \frac{1}{2} \max \{ \frac{1}{2n^2}, \frac{1}{2m + 1} \}$$

$$= \frac{1}{2} \max \{ p(x, fx), p(y, gy) \},$$

for any $x \in A, y \in B$. Also, for $x, y = 0, 1$ the above inequality obviously holds. This shows that the contractive condition of Corollary 27 is satisfied and 0 is a common fixed point of $f$ and $g$.

**Conflict of Interests**

The authors declare that there is no conflict of interests regarding the publication of this paper.

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**References**


