Research Article
On Retarded Integral Inequalities for Dynamic Systems on Time Scales

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The object of this paper is to establish some nonlinear retarded inequalities on time scales which can be used as handy tools in the theory of integral equations with time delays.

1. Introduction
Integral inequalities play an important role in the qualitative analysis of differential and integral equations. The well-known Gronwall inequality provides explicit bounds for solutions of many differential and integral equations. On the basis of various initiatives, this inequality has been extended and applied to various contexts (see, e.g., [1–4]), including many retarded ones (see, e.g., [5–9]).

Recently, Ye and Gao [7] obtained the following.

Theorem A. Let $I = [t_0, T) \subset \mathbb{R}$, $a(t), b(t) \in C(I, \mathbb{R}^+)$, $\phi(t) \in C([t_0 - r, t_0], \mathbb{R}^+)$, $a(t_0) = \phi(t_0)$, and $u(t) \in C([t_0 - r, T), \mathbb{R}^+)$

\[
\begin{align*}
\int_{t_0}^{t} u(s) ds & \leq a(t), & t & \in [t_0, T) \\
& \leq \phi(t), & t & \in [t_0 - r, t_0), \\
\end{align*}
\]

where $\beta > 0$. Then, the following assertions hold.

(i) Suppose that $\beta > 1/2$. Then,

\[
\begin{align*}
& u(t) \leq e^{\beta t} \left[ w_1(t) + y_1(t) \right]^{1/\beta}, & t & \in [t_0 + r, T), \\
& u(t) \leq a(t) + \int_{t_0}^{t} \left(t - s\right)^{\beta - 1} b(s) \phi(s - r) ds, & t & \in [t_0, t_0 + r), \\
\end{align*}
\]

where $K_1 = \Gamma(2\beta - 1) e^{2r}/4^{\beta - 1}$, $C_1 = \max\{2, e^{2r}\}$, $w_1(t) = C_1 e^{-2r} \phi^2(t)$, and

\[
y_1(t) = \left( \int_{t_0}^{t} K_1 b^2(\tau) \phi_1(\tau - r) d\tau \right)^{\frac{1}{\beta}}
\]

If, in addition, $a(t)$ and $\phi(t)$ are nondecreasing $C^1$-functions, then

\[
\begin{align*}
& u(t) \leq \left( \frac{K_1}{2} \int_{t_0}^t b^2(s) ds \right)^{\frac{1}{\beta}} , & t & \in [t_0, T). \\
\end{align*}
\]

(ii) Suppose that $0 < \beta \leq 1/2$. Then,

\[
\begin{align*}
& u(t) \leq e^{\beta t} \left[ w_2(t) + y_2(t) \right]^{1/\beta}, & t & \in [t_0 + r, T), \\
& u(t) \leq a(t) + \int_{t_0}^{t} \left(t - s\right)^{\beta - 1} b(s) \phi(s - r) ds, & t & \in [t_0, t_0 + r), \\
\end{align*}
\]

where $K_1 = \Gamma(2\beta - 1) e^{2r}/4^{\beta - 1}$, $C_1 = \max\{2, e^{2r}\}$, $w_1(t) = C_1 e^{-2r} \phi^2(t)$, and

\[
y_1(t) = \left( \int_{t_0}^{t} K_1 b^2(\tau) \phi_1(\tau - r) d\tau \right)^{\frac{1}{\beta}}
\]

If, in addition, $a(t)$ and $\phi(t)$ are nondecreasing $C^1$-functions, then

\[
\begin{align*}
& u(t) \leq \left( \frac{K_1}{2} \int_{t_0}^t b^2(s) ds \right)^{\frac{1}{\beta}} , & t & \in [t_0, T). \\
\end{align*}
\]
where \( K_2 = \left[ (\Gamma(1 - (1 - \beta)p)) / p^{1-n(1-\beta)} \right]^{1/p} \), \( C_2 = \max \{ 2^{q-1}, e^p \} \), \( w_2(t) = C_2 e^{-q\phi(t)}, \phi_3(t) = C_2 e^{-q\phi(t)}, \psi(t) = 2^{-1} K_2^p e^{-p^s b_0(t)} \), and

\[
y_2(t) = \int_{t_0}^{t} \psi(s) \phi_3(r - s) \, ds \cdot \exp \left( \int_{t_0}^{t} \psi(r) \, dr \right)
+ \int_{t_0}^{t} w_2(s - r) \psi(s) \exp \left( \int_{t_0}^{t} \psi(r) \, dr \right) \, ds.
\]

If, in addition, \( a(t) \) and \( \phi(t) \) are nondecreasing \( C^1 \)-functions, then

\[
u(t) \leq C_2^{1/q} a(t) \exp \left( t - t_0 + \frac{1}{q} \int_{t_0}^{t} \psi(s) \, ds \right),
\]

where \( \nu(t) \) holds, for any \( \nu(t) \). Multiplying (16) for \( a(t) \) and \( \phi(t) \), we get

\[
\Delta z(t) \leq p(t) a(t - r) e^{-p(t) s} + m \leq n \, p(t) e^{-p(t) s}.
\]

Lemma 3. Let \( a(t) \geq 0, b(t) > 0, p(t) := nb(t)/m, -b \in \mathbb{R}^r := \{ f \in \mathbb{R} : 1 + \mu(t) f(t) > 0, \text{ for all } t \in \mathbb{T} \}, \phi(t) \geq 0 \) is rd-continuous on \( [t_0 - r, t_0] \), and \( r \geq 0 \) and \( p \geq n > 0 \) are real constants. If \( u(t) \geq 0 \) is rd-continuous and

\[
u(t) \leq a(t) + \int_{t_0}^{t} b(s) u^n(s - r) \, ds, \quad t \in [t_0, T] \gamma,
\]

\[
u(t) \leq \phi(t), \quad t \in [t_0 - r, t_0] \gamma,
\]

where \( \Delta \) is any time scale, \( u(t), a(t), b(t), c(t), \) and \( \phi(t) \) are real-valued nonnegative rd-continuous functions defined on \( \mathbb{T}, m \) and \( n \) are positive constants, \( m \geq n, m \geq 1, (1/p) + (1/m) = 1, \beta > (p - 1)/p, \) and \( [t_0, T] \gamma := [t_0, T] \cap \mathbb{T} \).

First, we make a preliminary definition.

Definition 1. We say that a function \( p : \mathbb{T} \to \mathbb{R} \) is regressive provided that

\[
1 + \mu(t) p(t) \neq 0, \quad \forall t \in \mathbb{T}
\]

holds, where \( \mu(t) \) is graininess function; that is, \( \mu(t) := \sigma(t) - t \). The set of all regressive and rd-continuous functions \( f : \mathbb{T} \to \mathbb{R} \) will be denoted by \( \mathcal{R} \).

2. Main Results

For convenience, we first cite the following lemma.

Lemma 2 (see [10]). Let \( a \geq 0, p \geq q \geq 0, p \neq 0; \) then

\[
a^{q/p} \leq a^{q/p} K^{(a-p)/p} + p \cdot a^{q/p} \]

for any \( K > 0 \).
Integrating both sides from \( t_0 + r \) to \( t \), we obtain
\[
    z(t) \leq e^{-p(t_0 + r, t)} z(t_0 + r) + e^{-p(t_0 + r, t)} \int_{t_0 + r}^{t} p(s) (s) e^{-p(s, t) \Delta s} + \frac{m-n}{n} (e^{-p(t_0 + r, t)} - 1) .
\]
(18)

For \( t \in [t_0, t_0 + r, T] \), \( z^{\Delta}(t) \leq b(t) \phi^{\Delta}(t - r) \), so
\[
    z(t) \leq \int_{t_0}^{t} b(s) \phi^{\Delta}(s - r) \Delta s .
\]
(19)

Using (18) and (19), we get
\[
    z(t) \leq e^{-p(t_0 + r, t)} z(t_0 + r) + e^{-p(t_0 + r, t)} \int_{t_0 + r}^{t} p(s) (s) a(s, t) e^{-p(s, t) \Delta s} + \frac{m-n}{n} (e^{-p(t_0 + r, t)} - 1) .
\]
(20)

for \( t \in [t_0 + r, T] \).

Noting that \( u^{m}(t) \leq a(t) + z(t) \), inequalities (13) and (14) follow.

Finally, if \( a(t) \) and \( \phi(t) \) are nondecreasing, then for \( t \in [t_0, t_0 + r, T] \), by (14), we have
\[
    u^{m}(t) \leq a(t) + \phi^{\Delta}(t - r) \int_{t_0}^{t} b(s) \Delta s
\]
(21)

\[
    \leq a(t) \left( 1 + \int_{t_0}^{t} b(s) \Delta s \right) \leq c(t) e^{-b(t_0, t)} .
\]

If \( t \in [t_0 + r, T] \), by (13),
\[
    u^{m}(t) \leq a(t) + e^{-p(t_0 + r, t)} a(t) \int_{t_0 + r}^{t} b(s) \Delta s
\]
(22)

\[
    + a(t) \int_{t_0 + r}^{t} p(s) e^{-p(s, t) \Delta s} + \frac{m-n}{n} \int_{t_0 + r}^{t} p(s) e^{-p(s, t) \Delta s} \leq c(t) + e^{-p(t_0 + r, t)} c(t) \int_{t_0 + r}^{t} b(s) \Delta s
\]
\[
    + c(t) \int_{t_0 + r}^{t} p(s) e^{-p(s, t) \Delta s} = c(t) e^{-p(t_0 + r, t)} \left( 1 + \int_{t_0 + r}^{t} b(s) \Delta s \right) \leq c(t) e^{-b(t_0, t)} .
\]

The proof is complete. □
Hence,

\[
\begin{align*}
    u^m(t) &\leq 2^{m-1} a^m(t) + 2^{m-1} e^{\alpha t} \left( p\beta - p + 1 \right) \\
    &\quad \times \left( \frac{m}{p^n} \right)^{\beta m-1} \int_{t_0}^{t} b^m(s) e^{-n} u''(s-r) \Delta s
\end{align*}
\]

and so

\[
\left( u(t) e^{-t} \right)^m
\leq 2^{m-1} a^m(t) e^{-m\alpha} + 2^{m-1} m^{-1} \left( p\beta - p + 1 \right) \left( \frac{m}{p^n} \right)^{\beta m-1} \int_{t_0}^{t} b^m(s) e^{-n} u''(s-r) \Delta s.
\]

Let \( v(t) := e^{-t} u(t) \); then we have

\[
\begin{align*}
v^m(t) &\leq \omega_1(t) + K \int_{t_0}^{t} b^m(s) v^n(s-r) \Delta s, \quad t \in [t_0, T].
\end{align*}
\]

For \( t \in [t_0, t_0 + r, T] \), we have \( e^{-t} u(t) \leq e^{-\alpha}(t) \leq e^{-t^2} \phi(t) \); that is, \( v(t) \leq \phi^1(t) \). By Lemma 3, we get

\[
\begin{align*}
v^m(t) &\leq \omega_1(t) + \int_{t_0}^{t} b_1(s) \omega_1(s-r) e^{-\alpha}(s,t) \Delta s \\
    &\quad + e^{-\alpha}(t_0 + r,t) \int_{t_0}^{t+r} K b^m(s) \phi^n(s-r) \Delta s \\
    &\quad + \left( e^{-\alpha}(t_0 + r,t) - 1 \right).
\end{align*}
\]

Hence, the first inequality in (23) follows.

Finally, if \( a(t) \) and \( \phi(t) \) are nondecreasing, and \( a^n(t_0) = 2^{1-n} e^{-m\alpha} \phi^n(t_0) e^{-r} \), by Lemma 3, we have

\[
\begin{align*}
    u(t) &\leq e^{\alpha(t)} e^{-\lambda_0(t_0,t)} \left( t_0, T \right), \quad t \in [t_0, T].
\end{align*}
\]

The proof is complete. \( \square \)

**Lemma 5.** Let \( a(t) \geq 0, b(t) > 0, c(t) > 0, p(t) := (n b(t) / m), q(t) := (n c(t) / m), \gamma(t) := a(t) + (m-n)/m \) and \( -p, -q, -p+c+q \in \mathbb{R}^+ \) and let \( \phi(t) \geq 0 \) be rd-continuous on \([t_0, T], \) where \( r \geq 0 \) and \( m \geq n > 0 \) are real constants. If \( u(t) \geq 0 \) is rd-continuous and

\[
\begin{align*}
    u^m(t) &\leq a(t) + \int_{t_0}^{t} b(s) u^n(s) + c(s) u''(s-r) \Delta s, \quad t \in [t_0, T],
\end{align*}
\]

then

\[
\begin{align*}
    u^m(t) &\leq a(t) \\
    &\quad + \int_{t_0}^{t} \left[ p(s) \gamma(s) + q(s) \gamma(s-r) \right] e^{-p(t-s)} (t, s) \Delta s \\
    &\quad + e^{-p(t-t_0)} (t_0 + r, t) \\
    &\quad \times \int_{t_0}^{t+r} \left[ p(s) \gamma(s) + c(s) \phi^n(s-r) \right] e^{-p}(s, t) \Delta s
\end{align*}
\]

for \( t \in [t_0 + r, T] \) and

\[
\begin{align*}
    u^m(t) &\leq a(t) + \int_{t_0}^{t} \left[ p(s) \gamma(s) + c(s) \phi^n(s-r) \right] e^{-p}(s, t) \Delta s
\end{align*}
\]

for \( t \in [t_0 + r, T] \). Furthermore, if \( a(t) \) and \( \phi(t) \) are nondecreasing with \( a(t_0) = \phi(t_0) \), then

\[
\begin{align*}
    u^m(t) &\leq \gamma(t) e^{-p(t-r)} (t_0, t), \quad t \in [t_0, T].
\end{align*}
\]

**Proof.** Let \( z(t) = \int_{t_0}^{t} \left[ b(s) u^n(s) + c(s) u''(s-r) \right] \Delta s \). Then, \( z(t_0) = 0 \), \( u^m(t) \leq a(t) + z(t) \), \( z(t) \) is positive and nondecreasing for \( t \in [t_0, T] \). Further, we have

\[
\begin{align*}
    z^\Delta(t) &\leq b(t) u^n(t) + c(t) u''(t-r).
\end{align*}
\]

For \( t \in [t_0 + r, T] \), using Lemma 2, we have

\[
\begin{align*}
    z^\Delta(t) &\leq b(t) \left( a(t) + z(t) \right) u^n(t) + c(t) \phi^n(t-r) \\
    &\leq b(t) \left[ \frac{n}{m} \left( a(t) + z(t) \right) + \frac{m-n}{m} \right] + c(t) \phi^n(t-r) \\
    &\leq p(t) \gamma(t) + p(t) z(t) \sigma(t) + c(t) \phi^n(t-r), \\
    \left( e^{-p(t)} (t, t_0) z(t) \right)^\Delta &\leq \left( p(t) \gamma(t) + c(t) \phi^n(t-r) \right) e^{-p(t, t_0)}.
\end{align*}
\]

Integrating both sides from \( t_0 \) to \( t \), we obtain

\[
\begin{align*}
    z(t) &\leq \int_{t_0}^{t} \left[ p(s) \gamma(s) + c(s) \phi^n(s-r) \right] e^{-p(s, t)} \Delta s.
\end{align*}
\]
For $t \in [t_0 + r, T)$,
\[
z(t) \leq b(t) [a(t) + z(t)]^{n/m} + c(t) [a(t - r) + z(t - r)]^{n/m}
\leq b(t) \left( \frac{n}{m} (a(t) + z(t)) + \frac{m - n}{m} \right)
+ c(t) \left( \frac{n}{m} (a(t - r) + z(t - r)) + \frac{m - n}{m} \right)
\leq \left( \frac{n}{m} b(t) + \frac{n}{m} c(t) \right) z(\sigma(t))
+ \left( \frac{m - n}{m} b(t) + \frac{m - n}{m} c(t) \right)
\leq (p(t) + q(t)) z(\sigma(t)) + p(t) \gamma(t) + q(t) \gamma(t - r).
\]

Hence, we get
\[
(e - (p + q)(t_0 + r, t)) z(t_0 + r) \leq (p(t) \gamma(t) + q(t) \gamma(t - r)) e - (p + q)(t_0 + r, t).
\]

Integrating both sides from $t_0 + r$ to $t$, we obtain
\[
z(t) \leq e - (p + q)(t_0 + r, t) z(t_0 + r)
+ e - (p + q)(t_0 + r, t)
\times \int_{t_0 + r}^{t} [p(s) \gamma(s) + q(s) \gamma(s - r)] e - (p + q)(s, t_0 + r) \Delta s
\leq e - (p + q)(t_0 + r, t)
\times \int_{t_0 + r}^{t} \left[ p(s) \gamma(s) + q(s) \phi^n(s - r) \right] e - p(s, t_0 + r) \Delta s
+ \int_{t_0 + r}^{t} \left[ p(s) \gamma(s) + q(s) \gamma(s - r) \right] e - (p + q)(s, t) \Delta s.
\]

Using $u^m(t) \leq a(t) + z(t)$, we get inequalities (34) and (35). Finally, if $a(t)$ and $\phi(t)$ are nondecreasing, then, by (35),
\[
u^m(t) \leq \gamma(t) \left( 1 + \int_{t_0}^{t} (p(s) + c(s)) e - p(s, t) \Delta s \right)
\leq \gamma(t) \left( 1 + \int_{t_0}^{t} (p(s) + c(s)) e - (p + \phi)(s, t) \Delta s \right)
\leq \gamma(t) e - (p + \phi)(t_0, t)
\]
for $t \in [t_0, t_0 + r)$. Furthermore, by (34),
\[
u^m(t) \leq \gamma(t) + \gamma(t) e - (p + q)(t_0 + r, t)
\times \int_{t_0}^{t} \left[ p(s) + c(s) \right] e - p(s, t_0 + r) \Delta s
\leq \gamma(t) e - (p + q)(t_0 + r, t)
\times \int_{t_0}^{t} \left[ p(s) + c(s) \right] e - (p + q)(s, t) \Delta s
\leq \gamma(t) e - (p + q)(t_0, t).
\]

Theorem 6. Assume that $u(t)$ satisfies condition (9), $a(t) \geq 0$, $K := 3^{m-1} \Gamma^{m-1} (p \beta - p + 1)(m/\pi n)^{\beta m - 1}$, $p(t) := n K b(t)/m$, $c_1(t) := K e - n \phi^n(t_0)$, $q(t) := (n/m) c_1(t)$, $-p, -(p + c_1) \in \mathbb{R}^+$. If, in addition, $a(t)$ and $\phi(t)$ are nondecreasing, and $a^m(t_0) = 3^{1-m} e^{(m-n) p \beta + (m-n)/n}$, then
\[
u^m(t) \leq \gamma(t) e - (p + \phi)(t_0, t)^{1/m}, \quad t \in [t_0, T)\]
where $\gamma(t) = 3^{m-1} \Gamma^{m-1} e^{-n \phi^n(t_0) + (m-n)/n}$.

Proof. For $t \in [t_0, T)$, using Hölder’s inequality with indices $p$ and $m$, we obtain from (9) that
\[
u^m(t) \leq a(t) + \int_{t_0}^{t} (t - s)^{p-1} e^{n \phi^n(s) u^m(s)} \Delta s
+ \int_{t_0}^{t} (t - s)^{1-p} e^{-n \phi^n(s) u^m(s)} \Delta s
\leq a(t) + \left( \int_{t_0}^{t} (t - s)^{p-1} e^{n \phi^n(s) u^m(s)} \Delta s \right)^{1/p}
\times \int_{t_0}^{t} e^{-n \phi^n(s) \Delta s}^{1/m}
+ \left( \int_{t_0}^{t} (t - s)^{1-p} e^{-n \phi^n(s) u^m(s)} \Delta s \right)^{1/p}
\times \int_{t_0}^{t} e^{n \phi^n(s) u^n(s)} \Delta s^{1/m}
\]
\[
\begin{align*}
\leq a(t) + e^{nt/m} \left( \frac{m}{pn} \right)^{(\beta-1)/p} \Gamma^{1/p} \left( p\beta - p + 1 \right) \\
\times \left[ \left( \int_{t_0}^{t} b^m(s) e^{-u^n(s) \Delta s} \right)^{1/m} \\
+ \left( \int_{t_0}^{t} c^m(s) e^{-u^n(s-r) \Delta s} \right)^{1/m} \right]
\end{align*}
\]
(46)

By Jensen's inequality \((\sum_{i=1}^{n} x_i^p)^{1/p} \leq \sum_{i=1}^{n} x_i^q\), we get

\[
\begin{align*}
u^m(t) &\leq 3^{m-1} a^m(t) + 3^{m-1} e^{nt/m} \left( \frac{m}{pn} \right)^{(\beta-1)/p} \Gamma^{1/p} \left( p\beta - p + 1 \right) \\
&\times \left[ \left( \int_{t_0}^{t} b^m(s) e^{-u^n(s) \Delta s} \right)^{1/m} \\
&+ \left( \int_{t_0}^{t} c^m(s) e^{-u^n(s-r) \Delta s} \right)^{1/m} \right],
\end{align*}
\]
(47)

Let \(v(t) := e^{-t} u(t), w_2(t) := 3^{m-1} a^m(t) e^{-mt}, \) we have

\[
\begin{align*}
v^m(t) &\leq w_2(t) + \int_{t_0}^{t} K b^m(s) v^n(s) \Delta s \\
&+ \int_{t_0}^{t} K e^{-mr} c^m(s) v^n(s-r) \Delta s,
\end{align*}
\]
(49)

for \(t \in [t_0, T]_{\mathbb{T}}\). For \(t \in [t_0 - r, t_0]_{\mathbb{T}}\), we have \(e^{-t} u(t) \leq e^{-t} v(t) \leq \phi(t)\); that is, \(u(t) \leq \phi(t)\). By Lemma 5, we get

\[
\begin{align*}
u(t) &\leq e^{t} v(t) e^{-\Delta t_0} \left( t_0, t \right)^{1/m}, \quad t \in [t_0, T]_{\mathbb{T}}.
\end{align*}
\]
(50)

The proof is complete.

The following is a simple consequence of Theorem 4.

**Corollary 7.** Suppose that \(m = n = 2\),

\[
\begin{align*}
u(t) &\leq a(t) + \int_{t_0}^{t} (t-s)^{\beta-1} b(s) u(s-r) \Delta s, \\
&\quad t \in [t_0, T]_{\mathbb{T}},
\end{align*}
\]
(51)

\[
\begin{align*}
u(t) &\leq \phi(t), \quad t \in [t_0 - r, t_0]_{\mathbb{T}}.
\end{align*}
\]

then

\[
\begin{align*}
u(t) &\leq e^{t} \left[ w_1(t) + \int_{t_0 + r}^{t} K b^2(s) w_1(s-r) e^{-K b^2(s) \Delta s} \right] \\
&+ e^{-K b^2(t_0 + r, t)} \times \left( \int_{t_0}^{t} K b^2(s) \Delta s \right)^{1/2},
\end{align*}
\]
(52)

where \(K := \Gamma(2\beta - 1) e^{-2r} \cdot (1/4^{\beta-1}), w_1(t) := 2e^{r(t)} e^{-2r}, \phi(t) := e^{-r/2} \phi(t)\).

If \(\mathbb{T} = \mathbb{R}\), then the conclusion reduces to that of Theorem A for \(\beta > 1/2\).

**Conflict of Interests**

The authors declare that there is no conflict of interests regarding the publication of this paper.

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