Research Article

Sharp Power Mean Bounds for Sándor Mean

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We prove that the double inequality $M_p(a, b) < X(a, b) < M_q(a, b)$ holds for all $a, b > 0$ with $a \neq b$ if and only if $p \leq 1/3$ and $q \geq \log 2 / (1 + \log 2) = 0.4093 \dots$, where $X(a, b)$ and $M_r(a, b)$ are the Sándor and r th power means of a and b , respectively.

1. Introduction

Let $p \in \mathbb{R}$ and $a, b > 0$ with $a \neq b$. Then the p th power mean $M_p(a, b)$ of a and b is given by

$$M_p(a, b) = \left(\frac{a^p + b^p}{2} \right)^{1/p} \quad (p \neq 0), \quad M_0(a, b) = \sqrt{ab}. \quad (1)$$

The main properties for the power mean are given in [1]. It is well known that $M_p(a, b)$ is strictly increasing with respect to $p \in \mathbb{R}$ for fixed $a, b > 0$ with $a \neq b$. Many classical means are the special cases of the power mean; for example, $M_{-1}(a, b) = 2ab/(a+b) = H(a, b)$ is the harmonic mean, $M_0(a, b) = \sqrt{ab} = G(a, b)$ is the geometric mean, $M_1(a, b) = (a+b)/2 = A(a, b)$ is the arithmetic mean, and $M_2(a, b) = \sqrt{(a^2 + b^2)/2} = Q(a, b)$ is the quadratic mean.

Let $L(a, b) = (a-b)/(\log a - \log b)$, $P(a, b) = (a-b)/[2 \arcsin((a-b)/(a+b))]$, $I(a, b) = (a^a/b^b)^{1/(a-b)}/e$, $M(a, b) = (a-b)/[2 \sinh^{-1}((a-b)/(a+b))]$, and $T(a, b) = (a-b)/[2 \arctan((a-b)/(a+b))]$ be the logarithmic, first Seiffert, identric, Neuman-Sándor, and second Seiffert means of two distinct positive real numbers a and b , respectively. Then it is well known that the inequalities

$$\begin{aligned} H(a, b) &< G(a, b) < L(a, b) < P(a, b) \\ &< I(a, b) < A(a, b) < M(a, b) \\ &< T(a, b) < Q(a, b) \end{aligned} \quad (2)$$

hold for all $a, b > 0$ with $a \neq b$.

Recently, the bounds for certain bivariate means in terms of the power mean have been the subject of intensive research. Seiffert [2] proved that the inequalities

$$\begin{aligned} \frac{2}{\pi} M_1(a, b) &< P(a, b) < M_1(a, b) \\ &< T(a, b) < M_2(a, b) \end{aligned} \quad (3)$$

hold for all $a, b > 0$ with $a \neq b$.

Jagers [3] proved that the double inequality

$$M_{1/2}(a, b) < P(a, b) < M_{2/3}(a, b) \quad (4)$$

holds for all $a, b > 0$ with $a \neq b$.

In [4, 5], Hästö established that

$$\begin{aligned} P(a, b) &> M_{\log 2 / \log \pi}(a, b), \\ P(a, b) &> \frac{2\sqrt{2}}{\pi} M_{2/3}(a, b) \end{aligned} \quad (5)$$

for all $a, b > 0$ with $a \neq b$.

Witkowski [6] proved that the double inequality

$$\frac{2\sqrt{2}}{\pi} M_2(a, b) < T(a, b) < \frac{4}{\pi} M_1(a, b) \quad (6)$$

holds for all $a, b > 0$ with $a \neq b$.

In [7], Costin and Toader presented that

$$\begin{aligned} M_{\log 2 / (\log \pi - \log 2)}(a, b) &< T(a, b) \\ &< M_{5/3}(a, b) \end{aligned} \quad (7)$$

for all $a, b > 0$ with $a \neq b$.

Chu and Long [8] proved that the double inequality

$$M_p(a, b) < M(a, b) < M_q(a, b) \quad (8)$$

holds for all $a, b > 0$ with $a \neq b$ if and only if $p \leq \log 2 / \log[2 \log(1 + \sqrt{2})] = 1.224 \dots$ and $q \geq 4/3$.

The following sharp bounds for the logarithmic and identric means in terms of the power means can be found in the literature [9–16]:

$$\begin{aligned} M_0(a, b) &< L(a, b) < M_{1/3}(a, b), \\ M_{2/3}(a, b) &< I(a, b) < M_{\log 2}(a, b), \\ M_0(a, b) &< L^{1/2}(a, b) I^{1/2}(a, b) < M_{1/2}(a, b), \\ M_{\log 2/(1+\log 2)}(a, b) &< \frac{L(a, b) + I(a, b)}{2} < M_{1/2}(a, b) \end{aligned} \quad (9)$$

for all $a, b > 0$ with $a \neq b$.

Recently, Sándor [17] introduced the Sándor mean $X(a, b)$ of two positive real numbers a and b , which is given by

$$X(a, b) = A(a, b) e^{(G(a, b)/P(a, b))^{-1}}. \quad (10)$$

In [18], Sándor proved that

$$\begin{aligned} X(a, b) &< \frac{P^2(a, b)}{A(a, b)}, \\ \frac{A(a, b) G(a, b)}{P(a, b)} &< X(a, b) < \frac{A(a, b) P(a, b)}{2P(a, b) - G(a, b)}, \\ X(a, b) &> \frac{A(a, b) L(a, b)}{P(a, b)} e^{(G(a, b)/L(a, b))^{-1}}, \\ X(a, b) &> \frac{A(a, b) [P(a, b) + G(a, b)]}{3P(a, b) - G(a, b)}, \\ \frac{A^2(a, b) G(a, b)}{P(a, b) L(a, b)} e^{(L(a, b)/A(a, b))^{-1}} &< X(a, b) \\ &< A(a, b) \left[\frac{1}{e} + \left(1 - \frac{1}{e}\right) \frac{G(a, b)}{P(a, b)} \right], \\ A(a, b) + G(a, b) - P(a, b) &< X(a, b) \\ &< A^{-1/3}(a, b) \left[\frac{A(a, b) + G(a, b)}{2} \right]^{4/3}, \\ P^{1/(\log \pi - \log 2)}(a, b) A^{1-1/(\log \pi - \log 2)}(a, b) \\ &< X(a, b) < P^{-1}(a, b) \left[\frac{A(a, b) + G(a, b)}{2} \right]^2 \end{aligned} \quad (11)$$

for all $a, b > 0$ with $a \neq b$.

In the Introduction we cite only a minor part of the existing literature on the considered means. For example, an important paper on the first Seiffert mean $P(a, b)$ is again due to Sándor [19].

The main purpose of this paper is to present the best possible parameters p and q such that the double inequality $M_p(a, b) < X(a, b) < M_q(a, b)$ holds for all $a, b > 0$ with $a \neq b$.

2. Lemmas

In order to prove our main results we need several lemmas, which we present in this section.

Lemma 1. Let $g_1 : (0, 1) \times \mathbb{R} \rightarrow \mathbb{R}$ be defined by

$$g_1(x, p) = \frac{\sqrt{x}(x-1)(x^{p-1}+1)}{(x+1)(x^p+1)} - \arcsin \frac{x-1}{x+1}. \quad (12)$$

Then

- (1) $g_1(x, p)$ is strictly decreasing with respect to x on $(0, 1)$ if and only if $p \geq 1/2$;
- (2) $g_1(x, p)$ is strictly increasing with respect to x on $(0, 1)$ if and only if $p \leq 1/3$.

Proof. It follows from (12) that

$$\frac{\partial g_1(x, p)}{\partial x} = \frac{(1-x)x^{p-3/2}}{2(x+1)^2(x^p+1)^2} g_2(x, p), \quad (13)$$

where

$$\begin{aligned} g_2(x, p) &= -3x^{1-p} - x^{2-p} + x^p + 3x^{p+1} \\ &\quad + (2p-1)x^2 - 2p + 1. \end{aligned} \quad (14)$$

(1) If $g_1(x, p)$ is strictly decreasing with respect to x on $(0, 1)$, then (13) leads to the conclusion that $g_2(x, p) < 0$ for all $x \in (0, 1)$. In particular, we have $g_2(0^+, p) \leq 0$. We assert that $p \geq 1/2$. Indeed, from (14) we clearly see that $g_2(0^+, 0) = 2$, $g_2(0^+, p) = +\infty$ if $p < 0$, and $g_2(0^+, p) = 1 - 2p > 0$ if $0 < p < 1/2$.

If $p \geq 1/2$, then it follows from (14) that

$$\begin{aligned} \frac{\partial g_2(x, p)}{\partial p} &= (3x^{p+1} + 3x^{1-p} + x^{2-p} + x^p) \log x \\ &\quad - 2(1-x^2) < 0 \end{aligned} \quad (15)$$

for all $x \in (0, 1)$.

Equation (14) and inequality (15) lead to the conclusion that

$$g_2(x, p) \leq g_2\left(x, \frac{1}{2}\right) = -2\sqrt{x}(1-x) < 0 \quad (16)$$

for all $x \in (0, 1)$.

Therefore, $g_1(x, p)$ is strictly decreasing with respect to x on $(0, 1)$ which follows from (13) and (16).

(2) If $g_1(x, p)$ is strictly increasing with respect to x on $(0, 1)$, then (13) leads to the conclusion that $g_2(x, p) > 0$ for all $x \in (0, 1)$. In particular, we have

$$\lim_{x \rightarrow 1^-} \frac{g_2(x, p)}{1-x} = 4 - 12p \geq 0 \quad (17)$$

and $p \leq 1/3$.

If $p \leq 1/3$, then (14) and (15) lead to the conclusion that

$$g_2(x, p) \geq g_2\left(x, \frac{1}{3}\right) = \frac{1}{3} \left(1 + x^{1/3}\right) \left(1 + 5x^{1/3} + x^{2/3}\right) \times \left(1 - x^{1/3}\right)^3 > 0 \quad (18)$$

for all $x \in (0, 1)$.

Therefore, $g_1(x, p)$ is strictly increasing with respect to x on $(0, 1)$ which follows from (13) and (18). $\square$

Lemma 2. Let $g_1 : (0, 1) \times \mathbb{R} \rightarrow \mathbb{R}$ be defined by (12). Then there exists $x_0 \in (0, 1)$ such that $g_1(x, p)$ is strictly increasing with respect to x on $(0, x_0]$ and strictly decreasing with respect to x on $[x_0, 1)$ if $1/3 < p < 1/2$.

Proof. Let $p \in (1/3, 1/2)$ and $g_2(x, p)$ be defined by (14). Then (14) leads to

$$g_2(0, p) = 1 - 2p > 0, \quad g_2(1, p) = 0, \quad (19)$$

$$\begin{aligned} x^{1-p} \frac{\partial g_2(x, p)}{\partial x} &= 3(p-1)x^{1-2p} + (p-2)x^{2-2p} \\ &\quad + 2(2p-1)x^{2-p} + 3(p+1)x + p := g_3(x, p), \end{aligned} \quad (20)$$

$$g_3(0, p) = p > 0, \quad g_3(1, p) = 12p - 4 > 0, \quad (21)$$

$$\begin{aligned} x^{2p} \frac{\partial g_3(x, p)}{\partial x} &= 3(p+1)x^{2p} - 2(2p-1)(p-2)x^{1+p} \\ &\quad - 2(p-1)(p-2)x - 3(2p-1)(p-1) \\ &:= g_4(x, p), \end{aligned} \quad (22)$$

$$g_4(0, p) = -3(1-p)(1-2p) < 0, \quad (23)$$

$$g_4(1, p) = 4(3p-1)(2-p) > 0,$$

$$\begin{aligned} \frac{\partial^2 g_4(x, p)}{\partial x^2} &= -2p(1-2p)(p+1) \\ &\quad \times [3 + (2-p)x^{1-p}]x^{2p-2} < 0 \end{aligned} \quad (24)$$

for $x \in (0, 1)$.

Inequality (24) implies that $g_4(x, p)$ is strictly convex with respect to x on $(0, 1)$. From (22) and (23) together with the strict convexity of $g_4(x, p)$ with respect to x on $(0, 1)$ we clearly see that there exists $x_1 \in (0, 1)$ such that $g_3(x, p)$ is strictly decreasing with respect to x on $(0, x_1]$ and strictly increasing with respect to x on $[x_1, 1)$. We assert that

$$g_3(x_1, p) < 0. \quad (25)$$

Indeed, if $g_3(x_1, p) \geq 0$, then it follows from (20) and the piecewise monotonicity of $g_3(x, p)$ with respect to x on $(0, 1)$ that $g_2(x, p)$ is strictly increasing with respect to x on $(0, 1)$.

Hence, we get $g_2(x, p) < g_2(1, p) = 0$ for all $x \in (0, 1)$. This conjunction with Lemma 1 and (13) leads to the conclusion that $p \geq 1/2$, which contradicts with $1/3 < p < 1/2$.

From (20) and (21) together with (25) and the piecewise monotonicity of $g_3(x, p)$ with respect to x on $(0, 1)$ we clearly see that there exist $x_{11} \in (0, x_1)$ and $x_{12} \in (x_1, 1)$ such that $g_2(x, p)$ is strictly increasing with respect to x on $(0, x_{11}] \cup [x_{12}, 1)$ and strictly decreasing with respect to x on $[x_{11}, x_{12}]$.

Therefore, Lemma 2 follows easily from (13) and (19) together with the piecewise monotonicity of $g_2(x, p)$ with respect to x on $(0, 1)$. $\square$

Lemma 3. Let $g_1 : (0, 1) \times \mathbb{R} \rightarrow \mathbb{R}$ be defined by (12). Then the following statements are true:

- (1) $g_1(x, p) > 0$ for all $x \in (0, 1)$ if and only if $p \geq 1/2$;
- (2) $g_1(x, p) < 0$ for all $x \in (0, 1)$ if and only if $p \leq 1/3$;
- (3) if $1/3 < p < 1/2$, then there exists $\mu_0 \in (0, 1)$ such that $g_1(\mu_0, p) = 0$, $g_1(x, p) < 0$ for $x \in (0, \mu_0)$, and $g_1(x, p) > 0$ for $x \in (\mu_0, 1)$.

Proof. (1) If $g_1(x, p) > 0$ for all $x \in (0, 1)$, then $g_1(0^+, p) \geq 0$. Therefore, $p \geq 1/2$ follows from $g_1(0^+, p) = -\infty$ for $p < 1/2$.

If $p \geq 1/2$, then Lemma 1 (1) leads to the conclusion that $g_1(x, p) > g_1(1, p) = 0$ for all $x \in (0, 1)$.

(2) If $g_1(x, p) < 0$ for all $x \in (0, 1)$, then by making use of L'Hôpital's rules and (12) we get

$$\lim_{x \rightarrow 1^-} \frac{g_1(x, p)}{(1-x)^3} = \frac{1}{8} \left(p - \frac{1}{3}\right) \leq 0 \quad (26)$$

and $p \leq 1/3$.

If $p \leq 1/3$, then Lemma 1 (2) leads to the conclusion that $g_1(x, p) < g_1(1, p) = 0$ for all $x \in (0, 1)$.

(3) If $1/3 < p < 1/2$, then it follows from (12) that

$$g_1(0^+, p) = -\infty, \quad g_1(1, p) = 0. \quad (27)$$

Therefore, Lemma 3 (3) follows from Lemma 2 and (27). $\square$

Lemma 4. Let $g : (0, 1) \times (0, \infty) \rightarrow \mathbb{R}$ be defined by

$$\begin{aligned} g(x, p) &= \log \frac{X(1, x)}{M_p(1, x)} \\ &= \log \frac{x+1}{2} + \frac{2\sqrt{x}}{1-x} \arcsin \frac{1-x}{1+x} \\ &\quad - \frac{1}{p} \log \frac{x^p+1}{2} - 1. \end{aligned} \quad (28)$$

Then

- (1) $g(x, p)$ is strictly increasing with respect to x on $(0, 1)$ if and only if $p \geq 1/2$;
- (2) $g(x, p)$ is strictly decreasing with respect to x on $(0, 1)$ if and only if $p \leq 1/3$;
- (3) if $1/3 < p < 1/2$, there exists $\mu_0 \in (0, 1)$ such that $g(x, p)$ is strictly decreasing with respect to x on $(0, \mu_0]$ and strictly increasing with respect to x on $[\mu_0, 1)$.

Proof. It follows from (28) that

$$\frac{\partial g(x, p)}{\partial x} = \frac{1+x}{(1-x)^2 \sqrt{x}} g_1(x, p), \quad (29)$$

where $g_1(x, p)$ is defined by (12).

Therefore, Lemma 4 follows from Lemma 3 and (29). $\square$

3. Main Results

Theorem 5. *The double inequality*

$$M_p(a, b) < X(a, b) < M_q(a, b) \quad (30)$$

holds for all $a, b > 0$ with $a \neq b$ if and only if $p \leq 1/3$ and $q \geq \log 2/(1 + \log 2) = 0.4093 \dots$

Proof. Since both the Sándor mean $X(a, b)$ and r th power mean $M_r(a, b)$ are symmetric and homogeneous of degree 1, without loss of generality, we assume that $a = 1$ and $b = x \in (0, 1)$.

We first prove that the inequality $X(1, x) > M_p(1, x)$ holds for all $x \in (0, 1)$ if and only if $p \leq 1/3$.

If $p = 1/3$, then from (28) and Lemma 4 (2) we get

$$\log \frac{X(1, x)}{M_{1/3}(1, x)} = g\left(x, \frac{1}{3}\right) > g\left(1^-, \frac{1}{3}\right) = 0 \quad (31)$$

for all $x \in (0, 1)$.

Therefore, $X(1, x) > M_p(1, x)$ for all $x \in (0, 1)$ and $p \leq 1/3$ follows from (31) and the monotonicity of the function $p \rightarrow M_p(1, x)$.

If $X(1, x) > M_p(1, x)$, then (28) leads to $g(x, p) > 0$ for all $x \in (0, 1)$. In particular, we have

$$\lim_{x \rightarrow 1^-} \frac{g(x, p)}{(1-x)^2} = \frac{1}{8} \left(\frac{1}{3} - p \right) \geq 0 \quad (32)$$

and $p \leq 1/3$.

Next, we prove that the inequality $X(1, x) < M_q(1, x)$ holds for all $x \in (0, 1)$ if and only if $q \geq \log 2/(1 + \log 2)$.

If $X(1, x) < M_q(1, x)$ holds for all $x \in (0, 1)$, then (28) leads to $g(x, q) < 0$ for all $x \in (0, 1)$. In particular, we have

$$g(0, q) = \left(\frac{1}{q} - 1 \right) \log 2 - 1 \leq 0 \quad (33)$$

and $q \geq \log 2/(1 + \log 2)$.

If $q = \log 2/(1 + \log 2) \in (1/3, 1/2)$, then (28) leads to

$$g\left(0, \frac{\log 2}{1 + \log 2}\right) = g\left(1, \frac{\log 2}{1 + \log 2}\right) = 0. \quad (34)$$

It follows from (28) and (34) together with Lemma 4 (3) that

$$\log \frac{X(1, x)}{M_{\log 2/(1 + \log 2)}(1, x)} = g\left(x, \frac{\log 2}{1 + \log 2}\right) < 0 \quad (35)$$

for all $x \in (0, 1)$.

Therefore, $X(1, x) < M_q(1, x)$ for all $x \in (0, 1)$ and $q \geq \log 2/(1 + \log 2)$ follows from (35) and the monotonicity of the function $q \rightarrow M_q(1, x)$. $\square$

Theorem 6. *Let $a, b > 0$ with $a \neq b$. Then the double inequality*

$$\frac{2}{e} M_{1/2}(a, b) < X(a, b) < \frac{4}{e} M_{1/3}(a, b) \quad (36)$$

holds with the best possible constants $2/e$ and $4/e$.

Proof. Since both the Sándor mean $X(a, b)$ and r th power mean $M_r(a, b)$ are symmetric and homogeneous of degree 1, without loss of generality, we assume that $a = 1$ and $b = x \in (0, 1)$. It follows from Lemma 4 (1) and (2) together with (28) that

$$\begin{aligned} \log \frac{X(1, x)}{M_{1/2}(1, x)} &= g\left(x, \frac{1}{2}\right) > g\left(0, \frac{1}{2}\right) = \log \frac{2}{e}, \\ \log \frac{X(1, x)}{M_{1/3}(1, x)} &= g\left(x, \frac{1}{3}\right) < g\left(0, \frac{1}{3}\right) = \log \frac{4}{e} \end{aligned} \quad (37)$$

for all $x \in (0, 1)$.

Therefore, $2/e M_{1/2}(1, x) < X(1, x) < 4/e M_{1/3}(1, x)$ for all $x \in (0, 1)$ follows from (37), and the optimality of the parameters $2/e$ and $4/e$ follows from the monotonicity of the functions $g(x, 1/2)$ and $g(x, 1/3)$. $\square$

Remark 7. For all $a_1, a_2, b_1, b_2 > 0$ with $a_1/b_1 < a_2/b_2 < 1$. Then from Lemma 4 (1) and (2) together with (28) we clearly see that the Ky Fan type inequalities

$$\frac{M_p(a_2, b_2)}{M_p(a_1, b_1)} < \frac{X(a_2, b_2)}{X(a_1, b_1)} < \frac{M_q(a_2, b_2)}{M_q(a_1, b_1)} \quad (38)$$

hold if and only if $p \geq 1/2$ and $q \leq 1/3$.

Let $p \in \mathbb{R}$ and $L_p(a, b) = (a^{p+1} + b^{p+1})/(a^p + b^p)$ be the p th Lehmer mean of two positive real numbers a and b . Then the function $g_1(x, p)$ defined by (12) can be rewritten as

$$g_1(x, p) = \frac{1}{2} (1-x) \left[\frac{1}{P(1, x)} - \frac{G(1, x)}{A(1, x) L_{p-1}(1, x)} \right]. \quad (39)$$

From Lemma 3 and (39) we get Remark 8 as follows.

Remark 8. The double inequality

$$\frac{A(a, b)}{G(a, b)} L_{p-1}(a, b) < P(a, b) < \frac{A(a, b)}{G(a, b)} L_{q-1}(a, b) \quad (40)$$

holds for all $a, b > 0$ with $a \neq b$ if and only if $p \leq 1/3$ and $q \geq 1/2$.

From (5) and (9) together with Theorem 5 one has the following.

Remark 9. The inequalities

$$\begin{aligned} L(a, b) &< M_{1/3}(a, b) < X(a, b) < M_{\log 2/(1 + \log 2)}(a, b) \\ &< M_{\log 2/\log \pi}(a, b) < P(a, b) \end{aligned} \quad (41)$$

hold for all $a, b > 0$ with $a \neq b$.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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References

- [1] P. S. Bullen, D. S. Mitrinovic, and P. M. Vasić, *Means and Their Inequalities*, D. Reidel, Dordrecht, The Netherlands, 1988.
- [2] H.-J. Seiffert, "Aufgabe β_{16} ," *Die Wurzel*, vol. 29, pp. 221–222, 1995.
- [3] A. A. Jagers, "Solution of problem 887," *Nieuw Archief voor Wiskunde*, vol. 12, no. 4, pp. 230–231, 1994.
- [4] P. A. Hästö, "A monotonicity property of ratios of symmetric homogeneous means," *Journal of Inequalities in Pure and Applied Mathematics*, vol. 3, no. 5, article 71, 23 pages, 2002.
- [5] P. A. Hästö, "Optimal inequalities between Seiffert's mean and power means," *Mathematical Inequalities & Applications*, vol. 7, no. 1, pp. 47–53, 2004.
- [6] A. Witkowski, "Interpolations of Schwab-Borchardt mean," *Mathematical Inequalities & Applications*, vol. 16, no. 1, pp. 193–206, 2013.
- [7] I. Costin and G. Toader, "Optimal evaluations of some Seiffert-type means by power means," *Applied Mathematics and Computation*, vol. 219, no. 9, pp. 4745–4754, 2013.
- [8] Y.-M. Chu and B.-Y. Long, "Bounds of the Neuman-Sándor mean using power and identric means," *Abstract and Applied Analysis*, vol. 2013, Article ID 832591, 6 pages, 2013.
- [9] H. Alzer, "Ungleichungen für mittelwerte," *Archiv der Mathematik*, vol. 47, no. 5, pp. 422–426, 1986.
- [10] H. Alzer, "Ungleichungen für $(e/a)^a(b/e)^b$," *Elementary Mathematics*, vol. 40, pp. 120–123, 1985.
- [11] F. Burk, "The geometric, logarithmic, and arithmetic mean inequality," *The American Mathematical Monthly*, vol. 94, no. 6, pp. 527–528, 1987.
- [12] T. P. Lin, "The power mean and the logarithmic mean," *The American Mathematical Monthly*, vol. 81, pp. 879–883, 1974.
- [13] A. O. Pittenger, "Inequalities between arithmetic and logarithmic means," *Publikacije Elektrotehničkog Fakulteta. Serija Matematika i Fizika*, no. 678–715, pp. 15–18, 1980.
- [14] A. O. Pittenger, "The symmetric, logarithmic and power means," *Publikacije Elektrotehničkog Fakulteta. Serija Matematika i Fizika*, no. 678–715, pp. 19–23, 1980.
- [15] K. B. Stolarsky, "The power and generalized logarithmic means," *The American Mathematical Monthly*, vol. 87, no. 7, pp. 545–548, 1980.
- [16] H. Alzer and S.-L. Qiu, "Inequalities for means in two variables," *Archiv der Mathematik*, vol. 80, no. 2, pp. 201–215, 2003.
- [17] J. Sándor, "Two sharp inequalities for trigonometric and hyperbolic functions," *Mathematical Inequalities & Applications*, vol. 15, no. 2, pp. 409–413, 2012.
- [18] J. Sándor, "On two new means of two variables," *Notes on Number Theory and Discrete Mathematics*, vol. 20, no. 1, pp. 1–9, 2014.
- [19] J. Sándor, "On certain inequalities for means III," *Archiv der Mathematik*, vol. 76, no. 1, pp. 34–40, 2001.